THH of Waldhausen categories and the localization sequence for $THH(ku)$

Andrew J. Blumberg (blumberg@math.utexas.edu)

In order to study the localization sequence

$$K(k) \to K(A) \to K(K)$$

via trace methods, Hesselholt-Madsen established a localization cofiber sequence

$$THH(k) \to THH(A) \to THH(A|K),$$

- A be a discrete valuation ring,
- K its quotient field (with characteristic 0),
- k the residue field (with characteristic p),

which sits in a commutative diagram

$$\begin{tikzcd}
K(k) \arrow[r] \arrow[d] & K(A) \arrow[r] \arrow[d] & K(K) \arrow[d] \\
THH(k) \arrow[r] & THH(A) \arrow[r] & THH(A|K).
\end{tikzcd}$$

Question: what is $THH(A|K)$?

The localization sequence arises from the sequence of categories

$$(C_z^b(A))^q \to C_z^b(A) \to C_q^b(A),$$

- $C_z^b(A)$ is the Waldhausen category of bounded complexes of f.g. projective A-modules and quasi-isomorphisms,
- $C_q^b(A)$ denotes the same category with rational quasi-isomorphisms as weak equivalences,
- and $(C_z^b(A))^q$ is the rationally acyclic objects in $C_z^b(A)$; identified as $K(k)$ and $THH(k)$ via devissage.

Discrepancy: $K(C_q^b(A)) \simeq K(K)$ (approximation theorem), but "$THH(C_q^b(A))$" is something different — apparent failure of the approximation theorem.

Hard to reconcile with the general theory of localization in *THH*:

When

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

is a sequence of pre-triangulated spectral categories such that

$$\mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{B}) \to \mathrm{Ho}(\mathcal{C})$$

is a quotient sequence (i.e., the map from the Verdier quotient $\mathrm{Ho}(\mathcal{B})/\mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{C})$ is cofinal), then there exists a localization cofiber sequence

$$THH(\mathcal{A}) \to THH(\mathcal{B}) \to THH(\mathcal{C}).$$

The data for Waldhausen's localization theorem (i.e., two categories of weak equivalences $v\mathcal{C} \subset w\mathcal{C}$ and acyclics $\mathcal{C}^v$) encodes the same Bousfield localization (Neeman).

Clues and questions:

- The *THH* of spectral categories is a non-connective theory. (Must be, in order to have general localization sequences.)
- But the Hesselholt-Madsen construction (based on work of Dundas-McCarthy) is connective!
- Further, appears to be a mismatch of inputs — spectral categories vs. categories with weak equivalences.

Overview:

This talk describes joint work with Mike Mandell aimed at understanding what's going on here.

- Provide a theory of THH of Waldhausen categories, $WTHH$:
  1. Connective ($WTHH^{\Gamma}$) and non-connective ($WTHH$) variants.
  2. $WTHH^{\Gamma}$, $WTHH$ agrees with usual THH for rings and connective ring spectra.
  3. Cyclotomic trace $K(\mathcal{C}) \to WTHH^{\Gamma}(\mathcal{C})$.
  4. "Theorems of K-theory" hold for $WTHH^{\Gamma}$, $WTHH$.
- Two localization sequences:
  1. A connective localization sequence for $WTHH^{\Gamma}$: devissage sometimes allows interpretation of left-hand term. (Recovers Hesselholt-Madsen.)
  2. A non-connective localization sequence for $WTHH$ which is consistent with the "general theory of localization" in THH: allows interpretation of right-hand term.

Principal application:

Establish the THH localization sequences

$$THH(\mathbb{Z}) \to THH(\ell) \to WTHH^{\Gamma}(\ell|L)$$

and

$$THH(\mathbb{Z}) \to THH(ku) \to WTHH^{\Gamma}(ku|KU)$$

conjectured by Ausoni-Rognes and Hesselholt.

