Transport, Arnold Diffusion, Stability, and Negative Energy Modes

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Fields Institute, Toronto June14, 2011

Goal: Investigate Hamiltonian equilibria that are spectrally stable, but not energy stable.

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pde $\rightarrow$ ode $\rightarrow$ map $\rightarrow$ ode $\rightarrow$ map $\rightarrow$ pde $\rightarrow$ pde $\rightarrow$ ode $\rightarrow$ map

Why?

All interesting plasma magnetic confinement equilibria are either spectrally unstable or spectrally stable with indefinite linearized energy, i.e. have negative energy modes. Both are dangerous - the latter generically unstable because of nonlinearity? How fast?

PJM and D. Pfirsch (1990)

Program

- Do for infinite degree-of-freedom Hamiltonian systems that which can be done for finite. Krein-Moser theorem. Discrete spectrum pretty easy. Continuous spectrum? Not so easy. Analysis necessary. *G. Hagstrom and PJM (2011).*

- 'Real' problem pde vs. low dof models.

Kinetic Theory

Phase Space Density (main dynamical variable):

$$f : D \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}_+$$

$f(x, v, t)\Delta x \Delta v$= number of particles (probability density) in phase space volume $\Delta x \Delta v$ at time t.

Thermal equilibrium at Maxwell Distribution:

$$f_M = N e^{-v^2/2}$$

Nonthermal Relaxation:

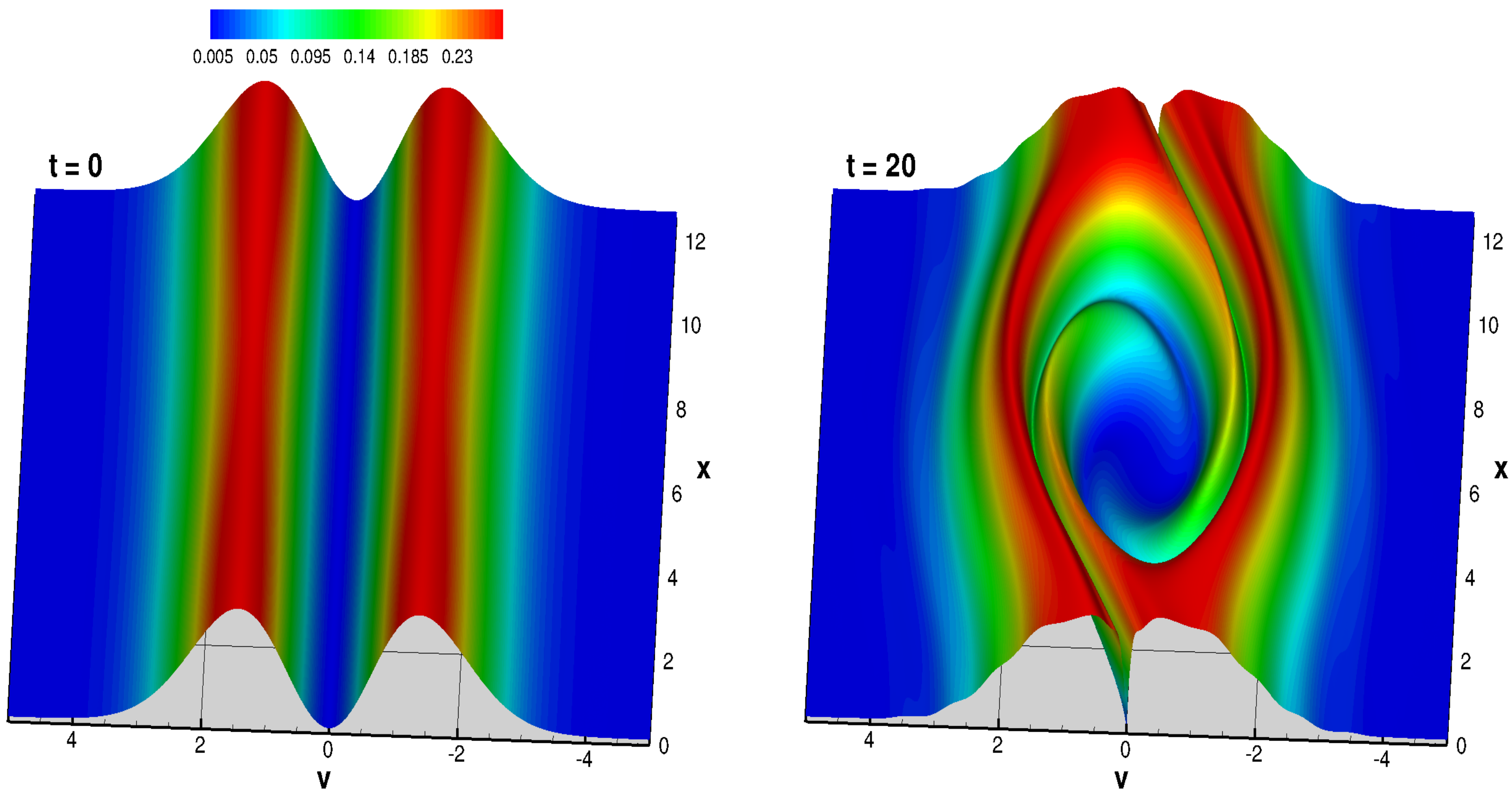
- Collisions $\Rightarrow$ asymptotic stability via Boltzman's H-theorem

- Long-range interactions $\rightarrow$ mean-field theory, i.e. Vlasov eqn.

spectral instability or <u>maybe something else</u>?

Two-Stream Instability

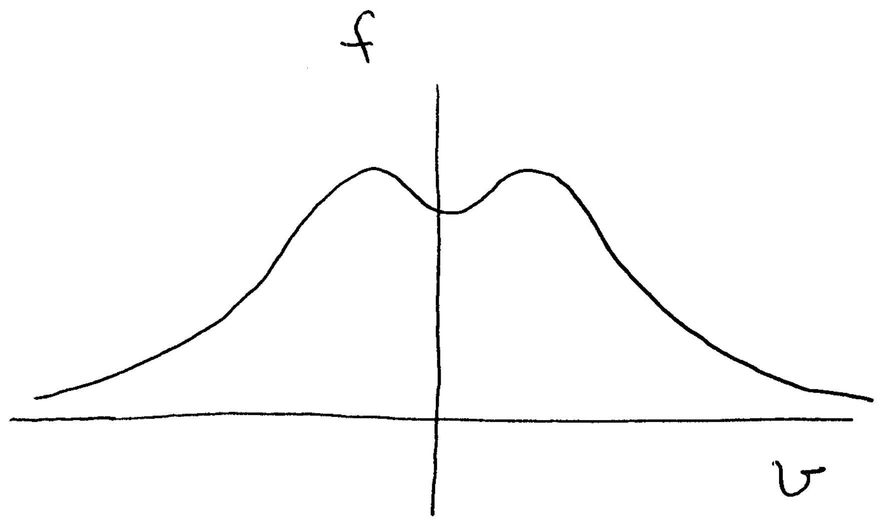
$$f_{TS} = N v^2 e^{-v^2/2}$$



DG code results with I. Gamba

Nonmonotonic or Anisotropic Equilibria

- Spectrally stable yet indefinite linearized energy!
- What happens nonlinearly?
- Something like Arnold diffusion $\rightarrow$ instability?
- Too slow to be important? Moser in celestial mech context.
- Nekhoroshev with n large $\rightarrow \infty$?



Stable

Comparison

Celestial Mechanics:

- basic time scale = 1 year
- solar system age = 5×10^9 years
- number of dof $n = 3 - 100$

Plasma Confinement Device:

- plasma or electron gyro frequency = $10^{12} - 10^{13}$ sec^{-1}
- confinement time of 100 sec (ITER burn flat top 400s)
- number of dof $n = 10^{23}$, but probably effectively much smaller?

