Secondary Invariants for String Bordism and Topological Modular Forms

Ulrich Bunke, Niko Naumann

http://arxiv.org/abs/0912.4875

B - closed oriented manifold of dimension 8.

$$(1)\ \mathrm{sign}(B) = \frac{1}{45} < 7p_2(TB) - p_1^2(TB), [B] > \in \mathbb{Z}.$$

$$(*)\ 2p_1^2(TB) - \mathrm{sign}(B) \equiv 0\,(7).$$

If $B = B_1 \amalg_M B_2$ with $H^3(M,\mathbb{Z}) = H^4(M,\mathbb{Z}) = 0$ then $\mathrm{sign}(B) = \mathrm{sign}(B_1) - \mathrm{sign}(B_2)$ and $p_1^2(TB) = \hat{p}_1^2(TB_1) - \hat{p}_1^2(TB_2)$ hence

$$\lambda(M) := [2 < \hat{p}_1^2(TB_1), [B_1, M] > - \mathrm{sign}(B_1)] \in \mathbb{Z}/7\mathbb{Z}$$

is independent of the choice of B_1 such that $M \simeq \partial B_1$ (note: $MSO_7 = 0$).
(*geometric* description of λ)

Intrinsic description of $\lambda(M)$ I

B - oriented of dimension 8 with boundary $M = \partial B$.

$$(1') \ \mathrm{sign}(B) = \frac{1}{45}\int_B (7p_2(\nabla^{TB}) - p_1^2(\nabla^{TB})) - \eta(M),$$

$\eta(M)$ denoting the eta-invariant of the signature operator on M.
Chosing $\rho \in \Omega^3(M)$ with $p_1(\nabla^{TB})|_M = d\rho$ and a cut-off function $\chi \in \mathcal{C}^\infty(B)$ for a collar of $M \subseteq B$, the form

$$p_1(\nabla^{TB}, \alpha) := p_1(\nabla^{TB}) - d(\chi\rho) \in \Omega^4_c(B)$$

represents $\hat{p}_1(TB) \in H^4(B, M; \mathbb{Z})$ and Stoke's theorem gives

$$\int_B p_1^2(\nabla^{TB}) =< \hat{p}_1^2(TB), [B, M] > + \int_M \rho \wedge p_1(\nabla^{TM})$$

and consequently

$$\lambda(M) = [\int_B 7p_2(\nabla^{TB}) - \int_M \rho \wedge p_1(\nabla^{TM}) - 45\eta(M)] \in \mathbb{Z}/7\mathbb{Z}.$$

Intrinsic description of $\lambda(M)$ II

Similarly, the first summand in

$$\lambda(M) = [\int_B 7p_2(\nabla^{TB}) - \int_M \rho \wedge p_1(\nabla^{TM}) - 45\eta(M)] \in \mathbb{Z}/7\mathbb{Z}$$

can be expressed intrinsically in terms of M leading to
$\lambda(M) = [\int_M (420\theta \wedge c(\nabla^{TM})^2 - 246\rho \wedge p_1(\nabla^{TM}) - 210c(\nabla^{TM})^2 \wedge \rho) - 45\eta(M) - 10080\eta(M^c)] \in \mathbb{Z}/7\mathbb{Z}$,
(*analytic* description of λ)
where:

- $c \in H^2(M, \mathbb{Z})$ is the characteristic class of a chosen *Spin*^{c}-structure on M.
- $\theta \in \Omega^3(M)$ solves $c(\nabla^{TM})^2|_M = d\theta$ and
- $\eta(M^c)$ is the eta-invariant of the *Spin*^{c}-Dirac operator on M.

Overview - Constructions

For all $m \geq 1$ we will define a group homomorphism

$$b^{an} : MString_{4m-1} \longrightarrow T_{2m}$$

using spectral invariants of twisted Dirac operators. b^{an} will be a secondary version of the Witten genus.
We will be able to evaluate b^{an} restricted to

$$A_{4m-1} := \ker(MString_{4m-1} \xrightarrow{j} MSpin_{4m-1}) \subseteq MString_{4m-1}$$

by equating it $b^{an}|_{A_{4m-1}} = b^{geom} = b^{top}$ with homomorphisms constructed using differential geometry and homotopy theory, respectively.

There will be a factorization

$$\begin{tikzcd} MString_{4m-1} \arrow[r, two heads, "\sigma"] & tmf_{4m-1} \arrow[d, dotted, "b^{tmf}"] \\ A_{4m-1} \arrow[u, hook] \arrow[r, "b^{top}"] & T_{2m} \end{tikzcd}$$

and b^{tmf} will be made explicit using the known structure of tmf_*. Recalling the Witten genus (Ando/Hopkins/Rezk) σ is onto (Hopkins/Mahowald) this will imply in particular non-triviality of b^{top} for infinitely many values of m.

