Numerical investigations of a conjecture by N.N. Nekhoroshev about stability in quasi–integrable systems

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In 1977 Nekhoroshev published his celebrated result about the exponential stability (Nekhoroshev theorem).

The theorem provides upper bounds to to the stability times of the action variables of quasi–integrable systems, with the following main hypotheses:

- analyticity of the Hamilton function
- a (weak) non–degeneracy of the integrable approximation: steepness
- the perturbation parameter is suitably small

In this talk we discuss the second point, relevant for applications; for example, in Celestial Mechanics strong non-degeneracy conditions are rarely satisfied.

Steepness, in its different forms, has been found in:

- normal forms of asteroids in the main belt (Morbidelli and Guzzo 1997, Pavlović and Guzzo 2008).
- Lagrangian points L4-L5 at different values of the mass ratio satisfy different steepness conditions (Benettin, Fassó and Guzzo 1998)
- Riemann ellipsoids (steady motions of incompressible self-gravitating fluids) (Fassò and Lewis, 2001)

and also, stability issues in non convex systems discussed for:

- Hamiltonians of particle accelerators (Laskar)

A synthetic statement of the theorem

NEKHOROSHEV THEOREM (1977)

Let us consider Hamiltonian Systems:

$$H(I,\varphi) = h(I) + \varepsilon\, f(I,\varphi) \quad ,$$

with $I = (I_1, \ldots, I_n)$, $\varphi = (\varphi_1, \ldots, \varphi_n)$ action-angle variables, and:

- h, f are analytic.
- $h(I)$ is steep

There exist positive constants $\varepsilon_0, a, T_0, \alpha, \beta$ such that: for any $|\varepsilon| \leq \varepsilon_0$ and for any initial condition it is:

$$|I(t) - I(0)| \leq a\, \varepsilon^{\alpha}$$

for any time t satisfying:

$$|t| \leq T_0 \exp\left(\frac{\varepsilon_0}{\varepsilon}\right)^{\beta}$$

The stability time is a stretched exponential, the value of the stretching exponent β depends on the specific steepness properties of h.

Already in Nekhoroshev 1977 paper:

- the value of the stretching exponent β depends only on the number of the degrees of freedom and on the so called steepness indices of h:

$$d_1 \ \ , \ \ d_2 \ \ , \cdots , \ \ d_{n-1} \in \{1, 2, 3, \ldots, ...\} \ \ ,$$

that is:

$$\beta(d_1, \ldots, d_{n-1}, n)$$

- β is a monotone decreasing function of the d_j: larger values of the indices correspond to shorter stability times.
- For convex and quasi–convex functions:

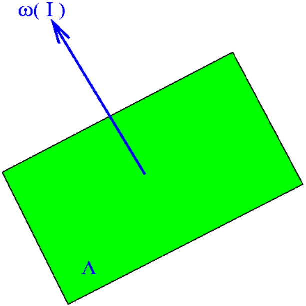
$$d_1 = d_2 = \ldots = d_{n-1} = 1$$

To recall the definition of the steepness index d_j let us fix:

- a point I in the action domain
- any plane Λ of dimension j, orthogonal to $\omega(I) = \nabla h(I)$

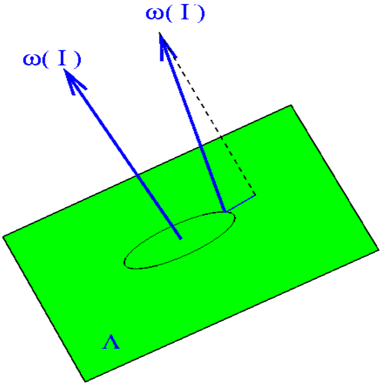
any integer vector $\lambda \in \Lambda$ may define a resonance:

$$\lambda \cdot \omega(I) = 0 \qquad \Pi_\Lambda \omega(I) = 0$$



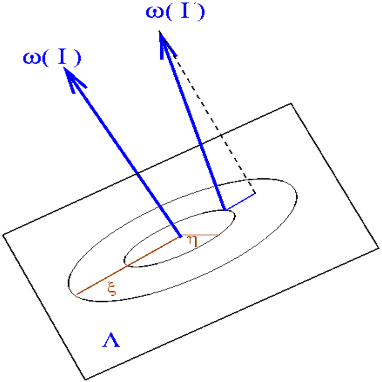
Ideally, one would require:

$$I' \in I + \Lambda \qquad \left|\Pi_\Lambda \omega(I')\right| \sim \left|I - I'\right|^{d_j}$$