$\boxed{\text{Plasma has million times more cycles and } n \text{ much bigger!}}$

Charged Particle on Quadratic Mountain

Simple model of FLR stabilization $\rightarrow$ mirror machine.

Lagrangian:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{eB}{2c}\left(\dot{y}x - \dot{x}y\right) + \frac{k}{2}\left(x^2 + y^2\right)$$

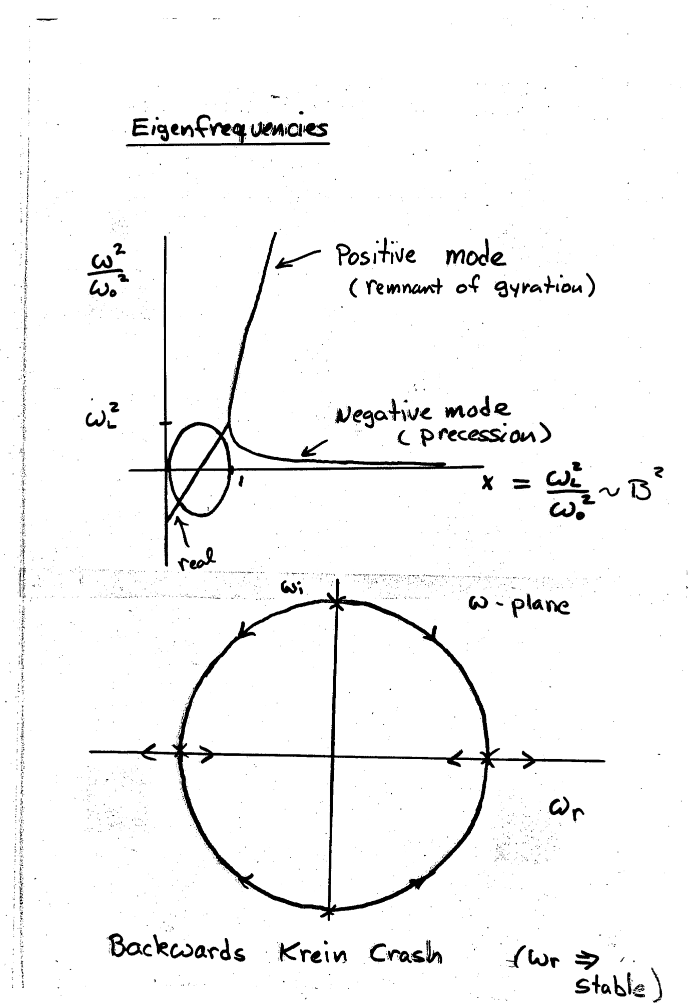
Hamiltonian:

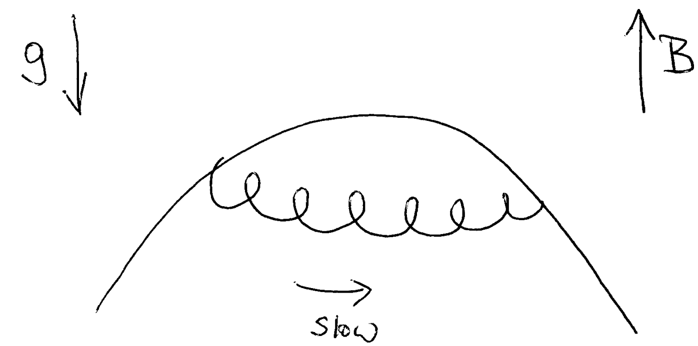
$$H = \frac{m}{2}\left(p_x^2 + p_y^2\right) + \omega_L\left(yp_x - xp_y\right) - \frac{m}{2}\left(\omega_L - \omega_0\right)\left(x^2 + y^2\right)$$

Two frequencies:

$$\omega_L = \frac{eB}{2mc} \qquad \text{and} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

Quadratic Mountain Krein





Quadratic Mountain Normal Form

For large enough B system is stable and $\exists$ a canonical transform to

$$H = |\omega_f| \left(P_f^2 + Q_f^2 \right) - |\omega_s| \left(P_s^2 + Q_s^2 \right)$$

Slow mode is negative energy mode.

Weierstrass (1894), Williamson (1936) ...

.

Charged Particle on Perturbed Integrable Mountain

$$H = \frac{m}{2}\left(p_x^2 + p_y^2\right) + \omega_L\left(yp_x - xp_y\right) - \frac{m}{2}\left(\omega_L - \omega_0\right)\left(x^2 + y^2\right) + ax^3 + \ldots$$

In terms of linear normal coords

$$H = \frac{|\omega_f|}{2}\left(P_f^2 + Q_f^2\right) - \frac{|\omega_s|}{2}\left(P_s^2 + Q_s^2\right) + \frac{\alpha}{2}\left[Q_s\left(Q_f^2 - P_f^2\right) - 2Q_f P_f P_s\right]$$

Assume 2:1, order three resonance, $\omega_f = 1/2$ and $\omega_s = 1$, averaging $\Rightarrow$ Cherry (1925):

$$Q_f = \frac{\sqrt{2}}{\alpha(t-\epsilon)}\sin(t+\gamma), \quad \text{and} \quad \text{etc.}$$

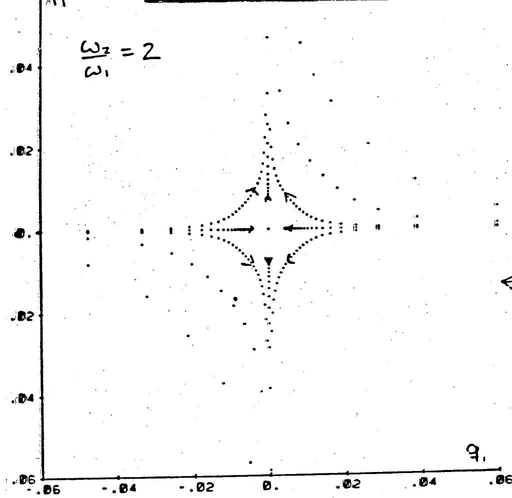
Explosive growth! So because of NEM have linear (spectral) stability but nonlinear instability (to infinitesimal perturbations).

Charged Particle on Perturbed Integrable Nonresonant Mountain

Cherry's System

Courtesy G Kuery

Surfaces of Section



$$H = \frac{\omega_1}{2}(p_1^2 + q_1^2) - \frac{\omega_2}{2}(p_2^2 + q_2^2) + \frac{\alpha}{2}\left[q_2(q_1^2 - p_1^2) - 2q_1 p_1 p_2\right]$$

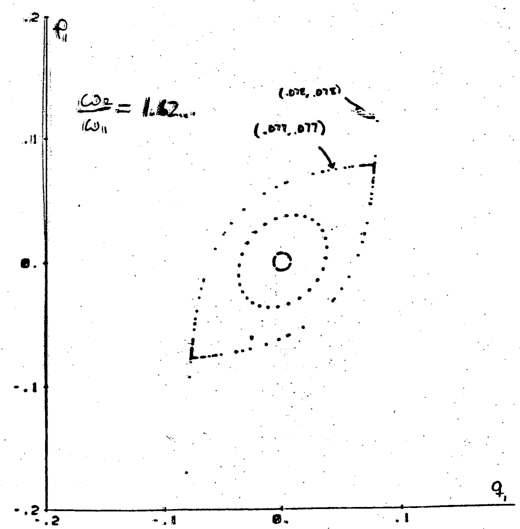
← Tuned O(3) resonance

Surfaces of section $q_2 = 0$

Detuned resonance →

Orbits outside eyelet → ∞

Both are integrable



Charged Particle on Perturbed Nonintegrable Nonresonant Mountain

$$H = |\omega_f| \left(P_f^2 + Q_f^2\right) - |\omega_s| \left(P_s^2 + Q_s^2\right) + \frac{\alpha}{2} \left[Q_s \left(Q_f^2 - P_f^2\right) - (1+\epsilon) Q_f P_f P_s\right]$$