Witten genus I

The Witten genus

$$R : MSpin_{4m} \to KO[[q]]_{4m} \simeq \mathbb{Z}[[q]]$$

is given by its characteristic series

$$\Phi \in \mathbb{Q}[[q]][[p_1, p_2, \ldots]]$$

as $R([M]) =$

$$\kappa_m \int_M \Phi(p_1(\nabla^{TM}), p_2(\nabla^{TM}), \ldots) \overset{(AS)}{=} \kappa_m \sum_{n \geq 0} q^n \cdot \mathrm{index}(D_M \otimes R_n(TM)),$$

where κ_m is 1 for m even and $\frac{1}{2}$ for m odd.

Witten genus II

There is a factorization

$$\begin{tikzcd}
 & & & \mathcal{M}_{2m}^{\mathbb{Z}} \arrow[d, hook, "\text{(q-exp.)}"] \\
MString_{4m} \arrow[r, "j"'] \arrow[urrr, dotted] & MSpin_{4m} \arrow[r, "R"'] & KO[[q]]_{4m} \arrow[r, "\cong"'] & \mathbb{Z}[[q]],
\end{tikzcd}$$

$\mathcal{M}_{2m}^{\mathbb{Z}}$ denoting modular forms of weight $2m$.
We have explicitely

$$\Phi(p_1,\dots) = \exp\left[\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k} N_{2k}(p_1,\dots)\right] e^{G_2 p_1} \in \mathbb{Q}[[q]][[p_1,\dots]]$$

and define

$$\tilde{\Phi}(p_1,\dots,) := \exp\left[\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k} N_{2k}(p_1,\dots)\right] \sum_{j=1}^{\infty} \frac{G_2^j p_1^{j-1}}{j!}.$$

target group

For all $m \geq 1$, the target of our invariants b^{an}, b^{geom} and b^{top} will be the quotient group

$$T_{2m} := \frac{\mathbb{R}[[q]]}{\mathbb{Z}[[q]] + \mathcal{M}_{2m}^{\mathbb{R}}}.$$

Interlude - structure of A_{4m-1}

For all $m \geq 1$,

- (Hovey) $MString_{4m-1}$ is finite consisting of 2- and 3-torsion.
- (Anderson/Brown/Peterson) $MSpin_{4m-1}$ is a finite $\mathbb{F}_2$-vector space and is zero for $m \leq 9$.
- The quotient $MString_{4m-1}/A_{4m-1}$ is finite 2-primary and zero for $m \leq 9$, i.e. up to dimension 35.

We will see that

$$b^{top} : A_{4m-1} \longrightarrow T_{2m}$$

detects all 3-torsion and part of the 2-torsion of A_{4m-1}.

Definition of b^{an}

Let $m \geq 1$ and M a $(4m-1)$-dimensional closed String manifold. Choose a metric on M and $H \in \Omega^3(M) : dH = \frac{p_1}{2}(\nabla^{TM})$.

Lemma

The following is well-defined and a group homomorphism:

$$b^{an} : MString_{4m-1} \longrightarrow T_{2m} \, , \; [M] \mapsto$$

$$[2\kappa_m \int_M H \wedge \tilde{\Phi}(\nabla^{TM}) + \kappa_m \sum_{n \geq 0} q^n \cdot \eta((\mathcal{M} \otimes R_n(TM \oplus \mathbb{R}))_t)]$$

This is essentially an applcation of the Atiyah/Patodi/Singer Index Theorem.

b^{geom} - using a Spin zero-bordism

Assuming $[M] \in A_{4m-1} \subseteq String_{4m-1}$ there is a Spin-manifold Z such that $M = \partial Z$.
An index formula shows that the η-invariants in the definition of b^{an} differ by integers from a characterisic class on Z, showing $b^{an}|_{A_{4m-1}} = b^{geom}$ defined by

$$b^{geom} : A_{4m-1} \longrightarrow T_{2m},$$

$$[M] \mapsto [2\kappa_m \int_M H \wedge \tilde{\Phi}(\nabla^{TM}) - \kappa_m \int_Z \Phi(\nabla^{TZ})].$$

Construction of b^{top} - motivating example

A very short summary of the construction of Adams' e-invariant:

$$
\begin{tikzcd}
 & \Sigma^{-1}\overline{MU} \arrow[r] \arrow[d] & \Sigma^{-1}K\mathbb{Q}/\mathbb{Z} \\
S^{2m-1} \arrow[r, "x"] \arrow[ur, dotted] \arrow[urr, dashed, "e(x)"'] \arrow[dr, "0"] & S \arrow[d] & \\
 & MU, &
\end{tikzcd}
$$

where the map $\Sigma^{-1}\overline{MU} \longrightarrow \Sigma^{-1}K\mathbb{Q}/\mathbb{Z}$ is derived from the complex orientation $MU \longrightarrow K$ of complex K-theory.