1. Prove a devissage theorem to identify left-hand term.
2. Provide a definition of right-hand term.

Right-hand terms should be connected to Rognes' "log THH".

Begin with *THH* of a spectral category:

Definition

A *spectral category* is a category enriched over symmetric spectra. Specifically, a spectral category $\mathcal{C}$ consists of:

1. A collection of objects $\mathrm{ob}\,\mathcal{C}$ (which may form a proper class),
2. A symmetric spectrum $\mathcal{C}(a, b)$ for each pair of objects $a, b \in \mathrm{ob}\,\mathcal{C}$,
3. A unit map $S \to \mathcal{C}(a, a)$ for each object $a \in \mathrm{ob}\,\mathcal{C}$, and
4. A composition map $\mathcal{C}(b, c) \wedge \mathcal{C}(a, b) \to \mathcal{C}(a, c)$ for each triple of objects $a, b, c \in \mathrm{ob}\,\mathcal{C}$,

satisfying the usual associativity and unit properties.

THH of a spectral category:

Define THH as the Hochschild-Mitchell cyclic bar construction:

Definition

For a small spectral category $\mathcal{C}$ and $(\mathcal{C},\mathcal{C})$-bimodule $\mathcal{M}$, let

$$N_q^{\mathrm{cy}}(\mathcal{C};\mathcal{M}) = \bigvee \mathcal{C}(c_{q-1},c_q)\wedge\cdots\wedge\mathcal{C}(c_0,c_1)\wedge\mathcal{M}(c_q,c_0),$$

where the sum is over the $(q+1)$-tuples $(c_0,\ldots,c_q)$ of objects of $\mathcal{C}$. This becomes a simplicial object using the usual cyclic bar construction face and degeneracy maps.
Define

$$THH(\mathcal{C}) = |N_\bullet^{\mathrm{cy}}(\mathcal{C};\mathcal{C})|.$$

(Really use Bökstedt's model, but we'll suppress that in this talk. Note though that all statements hold for TR, TC as well.)

Basic strategy for *THH* of Waldhausen categories:

- Let $\mathcal{C}$ be a Waldhausen category with weak equivalences $w\mathcal{C}$. Then $\mathcal{C}$ has coproducts.
- Dwyer-Kan simplicial localization builds "homotopically correct" mapping spaces $L^H\mathcal{C}(-,-)$ from $\mathcal{C}$ and $w\mathcal{C}$.

Then (morally) we can enrich $\mathcal{C}$ in Γ-spaces:

$$[n] \mapsto L^H\mathcal{C}(X, \coprod_n Y).$$

If $\mathcal{C}$ furthermore admits tensors with finite simplicial sets, then we can enrich in symmetric spectra:

$$n \mapsto L^H\mathcal{C}(X, \Sigma^n Y)$$

(Warning: this doesn't work as stated, coherence problems.)

Furthermore, for $\mathcal{C}$ a suitable Waldhausen category, we can form: $S_\bullet \mathcal{C}$ and $N_\bullet^w S_\bullet \mathcal{C}$ (nerve of weak equivalences) as spectral categories.

Then roughly (following Dundas-McCarthy):

$$WTTH^\Gamma(\mathcal{C}) = THH(N_\bullet^w S_\bullet \mathcal{C}^\Gamma)$$

and

$$WTHH(\mathcal{C}) = THH(N_\bullet^w S_\bullet \mathcal{C}^S).$$

Outline:

1. Pass from Waldhausen category $\mathcal{C}$ to spectral category.
2. Show how to construct $S_\bullet\mathcal{C}$ as spectral category.
3. Show how to construct $N^w_\bullet S_\bullet\mathcal{C}$ as spectral category.
4. Apply cyclic nerve.

Why the $N^w_\bullet$ and the $S_\bullet$?

- Needed for the cyclotomic trace.
- Also for the "theorems of K-theory", in the connective case.

So two theories: one connective, one non-connective. Will give rise to two localization sequences.