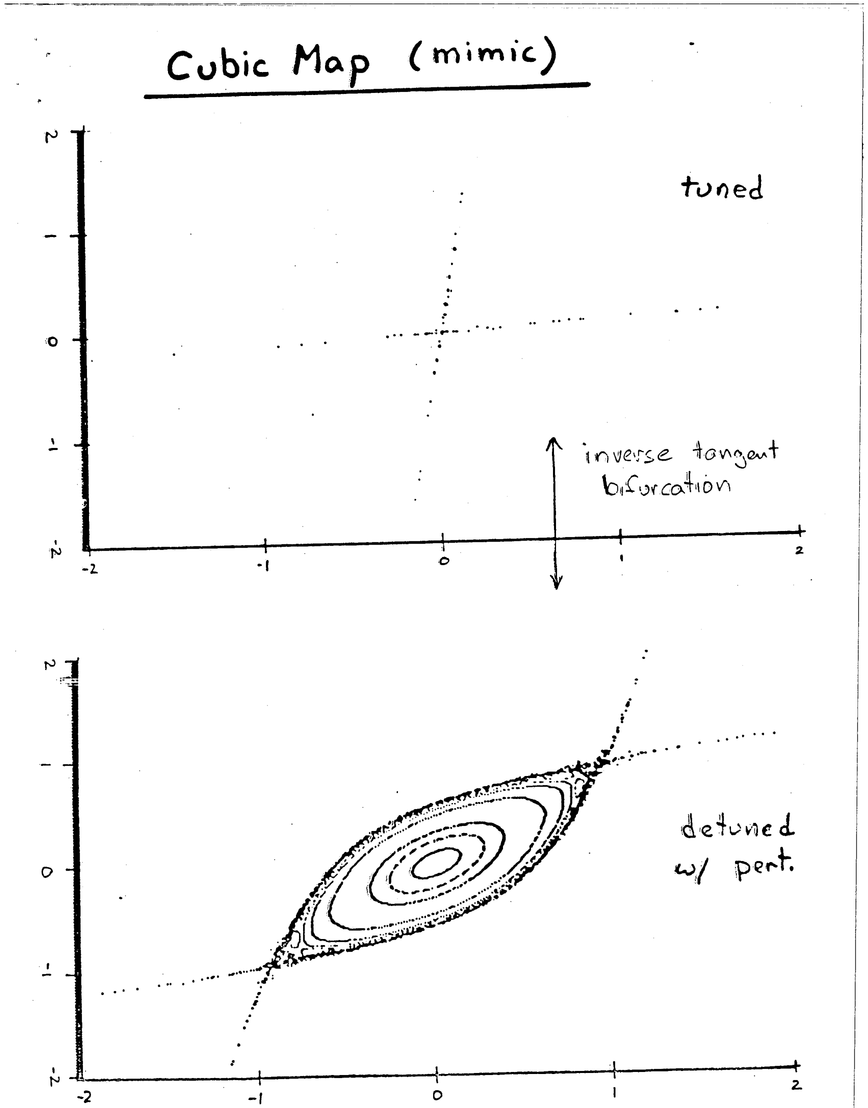
Despited ‘tangle’ system is stable because $\exists$ invariant tori close enough to central elliptic point. (Moser, ...)

Cubic Symplectic Map:

$$p' = -q \qquad\qquad q' = p + qt - q^3$$

Inverse tangent bifurcation at trace $t = -2$

Cubic Map



So, NEM system is stable. Tori near central elliptic point act as subneighborhoods in stability proof.

Charged Particle on Perturbed Nonintegrable Nonresonant Mountain with Earthquake

Instability by motion around invariant tori.

How fast?

What to study?

4D symplectic map

C. Kueny (1987) $\rightarrow$ Caroline Gameiro Lopes Martins (2011)

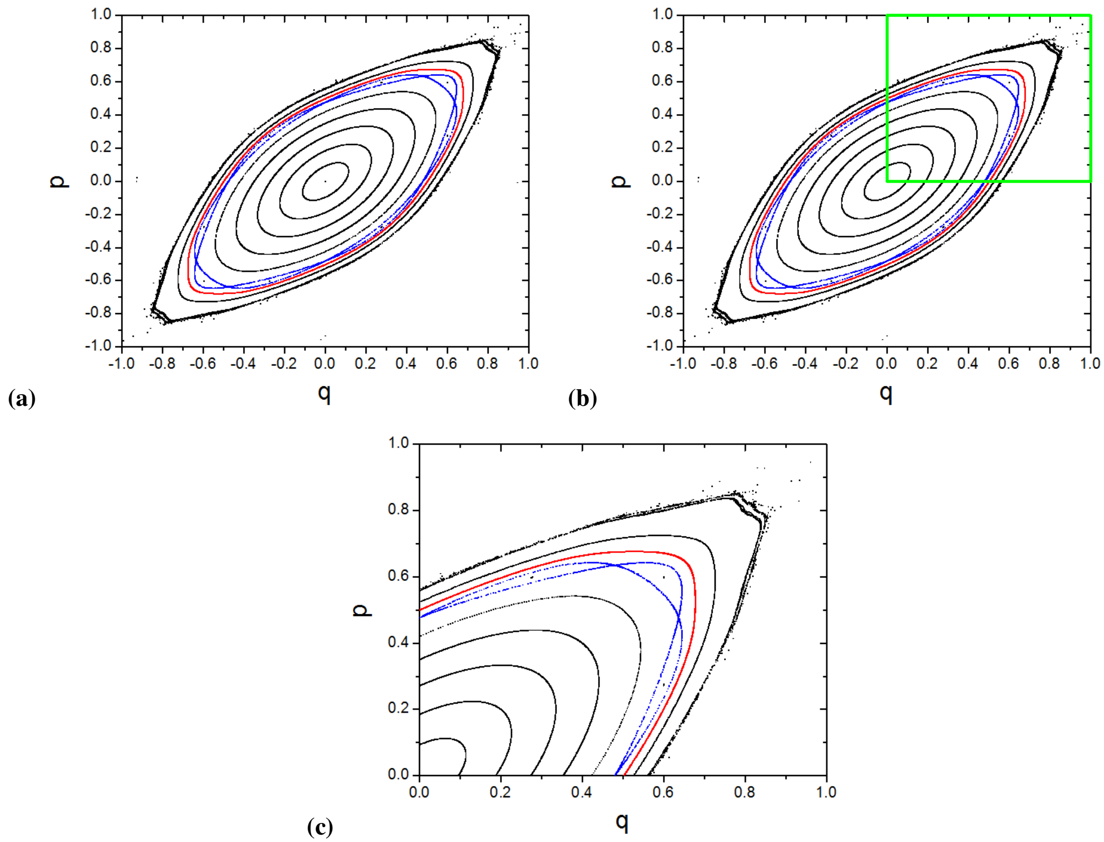


Fig. 6 **(a)** Phase space for the Cubic Map with t = -1.1 **(b)** Green square where we are going to focus **(c)** Zoom in at the green square, emphasizing the curves in **red** and **blue**.

Generating function:

$$F(q,q',Q,Q') = QQ' + qq' + \frac{\tau Q^2}{2} - \frac{tq^2}{2} + \frac{Q^3}{3} + \frac{q^4}{4} + aqQ$$

where, $P' = \dfrac{\partial F}{\partial Q'}$; $\quad -P = \dfrac{\partial F}{\partial Q}$; $\quad -p' = \dfrac{\partial F}{\partial q'}$; $\; p = \dfrac{\partial F}{\partial q}$.

