New Lax-Oleinik Type Operators in the Weak KAM Theory

—A joint work with Kaizhi Wang

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In this talk, we introduce some modifications of the Lax-Oleinik operators in the context of weak KAM theory, both in the autonomous and non-autonomous setting. These modified operators enjoy better properties of convergence: In the non autonomous case, they do converge, unlike the genuine L-O operators, and in the autonomous case, they converge more quickly. For example, when the Mather set is a quasiperiodic torus the new L-O operators converge faster than the genuine ones in the sense of order.

1. Background

Let M be a closed and connected smooth manifold.

Standard assumptions in Mather theory:

Consider a C^2 Lagrangian $L : TM \times \mathbb{R}^1 \to \mathbb{R}^1$, $(x, v, t) \mapsto L(x, v, t)$. We suppose that L satisfies the following conditions introduced by Mather:

(H1) **Periodicity**. L is 1-periodic in the $\mathbb{R}^1$ factor, i.e., $L(x, v, t) = L(x, v, t + 1)$ for all $(x, v, t) \in TM \times \mathbb{R}^1$.

(H2) **Positive Definiteness**. For each $x \in M$ and each $t \in \mathbb{R}^1$, the restriction of L to $T_xM \times t$ is strictly convex in the sense that its Hessian second derivative is everywhere positive definite.

(H3) **Superlinear Growth**. $\lim_{\|v\|_x \to +\infty} \frac{L(x,v,t)}{\|v\|_x} = +\infty$ uniformly on $x \in M$, $t \in \mathbb{R}^1$, where $\|\cdot\|_x$ denotes the norm on T_xM induced by a Riemannian metric. By the compactness of M, this condition is independent of the choice of the Riemannian metric.

(H4) **Completeness of the Euler-Lagrange Flow**. Every solution of the Euler-Lagrange equation, which in local coordinates is:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x,\dot{x},t) = \frac{\partial L}{\partial x}(x,\dot{x},t),$$

are defined on all of $\mathbb{R}^1$.

The corresponding Hamiltonian equations read

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x},$$

where $H(x,p) = p\dot{x} - L(x,\dot{x},t)$ and $p = \frac{\partial L}{\partial \dot{x}}$. The corresponding Hamilton-Jacobi equation is

$$u_t + H(x,u_x,t) = c, \tag{1}$$

where $c = c(L)$ is the Mañé critical value of the Lagrangian L.

For all $t_2 \geq t_1$ and $x,\, y \in M$, let

$$F_{t_1,t_2}(x,y) = \inf_{\gamma} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s), s)ds,$$

where the infimum is taken over the continuous and piecewise C^1 paths $\gamma : [t_1, t_2] \to M$ such that $\gamma(t_1) = x$ and $\gamma(t_2) = y$. Define the action potential and extended Peierls barrier as follows. For each $(t_1, t_2) \in \mathbb{S}^1 \times \mathbb{S}^1$, let

$$\Phi_{\tau_1,\tau_2}(x,y) = \inf F_{t_1,t_2}(x,y)$$

for all $(x,y) \in M \times M$, where the infimum is taken on the set of $(t_1, t_2) \in \mathbb{R}^2$ such that $\tau_1 = [t_1]$, $\tau_2 = [t_2]$ and $t_2 \geq t_1 + 1$. For each $(\tau_1, \tau_2) \in \mathbb{S}^1 \times \mathbb{S}^1$, let

$$h_{\tau_1,\tau_2}(x,y) = \liminf_{t_2 - t_1 \to +\infty} F_{t_1,t_2}(x,y)$$

for all $(x,y) \in M \times M$, where the liminf is restricted to the set of $(t_1, t_2) \in \mathbb{R}^2$ such that $\tau_1 = [t_1]$, $\tau_2 = [t_2]$.

Standard assumptions in the weak KAM theory:

Let $L_a : TM \to \mathbb{R}^1$, $(x, v) \mapsto L_a(x, v)$ be a C^2 Lagrangian satisfying the following two conditions:

(H2') **Positive Definiteness**. For each $(x, v) \in TM$, the Hessian second derivative $\frac{\partial^2 L_a}{\partial v^2}(x, v)$ is positive definite.