Construction of b^{top}

We contemplate

$$\begin{tikzcd}
 & \Sigma^{-1}F \arrow[r] \arrow[d] \arrow[rr, bend left, "a"] & \Sigma^{-1}G \arrow[r] \arrow[d] & \Sigma^{-1}G_{\mathbb{Q}} \arrow[d] \\
S^{4m-1} \arrow[r, "{[M]}"] \arrow[ur, dotted, "Z"] \arrow[dr, "0"'] & MString \arrow[r, "\sigma"] \arrow[d, "j"] & tmf \arrow[r] \arrow[d] & tmf_{\mathbb{Q}} \arrow[d] \\
 & MSpin \arrow[r] & ko[[q]] \arrow[r] & ko[[q]]_{\mathbb{Q}}
\end{tikzcd}$$

and see
that the following map $b^{top} : A_{4m-1} \longrightarrow T_{2m}$ is well-defined and a group homomorphism:

$$[M] \mapsto a \circ Z \in G_{\mathbb{Q},4m} \simeq \frac{\mathbb{Z}[[q]] \otimes \mathbb{Q}}{\mathcal{M}_{2m}^{\mathbb{Q}}} \xrightarrow{can} T_{2m}.$$

Comparison

For all $m \geq 1$ we can show

$$b^{geom} = b^{top} : A_{4m-1} \longrightarrow T_{2m}$$

and it is easy to construct

$$b^{tmf} : tmf_{4m-1} \longrightarrow T_{2m}$$

similarly as above factoring b^{top} through the (refined) Witten genus

$$\sigma : MString_{4m-1} \longrightarrow tmf_{4m-1}.$$

Computations at 3

m	*name*	ord	${}^{Y}E_2^{*,*}(tmf_{(3)})$	$b^{tmf}(\ldots)$
1	ν	3	$[\nu]_2$	$[\frac{2}{3}]$
7	$\nu\Delta$	3	$[\nu]_2\Delta_2$	$[\frac{2}{3}\Delta]$

The table gives the complete list of additive generators of $tmf_{(3),4m-1}$ for $4m-1 < 75$. It continues 72-periodicially under multiplication with Δ^3.
For all $m \geq 1$ $b^{tmf}: tmf_{(3),4m-1} \hookrightarrow T_{(3),2m}$ is injective.

- For all $m \geq 1$, $b^{tmf}(\mathrm{Filt}^4(tmf_{(2),4m-1})) = 0$.
- For all $m \geq 1$ such that $(m \bmod 48) \in \{1, 7, 13, 25, 31, 37\}$ we have an injection $\bar{b}^{tmf} : tmf_{(2),4m-1}/\mathrm{Filt}^4(tmf_{(2),4m-1}) \hookrightarrow T_{(2),2m}$.
- For all other $m \geq 1$ we have $b^{tmf}(tmf_{(2),4m-1}) = 0$.

Computations at 2 - II

m	$name$	ord	${}^{Y}E_2^{*,*}(tmf_{(2)})$	$b^{tmf}(\dots)$	$c \in \{\dots\}$
1	ν	8	$[\nu]_2$	$[\frac{3}{8}]$	
7	$2\nu\Delta$	4	$2[\nu]_2\Delta_2$	$[\frac{c}{4}\Delta]$	$1,3$
13	$\nu\Delta^2$	8	$[\nu]_2\Delta_2^2$	$[\frac{c}{8}\Delta^2]$	$1,5$
25	$\nu\Delta^4$	8	$[\nu]_2\Delta_2^4$	$[\frac{c}{8}\Delta^4]$	$1,5$
31	$2\nu\Delta^5$	4	$2[\nu]_2\Delta_2^5$	$[\frac{c}{4}\Delta^5]$	$1,3$
37	$\nu\Delta^6$	8	$[\nu]_2\Delta_2^6$	$[\frac{c}{8}\Delta^6]$	$1,5$
4	ηa_{14}	2	$[\eta]_2[a_{14}]_2$	0	
10	ηa_{38}	2	$[\eta]_2[a_{38}]_2$	0	
19	ηa_{74}	2	$[\eta]_2[a_{74}]_2$	0	
29	ηa_{110}	2	$[\eta]_2[a_{110}]_2$	0	
34	ηa_{134}	2	$[\eta]_2[a_{134}]_2$	0	

The second column is a complete list of additive generators of

$$tmf_{(2),4m-1}/\mathrm{Filt}^4(tmf_{(2),4m-1})$$

in this range of dimension and there is a 192-periodicity.

Closer look at A_{4m-1}

Recall $A_{4m-1} = \ker(MString_{4m-1} \xrightarrow{j} MSpin_{4m-1})$.

One has $\dim_{\mathbb{F}_2}(MSpin_{4m-1}) = 0, \ldots, 0\,(m = 9)$, $1\,(m = 10), 2, 4, 7, 22, \ldots, 17493\,(m = 32)$.

Using results of Mahowald/Gorbunov one can check $A_{39} = MString_{39}$ (the case $m = 10$) .

In general we have the following alternative.

Open problem (either way)

$$\begin{tikzcd}
MString_{4m-1} \arrow[r, "\sigma"] \arrow[dr, "b^{an}"'] & tmf_{4m-1} \arrow[d, "b^{tmf}"] \arrow[dl, phantom, "(commutes?)"] \\
A_{4m-1} \arrow[u, hook, "(equality?)"] \arrow[r, "b^{top}"'] & T_{2m}.
\end{tikzcd}$$