- The exponent (steepness index) d_j characterizes the lower bounded variation of j independent small denominators.
- The definition is complicated by a max/min condition

$$\max_{0\le\eta\le\xi}\ \min_{I'\in I+\Lambda:\ |I'-I|=\eta} \left|\Pi_\Lambda \omega(I')\right| > C\xi^{d_j} \quad \forall\, \xi\in(0,\tilde{\xi}]$$



DEFINITION

d_j is a steepness index (of order j) for h at the point I if there exist $C, \tilde{\xi} > 0$ such that for any j-dimensional linear space Λ orthogonal to $\omega(I) = \nabla h(I)$ it is:

$$\max_{0 \leq \eta \leq \xi} \quad \min_{I' \in I + \Lambda:\ |I - I'| = \eta} \left| \Pi_\Lambda \omega(I') \right| > C \xi^{d_j} \quad \forall\, \xi \in (0, \tilde{\xi}]$$

where Π_λ denotes the euclidean projection over Λ.

The functions with only positive steepness indices at any I are called steep functions.

The Nekhoroshev theorem provides an upper bound to the stability time: we do not know if, and in which other stronger sense (i.e. with some lower bound), the stretched exponential really characterizes the stability times.

A conjecture by N.N. Nekhoroshev

"The author conjectures that, in fact, if we compare system .... with the same number s of frequencies, then those for which h has smaller steepness indices are in a certain sense more stable than the systems with larger indices. In particular, systems with quasi–convex unperturbed Hamiltonian .... are the most stable." ..... "It would be interesting to verify this dependence somehow, for example, on a computer".

N.N. Nekhoroshev, Russian Mathematical Surveys, 1977.

But, how the conjecture can be investigated numerically?
Can we isolate the role of steepness in the instability properties of a system? We now that many other things can influence the long term stability:

- the local resonance properties of the initial conditions
- the for different systems, the analyticity properties of the Hamiltonian are different

Therefore: we need to compare the stability properties of initial conditions characterized by the same resonances, same Hamiltonian, but only with different steepness indices.

Is it possible?

Yes it is, because for non quadratic Hamiltonians $h(I)$, the frequency map:

$$\omega(I) = \nabla h(I)$$

is not one-to-one, and therefore: there is the possibility of choosing different points with the same frequencies, but different steepness indices. (Guzzo, Lega, Froeschlé, to appear on Chaos, 2011)

The simplest examples of steep functions which are not quasi–convex are found within the cubic ones.

In fact, the following three–jet non–degeneracy condition:

$$(h' \cdot u = 0 \ , \ h''u \cdot u = 0 \ , \ \sum_{i,j,k} \frac{\partial^3 h}{\partial I_i \partial I_j \partial I_k} u_i u_j u_k = 0) \Rightarrow u = 0 \ .$$

is sufficient for steepness.

To be more specific, let us consider the $n = 3$ integrable Hamiltonian:

$$h(I) = \frac{I_1^2}{2} - \frac{I_2^2}{2} + m\,\frac{I_2^3}{3} + 2\pi I_3$$

whose time-1 flow is the integrable map:

$$\boxed{\begin{gathered} \varphi_1' = \varphi_1 + I_1 \quad , \quad \varphi_2' = \varphi_2 - I_2 + m\, I_2^2 \\ I_1' = I_1 \quad , \quad I_2' = I_2 \end{gathered}}$$

For $m = 0$, it is not convex (nor steep) and the Nekhoroshev theorem does not apply.