(H3') **Superlinear Growth**. $\lim_{\|v\|_x \to +\infty} \frac{L_a(x,v)}{\|v\|_x} = +\infty$ uniformly on $x \in M$.

The corresponding Hamilton-Jacobi equation is

$$H_a(x, u_x) = c(L_a). \tag{2}$$

Definition (Lax-Oleinik semigroup) For each $u \in C(M, \mathbb{R}^1)$ and each $t \geq 0$, let

$$T_t^a u(x) = \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^t L_a(\gamma(s), \dot{\gamma}(s)) ds \right\} \tag{3}$$

for all $x \in M$, and

$$T_t u(x) = \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma}(s), s) ds \right\} \tag{4}$$

for all $x \in M$, where the infimums are taken among the continuous and piecewise C^1 paths $\gamma : [0, t] \to M$ with $\gamma(t) = x$. In view of (3) and (4), for each $t \geq 0$, T_t^a and T_t are operators from $C(M, \mathbb{R}^1)$ to itself. It is not difficult to check that $\{T_t^a\}_{t \geq 0}$ and $\{T_n\}_{n \in \mathbb{N}}$ are one-parameter semigroups of operators. $\{T_t^a\}_{t \geq 0}$ and $\{T_n\}_{n \in \mathbb{N}}$ are called the L-O semigroup associated with L_a and L, respectively.

Definition (weak KAM solution—time-independent case) A weak KAM solution of the Hamilton-Jacobi equation (2) is a function $u : M \to \mathbb{R}^1$ such that

(1) u is dominated by L, i.e.,

$$u(x) - u(y) \leq \Phi_{0,0}(y, x), \quad \forall x,\ y \in M.$$

(2) For every $x \in M$ there exists a curve $\gamma : (-\infty, 0] \to M$ with $\gamma(0) = x$ such that

$$u(x) - u(\gamma(t)) = \int_t^0 L_a(\gamma(s), \dot\gamma(s))ds, \quad \forall t \in (-\infty, 0].$$

Theorem (Fathi)

(1) For every $u \in C(M, \mathbb{R}^1)$, the uniform limit $\lim_{t\to+\infty} T_t^a u = \bar{u}$ exists and $\bar{u}$ is a weak KAM solution of (2).

(2) The weak KAM solutions and viscosity solutions are the same.

Fathi (1998) raised the question as to whether the analogous result holds in the time-periodic case. This would be the convergence of $T_n u$, $\forall u \in C(M, \mathbb{R}^1)$, as $n \to +\infty$, $n \in \mathbb{N}$.

Fathi and Mather (2000) gave a negative answer to the question.

2. New Lax-Oleinik type operators

2.1 Time-periodic case

Definition (new L-O operator—time-periodic case) For each $\tau \in [0,1]$, each $n \in \mathbb{N}$ and each $u \in C(M, \mathbb{R}^1)$, let

$$\tilde{T}_n^\tau u(x) = \inf_{\substack{k \in \mathbb{N} \\ n \leq k \leq 2n}} \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^{\tau+k} L(\gamma(s), \dot{\gamma}(s), s) ds \right\}$$

for all $x \in M$, where the second infimum is taken among the continuous and piecewise C^1 paths $\gamma : [0, \tau + k] \to M$ with $\gamma(\tau + k) = x$.

For each $\tau \in [0,1]$ and each $n \in \mathbb{N}$, $\tilde{T}_n^\tau$ is an operator from $C(M, \mathbb{R}^1)$ to itself. We call *$\tilde{T}_n^\tau$ the new L-O operator associated with L*. For each $n \in \mathbb{N}$ and each $u \in C(M, \mathbb{R}^1)$, let $U_n^u(x, \tau) = \tilde{T}_n^\tau u(x)$ for all $(x, \tau) \in M \times [0,1]$. Then U_n^u is a continuous function on $M \times [0,1]$.