Coupled quadratic & cubic mapping:

$$p' = -q$$

$$q' = p + qt - q^3 - aQ$$

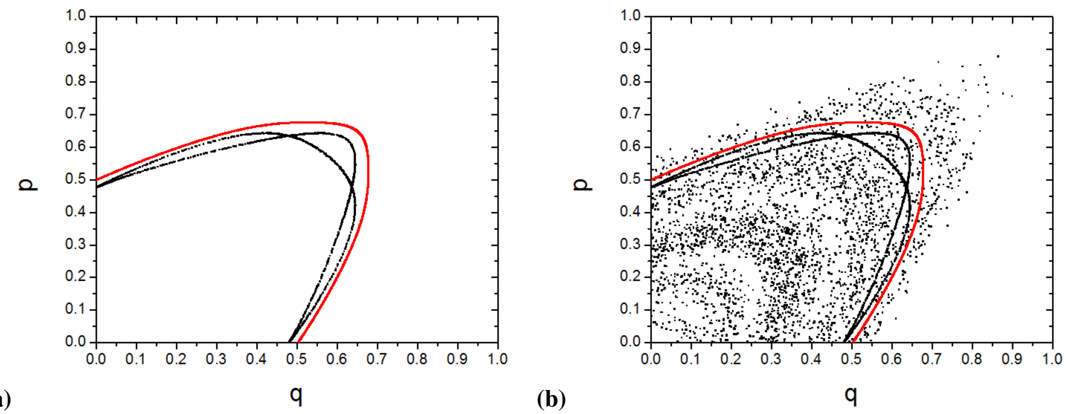
$$P' = Q$$

$$Q' = -P - Q\tau - Q^2 - aq$$

Constant values used: a = 0.01; τ = 0.9864; t = -1.1. Two orbits were iterated:

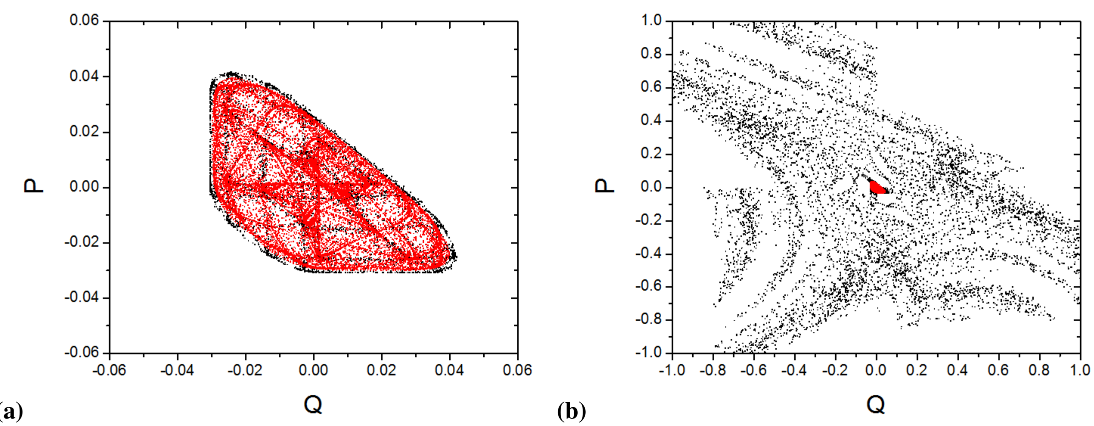
1) Chaotic orbit **in black** (q, p, Q, P) = (0.6253, 0.6230, 0.0000, 0.0000);

2) Invariant orbit **in red** (q, p, Q, P) = (0.65, 0.65, 0.00, 0.00), iterated with $n = 1x10^9$.



(a) **(b)**

Fig. 7 Phase space (q, p) for 2 orbits (described above) **(a)** Orbit **in black** iterated with $n = 5x10^3$ and orbit **in red** iterated with $n = 1x10^9$, but only $n = 1x10^4$ were plotted **(b)** Orbit **in black** iterated with $n = 2x10^4$ and orbit **in red** iterated with $n = 1x10^9$, but only $n = 1x10^4$ were plotted.



(a) **(b)**

Fig. 8 Phase space (Q, P) for 2 orbits (described above) **(a)** Orbit **in black** iterated with $n = 5x10^3$ and orbit **in red** iterated with $n = 1x10^9$, but only $n = 1x10^4$ were plotted **(b)** Orbit **in black** iterated with $n = 2x10^4$ and orbit **in red** iterated with $n = 1x10^9$, but only $n = 1x10^4$ were plotted.

Tools

For example:

- Gomez, Modelo, and Simo (2010)

- Huguet, de La Llave, and Sire (2011)

1D Vlasov-Poisson System - Prototype

Phase space density (1 + 1 + 1 field theory):

$$f(x, v, t) \geq 0$$

Conservation of phase space density:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \phi[x, t; f]}{\partial x}\frac{\partial f}{\partial v} = 0$$

Poisson's equation:

$$\phi_{xx} = 4\pi\left[e\int_{\mathbb{R}} f(x, v, t)\, dv - \rho_B\right]$$

Energy:

$$H = \frac{m}{2}\int_{\Pi}\int_{\mathbb{R}} v^2 f\, dxdv + \frac{1}{8\pi}\int_{\Pi}(\phi_x)^2\, dx$$

Noncanonical Hamiltonian Structure

Hamiltonian structure of media in Eulerian variables

Kinematic Commonality:

energy, momentum, Casimir conservation; dynamics is measure preserving rearrangement; continuous spectra; ... $\longrightarrow$ Krein's theorem

Noncanonical Poisson Bracket:

$$\{F,G\} = \int_{\mathcal{Z}} \zeta \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta}\right] dq\,dp = \int_{\mathcal{Z}} \frac{\delta F}{\delta \zeta} \mathcal{J} \frac{\delta G}{\delta \zeta} dq\,dp$$

Cosymplectic Operator:

$$\mathcal{J}\ \cdot = -\left(\frac{\partial \zeta}{\partial q}\frac{\partial\ \cdot}{\partial p} - \frac{\partial\ \cdot}{\partial q}\frac{\partial \zeta}{\partial p}\right)$$

Equation of Motion:

$$\frac{\partial \zeta}{\partial t} = \{\zeta, H\} = \mathcal{J}\frac{\delta H}{\delta \zeta} = -[\zeta, \mathcal{E}].$$

Organizing principle. Do one do all!

Linear Vlasov-Poisson System

Expand about Stable Homogeneous Equilibrium:

$$f = f_0(v) + \delta f(x, v, t)$$

Linearized EOM:

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial x} + \frac{e}{m}\frac{\partial \delta\phi[x, t; \delta f]}{\partial x}\frac{\partial f_0}{\partial v} = 0$$

$$\delta\phi_{xx} = 4\pi e \int_{\mathbb{R}} \delta f(x, v, t)\, dv$$

Linearized Energy (Kruskal-Oberman):

$$H_L = -\frac{m}{2}\int_{\Pi}\int_{\mathbb{R}} \frac{v\,(\delta f)^2}{f_0'}\, dv dx + \frac{1}{8\pi}\int_{\Pi}(\delta\phi_x)^2\, dx$$

Linear Hamiltonian PDE

• Because noncanonical must expand f-dependent Poisson bracket as well as Hamiltonian. $\Rightarrow$

Linear Poisson Bracket:

$$\{F,G\}_L = \int f_0 \left[\frac{\delta F}{\delta \delta f}, \frac{\delta G}{\delta \delta f}\right] dx\, dv \ ,$$

where δf is the new dynamical variable and the Hamiltonian is the Kruskal-Oberman energy, H_L. The LVP system has the following Hamiltonian form:

$$\frac{\partial \delta f}{\partial t} = \{\delta f, H_L\}_L \, ,$$

with variables <u>noncanonical</u> and H_L <u>not diagonal</u>.