Waldhausen categories:

A category $\mathcal{C}$ with a zero object $*$ equipped with a subcategory of weak equivalences $w\mathcal{C}$ and a collection of cofibrations $\mathrm{cof}(\mathcal{C})$ such that:

1. The isomorphisms are weak equivalences and cofibrations,
2. For all A, $* \to A$ is a cofibration,
3. Pushouts along cofibrations exist,
4. Glueing: if in the diagram

$$\begin{tikzcd} A \arrow[d] & B \arrow[l] \arrow[r] \arrow[d] & C \arrow[d] \\ A' & B' \arrow[l] \arrow[r] & C', \end{tikzcd}$$

   the maps $A \to A'$, $B \to B'$ and $C \to C'$ are weak equivalences, so is the induced map

$$A \coprod_B C \to A' \coprod_{B'} C'.$$

Simplicially enriched Waldhausen categories:

Definition

A *simplicially enriched Waldhausen category* consists of a category $\mathcal{C} = \mathcal{C}_\bullet$ enriched in simplicial sets together with a Waldhausen category structure on $\mathcal{C}_0$ such that:

1. The zero object $*$ in $\mathcal{C}_0$ is a zero object for $\mathcal{C}$,
2. Pushouts over cofibrations in $\mathcal{C}_0$ are pushouts in $\mathcal{C}$,
3. Cofibrations $x \to y$ induce Kan fibrations $\mathcal{C}(y,z) \to \mathcal{C}(x,z)$ for all objects z, and
4. A map $x \to y$ is a weak equivalence if and only if $\mathcal{C}(y,z) \to \mathcal{C}(x,z)$ is a weak equivalence for all objects z if and only if $\mathcal{C}(z,x) \to \mathcal{C}(z,y)$ is a weak equivalence for all objects z.

An *enriched exact functor* between such categories is a simplicial functor $\phi \colon \mathcal{C} \to \mathcal{D}$ that restricts to an exact functor of Waldhausen categories $\mathcal{C}_0 \to \mathcal{D}_0$.

Strong condition. But contains many interesting examples:

Example

An exact category becomes a simplicially enriched Waldhausen category by regarding its mapping sets as discrete simplicial sets.

We also have the following less trivial examples.

Example

Subcategories of cofibrant objects in simplicial model categories in which all objects are fibrant (with natural simplicial enrichment).

1. Finite cell R-modules for an EKMM S-algebra R,
2. The category of finite cell modules over a simplicial ring A,
3. The category of simplicial objects on an abelian category with the "split-exact" model structure (cofibrations are the levelwise split monomorphisms, weak equivalences are the simplicial homotopy equivalences).

Tensors:

Examples often have even more structure:

Definition

A simplicially tensored Waldhausen category is a simplicially enriched Waldhausen category in which tensors with finite simplicial sets exist and satisfy the pushout-product axiom.

A tensored exact functor between simplicially tensored Waldhausen categories is a enriched exact functor that preserves tensors with finite simplicial sets.

Pushout-product axiom implies that for objects x and y in $\mathcal{C}$, the simplicial set $\mathcal{C}(x, y) \cong \mathcal{C}_0(x \otimes \Delta[\cdot], y)$.

Sometimes need to pass to subcategories, for instance in the localization theorem. This motivates the following definition:

Definition

A *enhanced simplicially enriched Waldhausen category* is a pair $\mathcal{A} \subset \mathcal{C}$ where $\mathcal{C}$ is a simplicially tensored Waldhausen category and $\mathcal{A}$ is a full subcategory such that $\mathcal{A}_0$ is a closed Waldhausen subcategory.

For $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{B} \subset \mathcal{D}$ enhanced simplicially enriched Waldhausen categories, an *enhanced exact* functor $\mathcal{A} \to \mathcal{B}$ is a tensored exact functor of simplicially tensored Waldhausen categories $\mathcal{C} \to \mathcal{D}$ that restricts to a functor $\mathcal{A} \to \mathcal{B}$.