For $m > 0$, the system satisfies the three-jet condition, therefore the Nekhoroshev theorem applies, but with different steepness indices (GLF 2011):

- $d_1 = 1$, $d_2 = 1$ if $I_2 > \frac{1}{2m}$ (h is quasi-convex)
- $d_1 = 2$, $d_2 = 1$ if $I_2 < \frac{1}{2m}$
- $d_1 = 2$, $d_2 = 2$ if $I_2 = \frac{1}{2m}$

The steepness indices change at the line:

$$I_2 = \frac{1}{2m}$$

Comparing the frequencies

The frequency vector:

$$\omega(I_1, I_2) = (I_1, -I_2 + m\, I_2^2, 2\pi)$$

is symmetric with respect to the line:

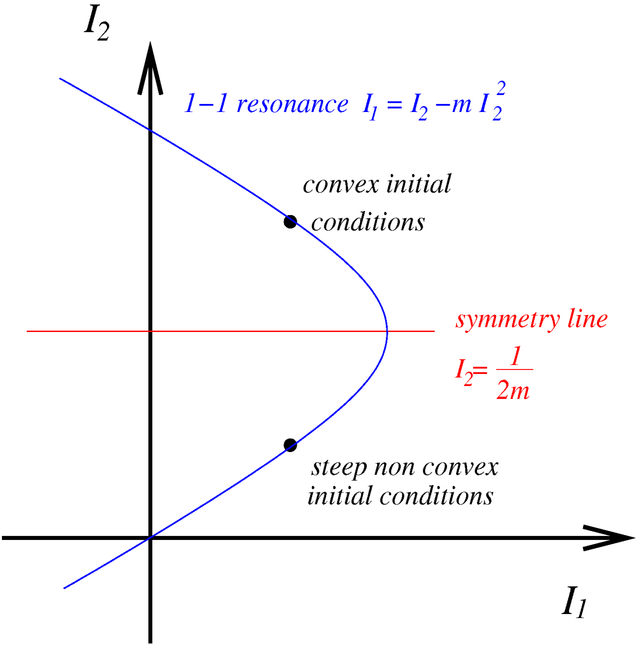
$$I_2 = \frac{1}{2m}$$

that is:

$$\omega(I_1, \frac{1}{2m} - x) = \omega(I_1, \frac{1}{2m} + x) \quad ,$$

Points which are symmetric with respect to $1/(2m)$ have the same frequencies, but different steepness indices.

The 1-1 resonance



A model of 4D steep maps

We consider perturbations of the previous map:

$$\boxed{\begin{array}{c} \varphi_1' = \varphi_1 + I_1 \quad , \quad \varphi_2' = \varphi_2 - I_2 + m\, I_2^2 \\ I_1' = I_1 - \varepsilon \dfrac{\partial f}{\partial \varphi_1}(\varphi') \quad , \quad I_2' = I_2 - \varepsilon \dfrac{\partial f}{\partial \varphi_2}(\varphi') \end{array}}$$

with a perturbation specifically designed to study the diffusion in the 1-1 resonance:

$$\varepsilon f = \varepsilon \cos(\varphi_1 + \varphi_2) + \frac{\varepsilon a}{\cos\varphi_1 + \cos\varphi_2 + 2.1}$$

where:

- $a = 10^{-6}$ is very small and fixed for all integrations
- ε changes

Comparing the perturbation on the two points

We chose two sets of points which are symmetric with respect to: $I_2 = \frac{1}{2m}$ and compute numerically the diffusion coefficient. The two sets are characterized by:

- the same resonances
- the same perturbation
- different steepness indices

Any eventual difference in the long term stability of the two sets, is due to the difference in the steepness indices.