Solution

Assume

$$\delta f = \sum_k f_k(v,t) e^{ikx}\,, \qquad \delta\phi = \sum_k \phi_k(t) e^{ikx}$$

Linearized EOM:

$$\frac{\partial f_k}{\partial t} + ikv f_k + ik\phi_k \frac{e}{m}\frac{\partial f_0}{\partial v} = 0\,, \qquad k^2\phi_k = -4\pi e \int_{\mathbb{R}} f_k(v,t)\,dv$$

Three methods:

1. Laplace Transforms (Landau and others 1946)
2. Normal Modes (Van Kampen, Case,... 1955)
3. Coordinate Change $\Longleftrightarrow$ Integral Transform (PJM, Pfirsch, Shadwick, Balmforth 1992)

Hamiltonian Spectrum

Hamiltonian Operator:

$$f_{kt} = -ikvf_k + \frac{if_0'}{k}\int_{\mathbb{R}} d\bar{v}\, f_k(\bar{v}, t) =: -T_k f_k\,,$$

Complete System:

$$f_{kt} = -T_k f_k \qquad \text{and} \qquad f_{-kt} = -T_{-k} f_{-k}\,, \qquad\qquad k \in \mathbb{R}^+$$

Lemma *If* λ *is an eigenvalue of the Vlasov equation linearized about the equilibrium* $f_0'(v)$*, then so are* $-\lambda$ *and* λ^* *. Thus if* $\lambda = \gamma + i\omega$*, then eigenvalues occur in the pairs,* $\pm\gamma$ *and* $\pm i\omega$*, for purely real and imaginary cases, respectively, or quartets,* $\lambda = \pm\gamma \pm i\omega$*, for complex eigenvalues.*

Spectral Theorem

Set $k = 1$ and consider $T\colon f \mapsto ivf - if_0' \int f$ in the space $W^{1,1}(\mathbb{R})$.

$W^{1,1}(\mathbb{R})$ is Sobolev space containing closure of functions $\|f\|_{1,1} = \|f\|_1 + \|f'\|_1 = \int_{\mathbb{R}} dv(|f| + |f'|)$. Contains all functions in $L^1(\mathbb{R})$ with weak derivatives in $L^1(\mathbb{R})$. T is densely defined, closed, etc.

Definition Resolvent of T is $R(T, \lambda) = (T - \lambda I)^{-1}$ and $\lambda \in \sigma(T)$. (i) λ in point spectrum, $\sigma_p(T)$, if $R(T, \lambda)$ not injective. (ii) λ in residual spectrum, $\sigma_r(T)$, if $R(T, \lambda)$ exists but not densely defined. (iii) λ in continuous spectrum, $\sigma_c(T)$, if $R(T, \lambda)$ exists, densely defined but not bounded.

Theorem *Let $\lambda = iu$. (i) $\sigma_p(T)$ consists of all points $iu \in \mathbb{C}$, where $\varepsilon = 1 - k^{-2} \int_{\mathbb{R}} dv\, f_0'/(u - v) = 0$. (ii) $\sigma_c(T)$ consists of all $\lambda = iu$ with $u \in \mathbb{R} \setminus (-i\sigma_p(T) \cap \mathbb{R})$. (iii) $\sigma_r(T)$ contains all the points $\lambda = iu$ in the complement of $\sigma_p(T)$ that satisfy $f_0'(u) = 0$.*

Note Penrose (1960) criterion and e.g. P. Degond (1986). Similar but different.

Canonization & Diagonalization

Fourier Linear Poisson Bracket:

$$\{F,G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{\mathbb{R}} f_0' \left(\frac{\delta F}{\delta f_k} \frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k} \frac{\delta F}{\delta f_{-k}} \right) dv$$

Linear Hamiltonian:

$$\begin{aligned} H_L = -\frac{m}{2} \sum_k \int_{\mathbb{R}} \frac{v}{f_0'} |f_k|^2 \, dv \ &+ \ \frac{1}{8\pi} \sum_k k^2 |\phi_k|^2 \\ &= \ \sum_{k,k'} \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(v)\, \mathcal{O}_{k,k'}(v|v')\, f_{k'}(v')\, dv dv' \end{aligned}$$

Canonization:

$$q_k(v,t) = f_k(v,t)\,, \qquad p_k(v,t) = \frac{m}{ikf_0'} f_{-k}(v,t) \qquad \Longrightarrow$$

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\frac{\delta F}{\delta q_k} \frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k} \frac{\delta F}{\delta p_k} \right) dv$$

Integral Transform

Definintion:

$$\boxed{f(v) = \mathcal{G}[g](v) \;\; := \varepsilon_R(v)\, g(v) + \varepsilon_I(v)\, H[g](v)\,,}$$

where

$$\varepsilon_I(v) = -\pi \frac{\omega_p^2}{k^2} \frac{\partial f_0(v)}{\partial v}\,, \qquad\qquad \varepsilon_R(v) = 1 + H[\varepsilon_I](v)\,,$$

and the Hilbert transform

$$H[g](v) := \frac{1}{\pi} ⨍ \frac{g(u)}{u-v}\, du\,,$$

with $⨍$ denoting Cauchy principal value of $\int_{\mathbb{R}}$.

Transform Properties

Theorem (G1) *$\mathcal{G}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$, is a bounded linear operator; i.e.*

$$\|\mathcal{G}[g]\|_p \leq B_p \, \|g\|_p \, ,$$

where B_p depends only on p.

Theorem (G2) *If $f_0' \in L^q(\mathbb{R})$, stable, Hölder decay, then $\mathcal{G}[g]$ has a bounded inverse,*

$$\mathcal{G}^{-1}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R}) \, ,$$

for $1/p + 1/q < 1$, given by

$$\begin{aligned} g(u) &= \mathcal{G}^{-1}[f](u) \\ &:= \frac{\varepsilon_R(u)}{|\varepsilon(u)|^2} \, f(u) - \frac{\varepsilon_I(u)}{|\varepsilon(u)|^2} \, H[f](u) \, . \end{aligned}$$

where $|\varepsilon|^2 := \varepsilon_R^2 + \varepsilon_I^2$.

Diagonalization

Mixed Variable Generating Functional:

$$\mathcal{F}[q, P'] = \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v)\, \mathcal{G}[P'_k](v)\, dv$$

Canonical Coordinate changes $(q, p) \longleftrightarrow (Q', P')$:

$$p_k(v) = \frac{\delta \mathcal{F}[q, P']}{\delta q_k(v)} = \mathcal{G}[P_k](v)\,, \qquad Q'_k(u) = \frac{\delta \mathcal{F}[q, P']}{\delta P_k(u)} = \mathcal{G}^{\dagger}[q_k](u)$$

New Hamiltonian:

$$H_L = \frac{1}{2} \sum_{k=1}^{\infty} \int_{\mathbb{R}} du\, \sigma_k(u) \omega_k(u) \left[Q_k^2(u) + P_k^2(u) \right]$$

where $\sigma_k(v) = -\mathrm{sgn}(v f'_0(v))$ and $\omega_k(u) = |ku|$

$(Q', P') \longleftrightarrow (Q, P)$ is trivial.