Consistent with simplicial localization:

Enhanced simplicially enriched Waldhausen categories are compatible with the Dwyer-Kan simplicial localization, in the following sense:

Definition

Let $\mathcal{C}$ be a simplicially enriched Waldhausen category. We say that $\mathcal{C}$ is *DK-compatible* if for every object x, y in $\mathcal{C}$, the maps

$$\mathcal{C}(x,y) \to \operatorname{diag} L\mathcal{C}_\bullet(x,y) \longleftarrow L\mathcal{C}_0(x,y)$$

are weak equivalences of simplicial sets.

In fact, any enhanced simplicially enriched Waldhausen category is DK-compatible. (In particular, our various examples.)

Connective spectral enrichment:

Definition

Let $\mathcal{C}$ be a simplicially enriched Waldhausen category. Define $\mathcal{C}^{\Gamma}$, the Γ-category associated to $\mathcal{C}$, by mapping Γ-spaces

$$\mathcal{C}^{\Gamma}_q(x, y) = \mathcal{C}(x, \bigvee_q y).$$

Here the composition

$$\mathcal{C}^{\Gamma}_r(y, z) \wedge \mathcal{C}^{\Gamma}_q(x, y) \to \mathcal{C}^{\Gamma}_{qr}(x, z).$$

comes from the $\Sigma_q \wr \Sigma_r$-equivariant map

$$\mathcal{C}(y, \bigvee_r z) \to \prod_q \mathcal{C}(y, \bigvee_r z) \to \mathcal{C}(\bigvee_q y, \bigvee_{rq} z)$$

and composition

$$\mathcal{C}(\bigvee_q y, \bigvee_{rq} z) \wedge \mathcal{C}(x, \bigvee_q y) \to \mathcal{C}(x, \bigvee_{rq} z).$$

Recovers Dundas-McCarthy construction:

Example

For $\mathfrak{E}$ be an exact category, simplicially enriched as in Example 1,

$$\mathfrak{E}^{\Gamma}_q(x, y) = \mathfrak{E}(x, \bigoplus_{i=1}^{q} y) \cong \prod_{i=1}^{q} \mathfrak{E}(x, y).$$

Prolonging to symmetric spectra, we get

$$\mathfrak{E}^{\Gamma}(x, y)(n) = \mathfrak{E}(x, y) \otimes \tilde{\mathbb{Z}}[S^n],$$

where $\tilde{\mathbb{Z}}[X] = \mathbb{Z}[X]/\mathbb{Z}[*]$.

Non-connective spectral enrichment:

When $\mathcal{C}$ is a simplicially tensored Waldhausen category, let Σx be the cofiber of the map $x \otimes \partial I \to x \otimes I$, ($I$ is unit interval).

Definition

Let $\mathcal{A} \subset \mathcal{C}$ be an enhanced simplicially Waldhausen category. Define $\mathcal{A}^S$ be the spectral category with objects the objects of $\mathcal{A}$ and mapping symmetric spectra

$$\mathcal{A}^S(x, y)(n) = \mathcal{C}(x, \Sigma^n y).$$

$$\mathcal{A}^S(y, z) \wedge \mathcal{A}^S(x, y) \to \mathcal{A}^S(x, z)$$

comes from the $\Sigma_n \times \Sigma_m$-equivariant maps

$$\mathcal{C}(y, \Sigma^m z) \wedge \mathcal{C}(x, \Sigma^n y) \to \mathcal{C}(\Sigma^n y, \Sigma^{m+n} z) \wedge \mathcal{C}(x, \Sigma^n y) \to \mathcal{C}(x, \Sigma^{m+n} z).$$

(Might depend on ambient category!)

Often, $\mathcal{C}^\Gamma$ is the connective cover of $\mathcal{C}^S$.
In general, $\mathcal{C}^S$ captures the stable homotopy theory of $\mathcal{C}$:

Proposition

Let $\mathcal{A} \subset \mathcal{C}$ be enhanced simplicially enriched Waldhausen category.