A set of variables adapted to the resonance

With the following action-angle variables:

$$J_1 = I_2 \;\;,\;\; J_2 = I_2 - I_1$$
$$\psi_1 = \varphi_1 + \varphi_2 \;\;,\;\; \psi_2 = -\varphi_1$$

the map takes the form:

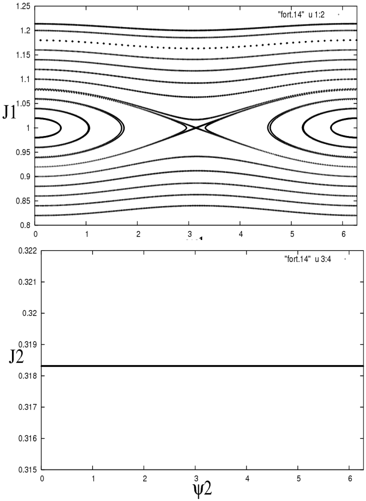
$$\begin{array}{rcl} \psi_1' & = & \psi_1 - J_2 + mJ_1^2 \\ \psi_2' & = & \psi_2 + J_2 - J_1 \\ J_1' & = & J_1 - \varepsilon \sin(\psi_1') + \mathcal{O}(\varepsilon a) \\ J_2' & = & J_2 + \mathcal{O}(\varepsilon a) \end{array}$$

The action J_2 may diffuse only if $a \neq 0$, and by Nekhoroshev theorem the diffusion is exponentially slow.

Case: $a = 0$

$$\psi_1' = \psi_1 - J_2 + mJ_1^2$$

$$J_1' = J_1 - \varepsilon \sin(\psi_1')$$

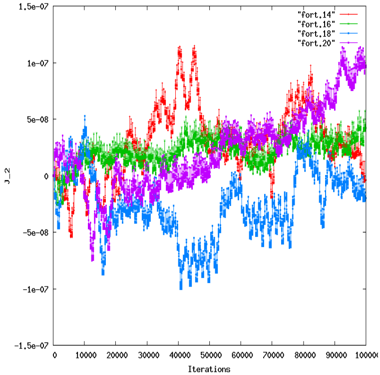


$$\psi_2' = \psi_2 + J_2 - J_1$$

$$J_2' = J_2$$

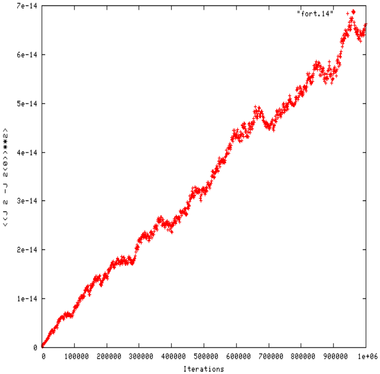
The resonance has an hyperbolic torus.

Numerical experiments confirm the existence of diffusion for J_2



Evolution of the action J_2 for four orbits near the hyperbolic torus

Parameters: $\varepsilon = 10^{-3}$, $a = 10^{-6}$, $c = 2.1$.



Evolution of the quadratic dispersion

$$< (J_2(t) - J_2(0))^2 >$$

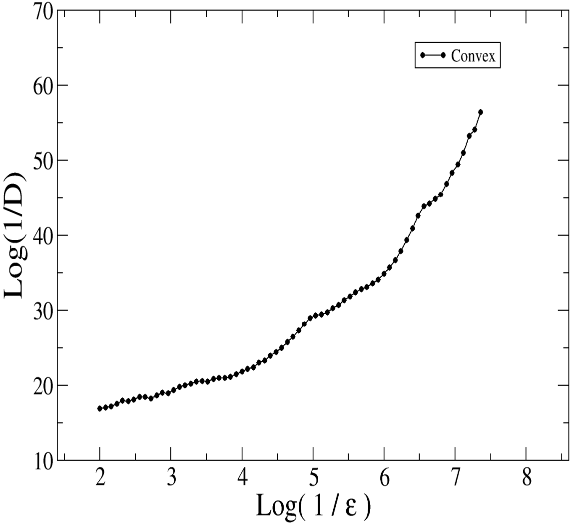
averaged over 1000 orbits.

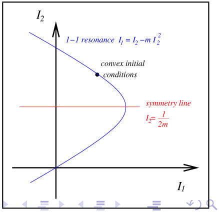
Comparison of the diffusion between the convex and non-convex initial conditions for many ε

With the help of normal form theory, for both convex and non-convex initial conditions:

- we select a point on an hyperbolic torus of the resonance
- we compute an approximation of the unstable local manifold
- we choose a set of 100 points aligned to the local unstable manifold
- we compute the diffusion coefficient of this set of points

Diffusion coefficients for convex initial conditions





Guzzo, Lega, Froeschlé, to appear on Chaos, 2011.