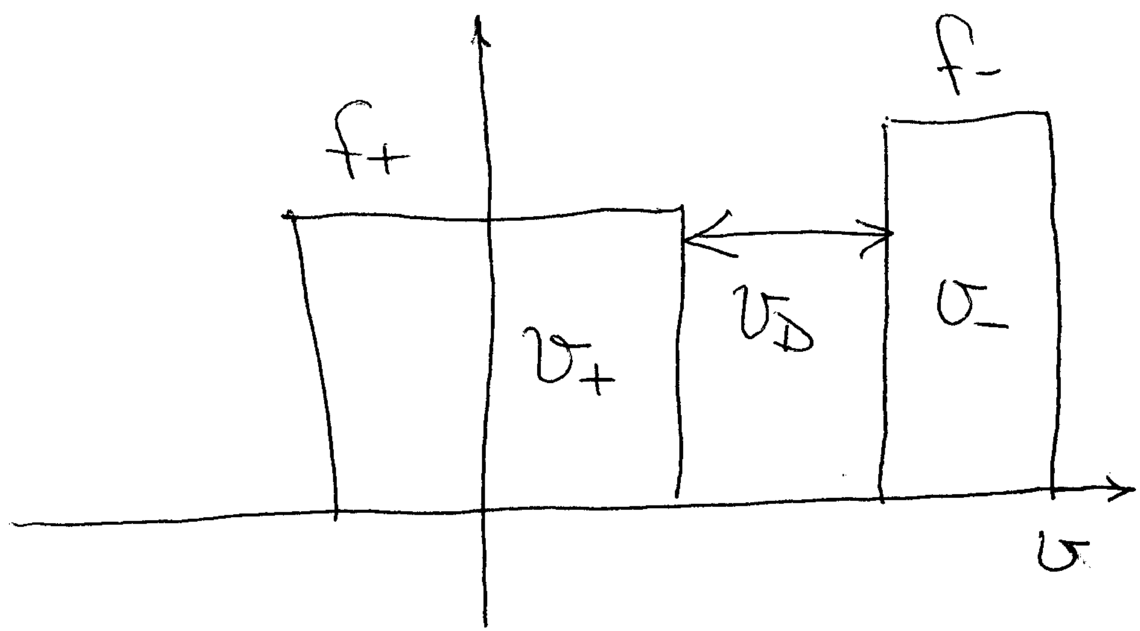
Krein-Like Theorem for VP

Theorem *Let f_0 be a stable equilibrium distribution function for the Vlasov equation. Then f_0 is structurally stable under dynamically accessible perturbations in $W^{1,1}$, if there is only one solution of $f_0'(v) = 0$. If there are multiple solutions, f_0 is structurally unstable and the unstable modes come from the roots of f_0' that satisfy $f_0''(v) < 0$.*

Remark A change in the signature of the continuous spectrum is a necessary and sufficient condition for structural instability. The bifurcations do not occur at <u>all</u> points where the signature changes, however. Only those that represent valleys of the distribution can give birth to unstable modes.

Fluid Two-Stream

Waterbag distribution function:



Two-Stream Instability (warm ions & electrons)

$$\frac{\partial v_\alpha}{\partial t} + v_\alpha \frac{\partial v_\alpha}{\partial x} = \frac{e_\alpha}{m_\alpha} E - \frac{1}{\rho_\alpha}\frac{\partial p_\alpha}{\partial x}$$

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(n_\alpha v_\alpha) = 0 \qquad \frac{\partial E}{\partial x} = 4\pi e (n_i - n_e)$$

equil. n_{0i}, n_{0e}, v_D ← drifting electrons

spectral stability condition given via

$$\boxed{0 = 1 - \frac{\omega_{pi}^2}{\omega^2 - k^2 v_{Ti}^2} - \frac{\omega_{pe}^2}{(\omega - k v_D)^2 - k^2 v_{Te}^2} = \varepsilon(k,\omega)}$$

Threshold: $v_D > v_{Ti} + v_{Te}$ $\Rightarrow$ instability

$\delta^2 F$:

Threshold: $v_D < v_{Te}$ $\Rightarrow$ $\delta^2 F$ positive definite

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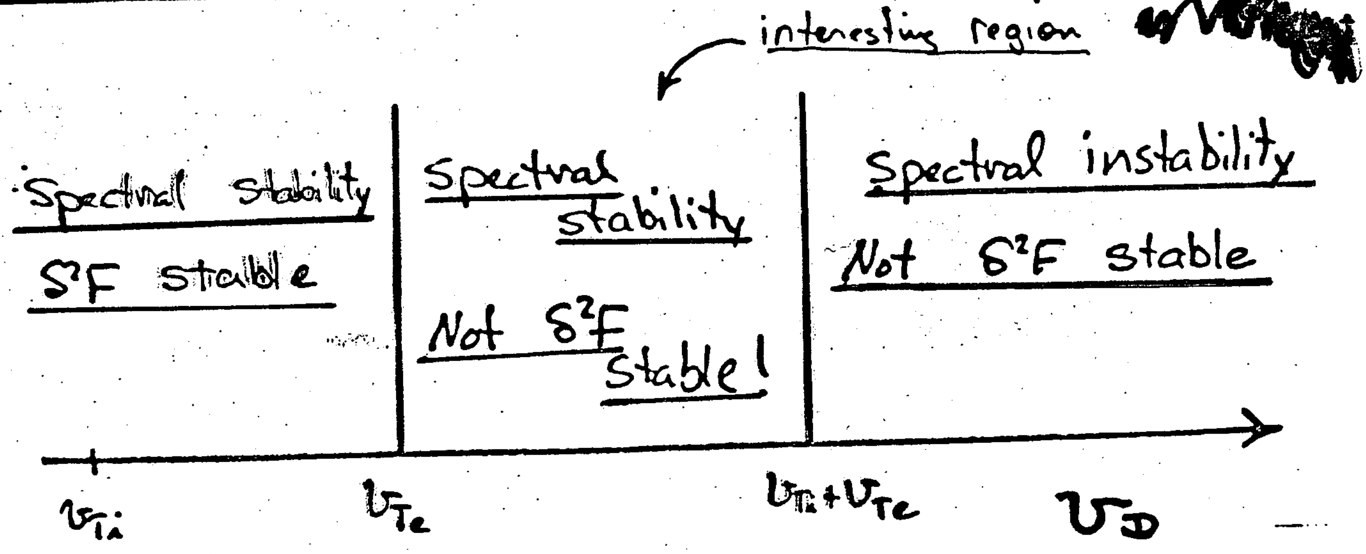


Noncanonical Variables $\longrightarrow$

Canonical Variables + Fourier Trans.

$\Longrightarrow$

$$H = \sum_{k}^{\infty} \omega_k J_k + \mathcal{O}(J^{3/2})$$

In the band $v_{Te} < v_D < v_{Ti} + v_{Te}$

$\exists$ ω_k's < 0.

Pick out "1" resonant triad + resonant driving term $\Longrightarrow$

$$J \sim \frac{1}{t_0 - t}$$

Explosive Growth.

Detune resonance $\Longrightarrow$?

Coherent 3-wave resonance (detuned)

"=" 4 dimensional Symplectic Map

2 Degree of freedom Autonomous

→ 1 degree of freedom Nonautonomous

= area preserving map

3 Degree of freedom autonomous

→ 2 degree of freedom nonautonomous

= 4 dim. symp. map

Generating function:

$$F = F_{\text{cubic}} + F_{?} + F_{\text{coupling}}$$

area preservers

4 Dimensional Symplectic Map (mimic)

(anharmonic moontain with earthquake)

Generating Function: $F = QQ' + qq' + \frac{\tau Q^2}{2} - \frac{t q^2}{2} + \frac{Q^3}{3} + \frac{q^4}{4}$

coupling : $\rightarrow \underline{+aqQ}$

coupled quadratic & cubic area preserving maps.

$$\frac{\partial F}{\partial Q'} = P' \qquad \frac{\partial F}{\partial Q} = -P \qquad \frac{\partial F}{\partial q'} = -p' \qquad \frac{\partial F}{\partial q} = p$$

$$P' = Q \qquad\qquad p' = -q$$

$$Q' = \tau Q + Q^2 + P + \underline{aq} \qquad\qquad q' = p + tq + q^3 + \underline{aQ}$$