1. *For any x, y in $\mathcal{A}$, the map $\mathcal{A}^S(x,y) \to \mathcal{A}^S(\Sigma x, \Sigma y)$ is a weak equivalence.*
2. *For a cofibration $f : a \to b$, cf the homotopy cofiber, and any object z the sequences*

$$\Omega\mathcal{A}^S(a,z) \longrightarrow \mathcal{A}^S(cf,z) \longrightarrow \mathcal{A}^S(b,z) \longrightarrow \mathcal{A}^S(a,z)$$
$$\mathcal{A}^S(z,a) \longrightarrow \mathcal{A}^S(z,b) \longrightarrow \mathcal{A}^S(z,cf) \longrightarrow \Sigma\mathcal{A}^S(z,a)$$

form a fiber sequence and a cofiber sequence in the stable category, respectively.

For instance, $\mathcal{C}^S$ is pre-triangulated if every object is equivalent to a suspension.

Mix in $S_\bullet$ construction:

Let $\mathrm{Ar}[n]$ denote lexicographically ordered pairs (i, j) where $0 \leq i \leq j \leq n$.

For a Waldhausen category $\mathcal{C}_0$, $S_n\mathcal{C}_0$ is the full subcategory of the category of functors $A\colon \mathrm{Ar}[n] \to \mathcal{C}_0$ such that for all i,

1. $a_{i,i} = *$,
2. $a_{i,j} \to a_{i,k}$ is a cofibration for all $j < k$, and
3. The square

$$\begin{tikzcd} a_{i,j} \arrow[r] \arrow[d] & a_{i,k} \arrow[d] \\ a_{j,j} \arrow[r] & a_{j,k} \end{tikzcd}$$

is cocartesian.

(A map in $S_\bullet\mathcal{C}_0$ is a natural transformation of functors.)

Becomes a Waldhausen category with pointwise weak equivalences and Reedy cofibrations.

Simplicially enriched $S_\bullet$ construction:

Definition

For a simplicially enriched Waldhausen category $\mathcal{C}$, let $S_n\mathcal{C}$ be the simplicial enriched category with objects the same as $S_n\mathcal{C}_0$ and simplicial sets of maps the simplicial set of natural transformations of functors $\mathrm{Ar}[n] \to \mathcal{C}$.

Simplicial mapping space computed via a limit (end) over the diagram.

Since the maps $a_{0,j} \to a_{0,j+1}$ are cofibrations and $a_{ij}/a_{ik} \cong a_{jk}$, we can identify the simplicial set of maps $S_n\mathcal{C}(A,B)$ as a pullback over fibrations

$$S_n\mathcal{C}(A,B) \cong \mathcal{C}(a_{0,1}, b_{0,1}) \times_{\mathcal{C}(a_{0,1},b_{0,2})} \cdots \times_{\mathcal{C}(a_{0,n-1},b_{0,n})} \mathcal{C}(a_{0,n}, b_{0,n}).$$

Proposition

Let $\mathcal{C}$ be a simplicially enriched Waldhausen category. Then:

1. *$S_n\mathcal{C}$ is a simplicially enriched Waldhausen category.*
2. *If $\mathcal{C}$ is is simplicially tensored or enhanced, then so is $S_n\mathcal{C}$.*
3. *The face and degeneracy maps $S_m\mathcal{C} \to S_n\mathcal{C}$ are enriched exact and are tensored exact or enhanced exact when $\mathcal{C}$ is simplicially tensored or enhanced.*

Moreover, S_n preserves enriched exact, tensored exact, and enhanced exact functors.