Definition (weak KAM solution—time-periodic case) A weak KAM solution of the Hamilton-Jacobi equation (1) is a function $u : M \times \mathbb{S}^1 \to \mathbb{R}^1$ such that

(1) u is dominated by L, i.e.,

$$u(x,\tau) - u(y,s) \le \Phi_{s,\tau}(y,x), \quad \forall (x,\tau),\ (y,s) \in M \times \mathbb{S}^1.$$

(2) For every $(x,\tau) \in M \times \mathbb{S}^1$ there exists a curve $\gamma : (-\infty, \tilde{\tau}] \to M$ with $\gamma(\tilde{\tau}) = x$ and $[\tilde{\tau}] = \tau$ such that

$$u(x,\tau) - u(\gamma(t),[t]) = \int_t^{\tilde{\tau}} L(\gamma(s), \dot{\gamma}(s), s)ds, \quad \forall t \in (-\infty, \tilde{\tau}].$$

Now we come to the main result:

Theorem (Wang and Yan, 2010) For each $u \in C(M, \mathbb{R}^1)$, the uniform limit $\lim_{n\to+\infty} U_n^u$ exists and

$$\lim_{n\to+\infty} U_n^u(x,\tau) = \inf_{y\in M} \left(u(y) + h_{0,[\tau]}(y,x)\right)$$

for all $(x,\tau) \in M \times [0,1]$, where $[\tau] = \tau \mod 1$, and h denotes the extended Peierls barrier. Furthermore, let $\bar{u}(x,[\tau]) = \inf_{y\in M} \left(u(y) + h_{0,[\tau]}(y,x)\right)$. Then $\bar{u} : M \times \mathbb{S}^1 \to \mathbb{R}^1$ is a weak KAM solution of the Hamilton-Jacobi equation (1)

$$u_s + H(x, u_x, s) = 0.$$

Another important result states as follows.

Theorem (Wang and Yan, 2010) Let $\bar{u} \in C(M \times \mathbb{S}^1, \mathbb{R}^1)$. Then the following three statements are equivalent.

- There exists $u \in C(M, \mathbb{R}^1)$ such that the uniform limit $\lim_{n\to+\infty} U_n^u = \bar{u}$.
- $\bar{u}$ is a weak KAM solution of (1).
- $\bar{u}$ is a viscosity solution of (1).

2.2 Time-independent case

Definition (new L-O operator—time-independent case) For each $u \in C(M, \mathbb{R}^1)$ and each $t \geq 0$, let

$$\tilde{T}_t^a u(x) = \inf_{t \leq \sigma \leq 2t} \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^{\sigma} L_a(\gamma(s), \dot{\gamma}(s)) ds \right\}$$

for all $x \in M$, where the second infimum is taken among the continuous and piecewise C^1 paths $\gamma : [0, \sigma] \to M$ with $\gamma(\sigma) = x$.

It is easy to check that $\{\tilde{T}_t^a\}_{t \geq 0} : C(M, \mathbb{R}^1) \to C(M, \mathbb{R}^1)$ is a one-parameter semigroup of operators. We call it *the new L-O semigroup associated with L_a*.

Theorem (Wang and Yan, 2010)

(1) For each $u \in C(M, \mathbb{R}^1)$, the uniform limit $\lim_{t\to+\infty} \tilde{T}^a_t u$ exists and

$$\lim_{t\to+\infty} \tilde{T}^a_t u = \lim_{t\to+\infty} T^a_t u = \bar{u}.$$

(2) For each $t \geq 0$ and each $u \in C(M, \mathbb{R}^1)$, $\|\tilde{T}^a_t u - \bar{u}\|_\infty \leq \|T^a_t u - \bar{u}\|_\infty$.

3. Rates of convergence of the L-O semigroup (time-independent case) and the family of the new L-O operators.

We believe that there is a deep relation between dynamical properties of the Aubry set (Mather set) and the rates of convergence of the L-O semigroup (time-independent case) and the family of the new L-O operators.