Orbit A

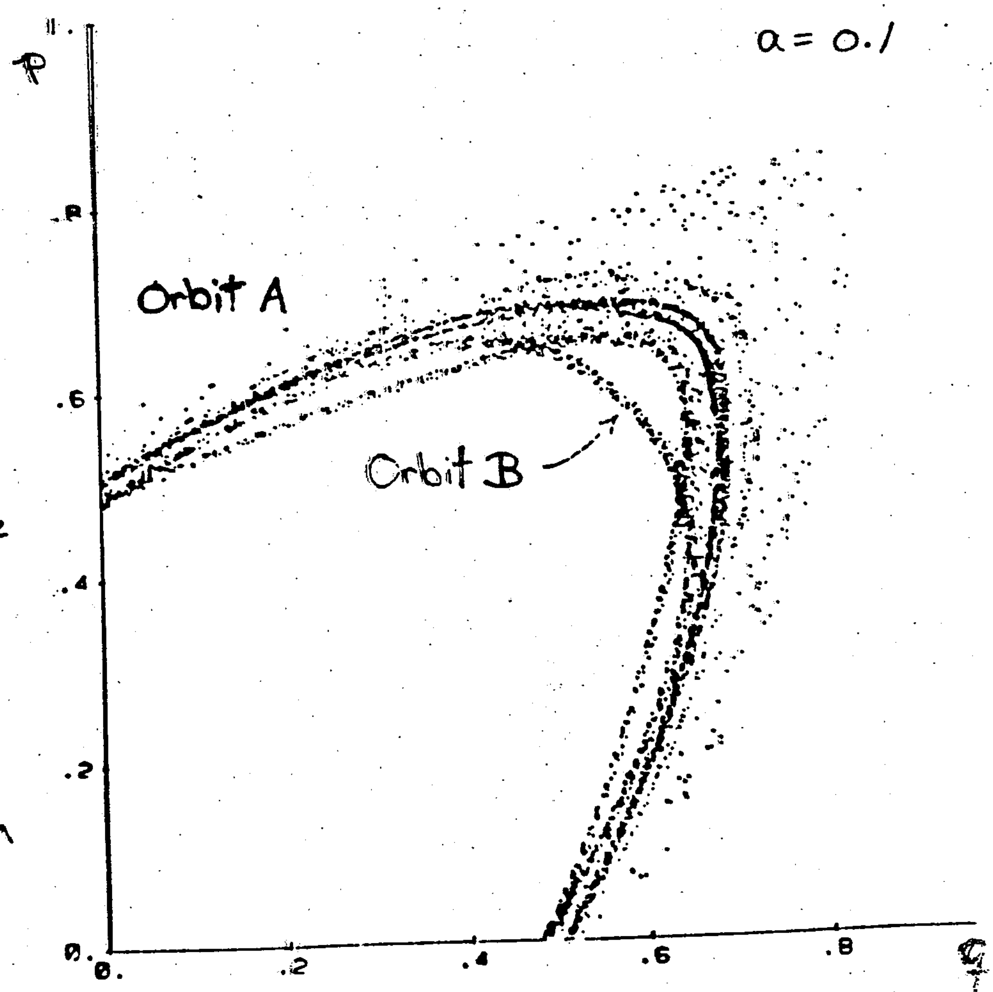
$(q,p,Q,P) = (.65, .65, 0, 0)$

No movement in 10 million iterations. (5×10^3 plotted)

Orbit B

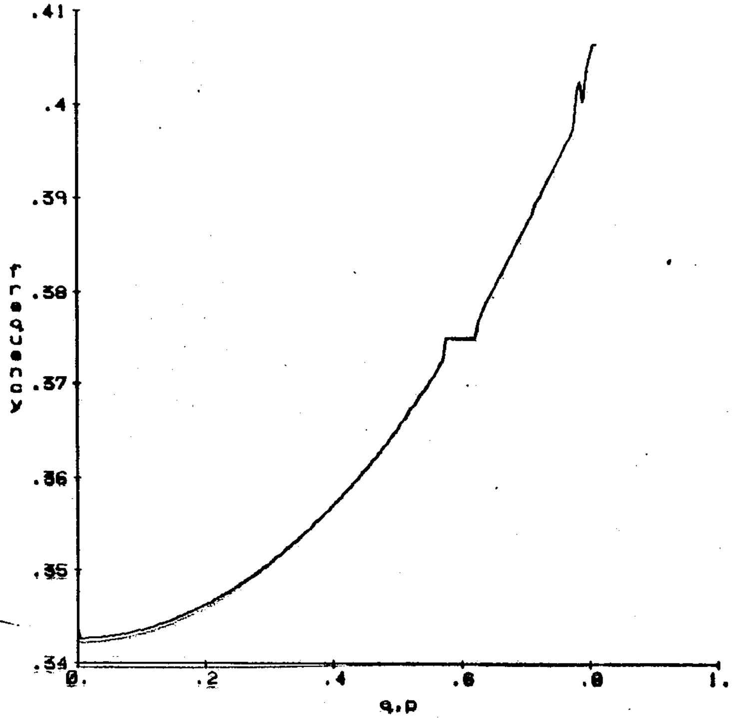
$(q,p,Q,P) = (.623\ , .623\ldots, 0, 0)$

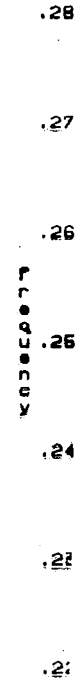
2 million iterations. The first 5×10^3 map out separatrix lying completely inside A. Suddenly the orbit jumps outside A, jumps again and then $\longrightarrow \infty$. The last 5×10^3 are plotted.