First definition of THH of Waldhausen categories:

Definition

For a simplicially enriched Waldhausen category $\mathcal{C}$, we define

$$WTHH^{\Gamma}(\mathcal{C}) = \Omega|THH(S_{\bullet}\mathcal{C}^{\Gamma})|$$

If $\mathcal{C}$ is a simplicially tensored Waldhausen category and $\mathcal{A} \subset \mathcal{C}$ is an enhanced simplicial Waldhausen category, then we define

$$WTHH(\mathcal{A}) = \Omega|THH(S_{\bullet}\mathcal{A}^{S})|$$

Mix in the nerve of the weak equivalences:

First guess: Define $N^w_\bullet \mathcal{C}$ as the nerve category (objects are sequences $C_0 \to C_1 \to \ldots$ with maps weak equivalences, morphisms all natural transformations).

Doesn't work.

- Harder than the $S_\bullet$ construction — there, cofibration condition ensures end computing mapping space is homotopy end (pullback is homotopy pullback). Here, wrong homotopy type.
- Idea: use Moore version (including the homotopies), via the Tot construction of McClure-Smith and prismatic subdivision.
- (Also works for $S'_\bullet$ construction, the homotopical variant of the $S_\bullet$ construction in which cocartesian is replaced by homotopy cocartesian.)

Finally:

Definition

Let $\mathcal{C}$ be a simplicially enriched Waldhausen category, and let $v\mathcal{C}_0$ be a subcategory of $\mathcal{C}_0$ containing all the isomorphisms and satisfying mild hypotheses, define

$$WTHH^{\Gamma}(\mathcal{C}|v) = \Omega|THH(v_{\bullet}^{M} S_{\bullet}\mathcal{C}^{\Gamma})|$$

and

$$WTHH(\mathcal{A}|v) = \Omega|THH(v_{\bullet}^{M} S_{\bullet}\mathcal{A}^{S})|$$

Consistency: For $\mathcal{C}$ a simplicially enriched Waldhausen category and $\mathcal{A}$ an enhanced simplicial enriched Waldhausen category, the maps

$$WTHH^{\Gamma}\mathcal{C} \to WTHH^{\Gamma}(\mathcal{C}|w) \qquad WTHH\mathcal{A} \to WTHH(\mathcal{A}|w)$$

are weak equivalences.

Payoff for all this work:

The theorems of K-theory:

- Additivity theorem: most important structural theorem about "theories like K-theory". Cofibration sequences split:

$$WTHH^{\Gamma}(S_2\mathcal{C}) \simeq WTHH^{\Gamma}(\mathcal{C}) \times WTHH^{\Gamma}(\mathcal{C}).$$

- Agreement with usual definitions for rings/ connective ring spectra.
- Approximation theorem (DK-equivalence induces equivalence of $WTHH$, $WTTH^{\Gamma}$),
- Cyclotomic trace map $K(\mathcal{C}) \to WTHH^{\Gamma}(\mathcal{C})$, constructed via Dundas-McCarthy "inclusion of the objects":

$$\mathrm{ob}\, N^w_\bullet S_\bullet \mathcal{C}^{\Gamma} \to THH(N^w_\bullet S_\bullet \mathcal{C}^{\Gamma})$$

via inclusion of x as identity morphism in $N^w_\bullet S_\bullet \mathcal{C}(x,x)$.

Localization theorem, statement:

Theorem

When $v\mathcal{C}$ is a second category of weak equivalences on $\mathcal{C}$ which contains $w\mathcal{C}$ and is nice, there are cofiber sequences

$$WTHH^{\Gamma}(\mathcal{A}^{v}) \to WTHH^{\Gamma}(\mathcal{A}) \to WTHH^{\Gamma}(\mathcal{A}|v) \to \Sigma WTHH^{\Gamma}(\mathcal{A}^{v})$$
$$WTHH(\mathcal{A}^{v}) \to WTHH(\mathcal{A}) \to WTHH(\mathcal{A}|v) \to \Sigma WTHH(\mathcal{A}^{v}).$$

Very different.

- Often can identify the relative term $WTHH(\mathcal{A}|v)$ in the second sequence above as the THH of the triangulated quotient $\mathcal{A}/\mathcal{A}^{v}$. The first term is hard to understand.
- But for $WTHH^{\Gamma}$, the first term is often identifiable via devissage, and the third term is unusual (Hesselholt-Madsen).