3.1. Results on the rate of convergence of the L-O semigroup $\{T_t^a\}_{t\geq 0}$

1. Iturriaga and Sánchez-Morgado (2009) proved that if the Aubry set consists in a finite number of hyperbolic periodic orbits or hyperbolic fixed points, the L-O semigroup converges exponentially.

2. Wang and Yan (2010) discussed the rate of convergence problem when the Mather set is a quasi-periodic invariant torus of the Euler-Lagrange flow. Consider a class of C^2 superlinear and strictly convex Lagrangians on $\mathbb{T}^n$

$$L_a^1(x,v) = \frac{1}{2}\langle A(x)(v-\omega),(v-\omega)\rangle + f(x,v-\omega), \quad x \in \mathbb{T}^n,\ v \in \mathbb{R}^n, \tag{5}$$

where $A(x)$ is an $n \times n$ matrix, $\omega \in \mathbb{S}^{n-1}$ is a given vector, and $f(x,v-\omega) = O(\|v-\omega\|^3)$ as $v-\omega \to 0$. It is clear that $c(L_a^1)=0$ and $\tilde{\mathcal{M}}_0 = \tilde{\mathcal{A}}_0 = \tilde{\mathcal{N}}_0 = \cup_{x\in\mathbb{T}^n}(x,\omega)$, which is a quasi-periodic invariant torus with frequency vector ω of the Euler-Lagrange flow associated to L_a^1. For (5), the authors showed that for each $u \in C(\mathbb{T}^n,\mathbb{R}^1)$, there is a constant $K_2>0$ such that

$$\|T_t^a u - \bar{u}\|_\infty \le \frac{K_2}{t}, \quad \forall t>0. \tag{6}$$

An example was provided to show that the above result is sharp in the sense of order.

An example.

Consider the following integrable Lagrangian

$$L = \frac{1}{2}\langle v-\omega, v-\omega\rangle.$$

Take

$$u(x) = \begin{cases} \delta - \mathrm{dist}(x, x_0), & \mathrm{dist}(x, x_0) \le \delta; \\ 0, & \text{otherwise.} \end{cases}$$

$$\lim_{t\to+\infty} T_t u(x) = \min_{x\in\mathbb{T}^n} u(x) \triangleq u_0.$$

There exist $t_m \to +\infty$ such that

$$|T^-_{t_m} u(x_0) - u_0(x_0)| \ge \frac{\delta^2}{32 t_m}.$$

3.2. Results on the rate of convergence of the new L-O semigroup $\{\tilde{T}_t^a\}_{t\geq 0}$

Recall the notations for Diophantine vectors: for $\varrho > n-1$ and $\alpha > 0$, let

$$\mathcal{D}(\varrho, \alpha) = \left\{ \beta \in \mathbb{S}^{n-1} \mid |\langle \beta, k \rangle| \geq \frac{\alpha}{|k|^\varrho}, \ \forall k \in \mathbb{Z}^n \backslash \{0\} \right\},$$

where $|k| = \sum_{i=1}^n |k_i|$. For (5), Wang and Yan (2010) proved that given any frequency vector $\omega \in \mathcal{D}(\varrho, \alpha)$, for each $u \in C(\mathbb{T}^n, \mathbb{R}^1)$, there is a constant $K_3 > 0$ such that

$$\|\tilde{T}_t^a u - \bar{u}\|_\infty \leq K_3 t^{-(1+\frac{4}{2\varrho+n})}, \quad \forall t > 0.$$

In view of (6) and (7), we conclude that the new L-O semigroup converges faster than the L-O semigroup in the sense of order when the Aubry set $\tilde{\mathcal{A}}_0$ of the Lagrangian system (5) is a quasi-periodic invariant torus with Diophantine frequency vector $\omega \in \mathcal{D}(\varrho, \alpha)$.

THANK YOU VERY MUCH!

Reference: A New Kind Of Lax-Oleinik Type Operator With Parameters For Time-Periodic Positive Definite Lagrangian Systems, accepted by CMP

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