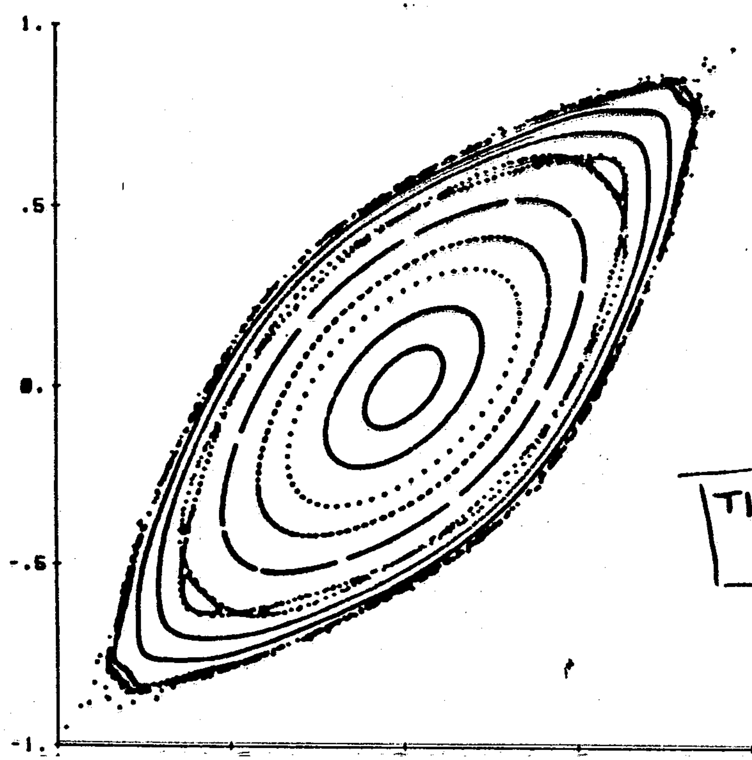
Courtesy C Kerry

ω_1





$t = -1.1$



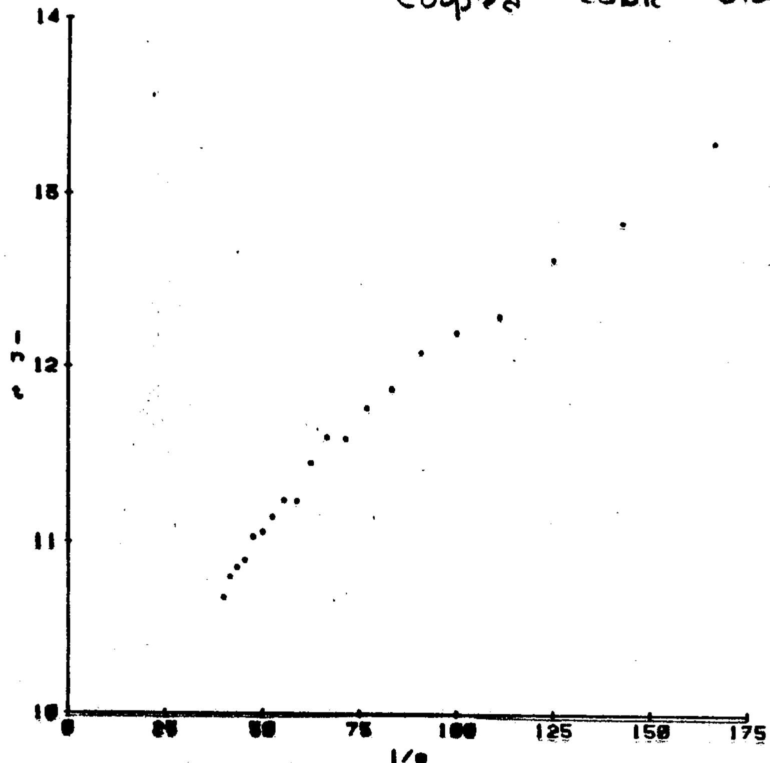
$t = .25$

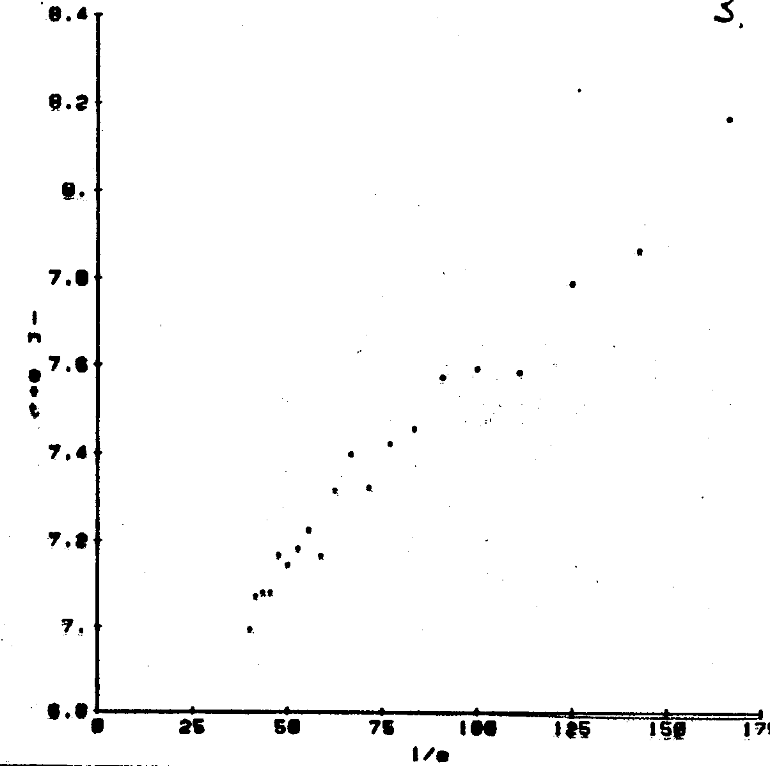
The Cubic Map

Coupled Cubic - Standard

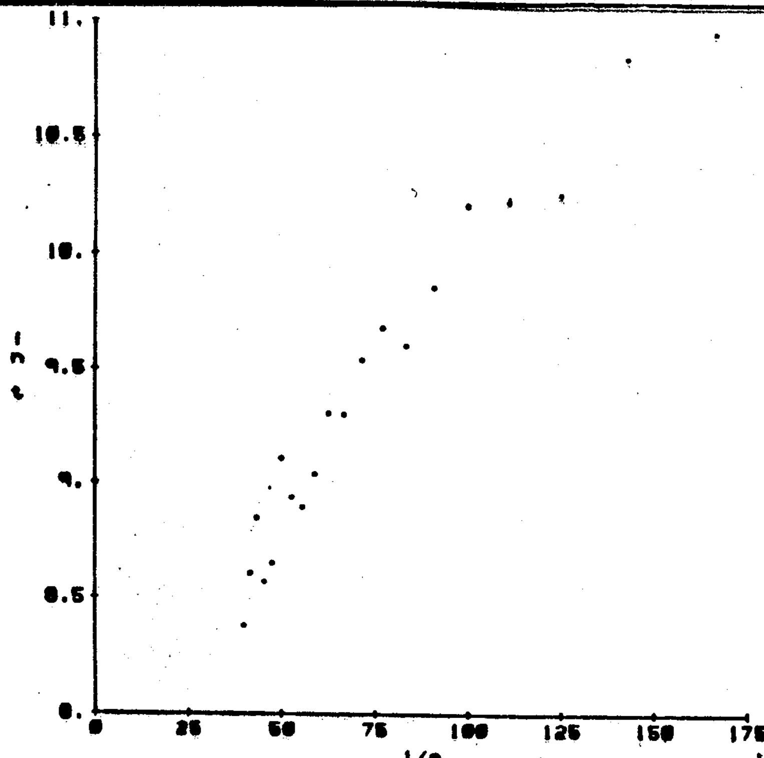
lnt
escape time

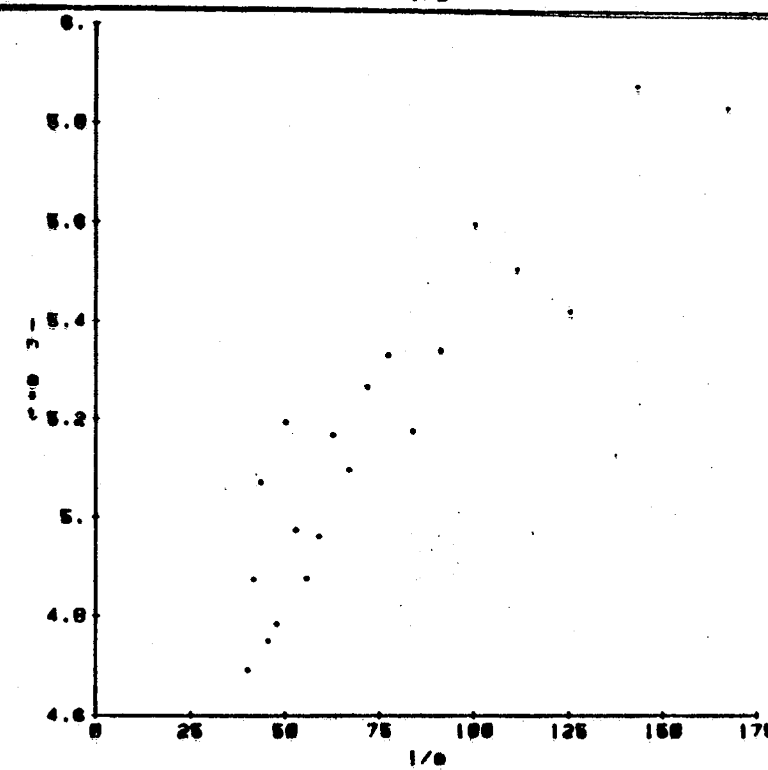
K=0.5





K=2.0





inverse coupling

Coupled Cubic-Standard
(k= 2.0)