Key point: $WTHH^{\Gamma}(\mathcal{A}|v)$ has mapping spaces compatible with $w\mathcal{C}$ but is built from nerve of $v\mathcal{C}$.

Localization sequence for $THH(ku)$:

Applying the connective result to the sequence of Waldhausen categories

$$\mathcal{C}_{ku}^{KU} \to \mathcal{C}_{ku} \to \mathcal{C}_{ku,KU}$$

- $\mathcal{C}_{ku}$ denotes the category of finite cell ku-modules,
- $\mathcal{C}_{ku}^{KU}$ the KU-acyclics,
- and $\mathcal{C}_{ku,KU}$ the category of finite cell ku-modules with the KU-equivalences.

This yields the localization sequence

$$THH(\mathbb{Z}) \to THH(ku) \to THH(ku|KU),$$

where $THH(\mathbb{Z}) \simeq WTHH^{\Gamma}(\mathcal{C}_{ku}^{KU})$ by a devissage result and $THH(ku|KU) = WTHH^{\Gamma}(\mathcal{C}_{ku}|KU)$.

Interest of this localization sequence:

The relationship between the homotopy groups of ku and ℓ,

$$\pi_* ku_{(p)} = \mathbb{Z}_{(p)}[u] = \pi_* \ell[u]/(v = u^{p-1}),$$

suggests that the map $\ell \to ku_{(p)}$ is a "tamely ramified" extension of ring spectra. Hesselholt observed that the calculations of $THH(ku)$ and $THH(\ell)$ are consistent with the existence of a commutative diagram of localization cofiber sequences

$$\begin{tikzcd}
THH(\mathbb{Z}_{(p)}) \arrow[r] \arrow[d] & THH(\ell) \arrow[r] \arrow[d] & THH(\ell|L) \arrow[d] \\
THH(\mathbb{Z}_{(p)}) \arrow[r] & THH(ku_{(p)}) \arrow[r] & THH(ku_{(p)}|KU_{(p)}),
\end{tikzcd}$$

and a "tamely ramified descent" result

$$ku_{(p)} \wedge_\ell THH(\ell|L) \to THH(ku_{(p)}|KU_{(p)}).$$

Clean-up — generality of this formulation:

Finally, justify our hypotheses on the Waldhausen category:

Theorem

Let $\mathcal{C}$ be a Waldhausen category that admits a homotopy calculus of left fractions. Then there exists a DK-compatible Waldhausen category $\tilde{\mathcal{C}}$ and a weakly exact functor $\mathcal{C} \to \tilde{\mathcal{C}}$ that is a DK-equivalence on simplicial localizations. Moreover:

1. *$WTHH^{\Gamma}(\tilde{\mathcal{C}})$ is a functor from the category of Waldhausen categories and weak exact maps to the homotopy category of cyclotomic spectra.*
2. *As a map in the stable category, $K(\mathcal{C}) \to K(\tilde{\mathcal{C}})$ is natural in exact functors of $\mathcal{C}$.*
3. *As a map in the stable category, the cyclotomic trace $K(\tilde{\mathcal{C}}) \to WTHH^{\Gamma}(\tilde{\mathcal{C}})$ is natural in weak exact functors of $\mathcal{C}$.*

Consistency checks:

- If $\mathcal{C}$ is stable and obtained from a simplicially enriched Waldhausen category, then $\mathcal{C} \to \tilde{\mathcal{C}}$ induces an equivalence.
- In many circumstances when $\mathcal{C}$ is not stable, $\tilde{\mathcal{C}}$ is equivalent to a stabilization of $\mathcal{C}$.

One issue: as the statement above implies, this rectification is not strictly functorial (although it is homotopy coherent). Similar problem arises already in context of $S'_\bullet$ construction (and weakly exact functors).