An exponential best fit with parameters

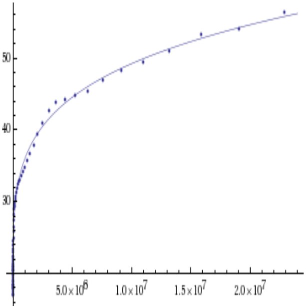
Fit of:

$$\log(1/D) = A + Bx + C10^{\beta x} \qquad x = \log(1/\varepsilon)$$

provides:

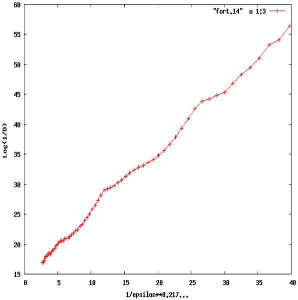
$$\beta \sim 0.2...$$

Log(1/D)



$1/\varepsilon$

Log(1/D)



$1/\varepsilon^{\beta}$

Guzzo, Lega, Froeschlé, to appear on Chaos.

The stability time of the actions for convex systems in a single resonance of a $n = 3$ system (Pöschel 1992) are:

$$|t| \leq \exp\left(\frac{\varepsilon_0}{\varepsilon}\right)^{\beta}$$

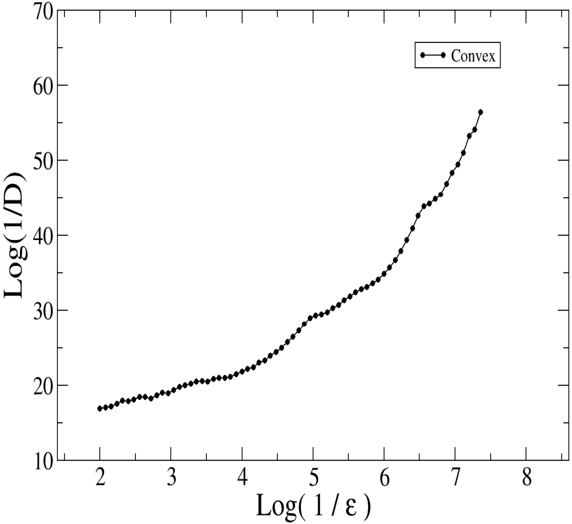
with

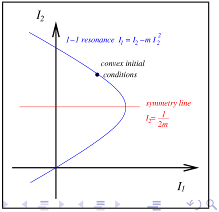
$$\beta = \frac{1}{2\,\nu} = \frac{1}{4}$$

which is compatible with $\beta = 0.2....$

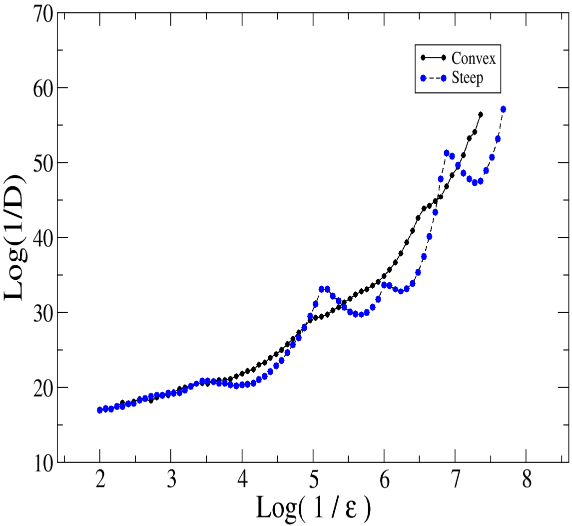
Semi-analytic computer estimates of normal forms reminders provided in similar conditions: $\beta \sim 0.21$ (Efthymiopoulos 2008).

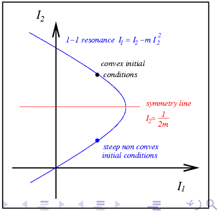
Diffusion coefficients





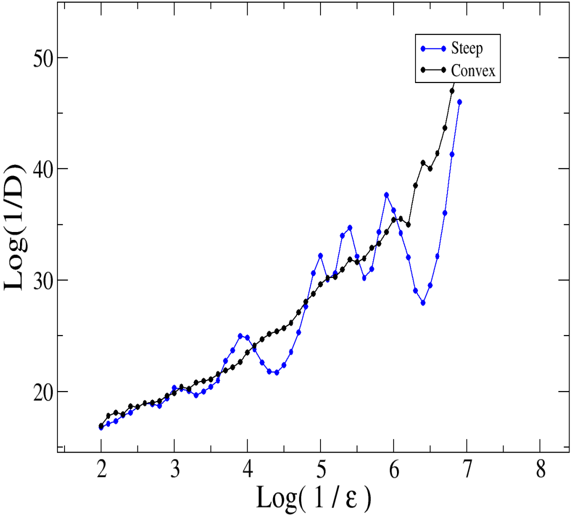
Diffusion coefficients

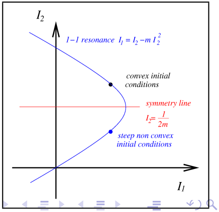




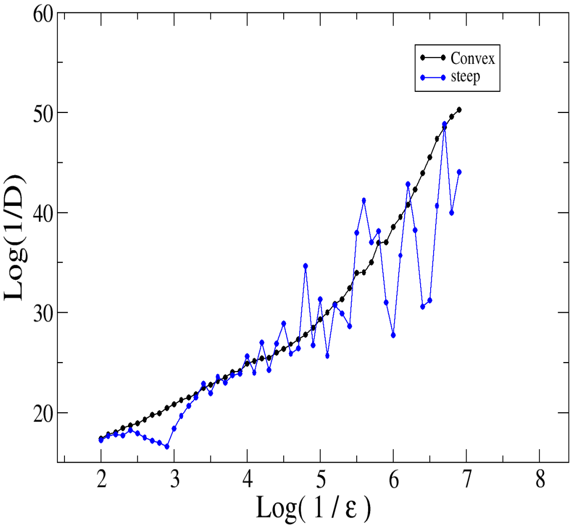
Guzzo, Lega, Froeschlé, to appear on Chaos, 2011.

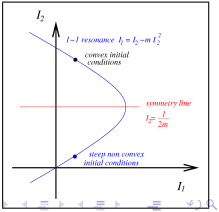
Diffusion coefficients





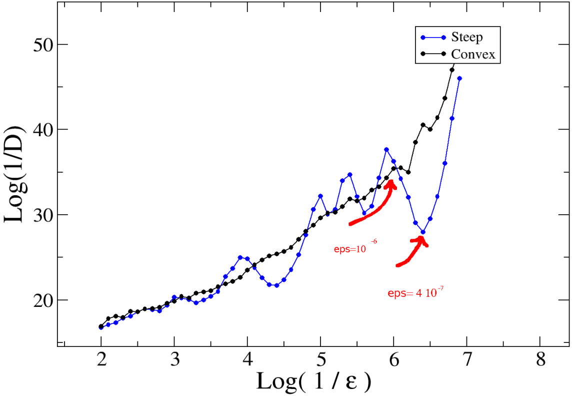
Diffusion coefficients



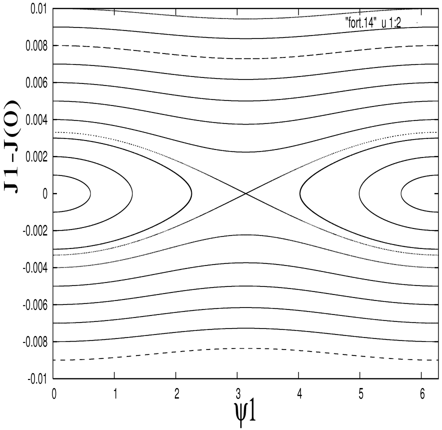


Unstable manifold of the resonant hyperbolic torus

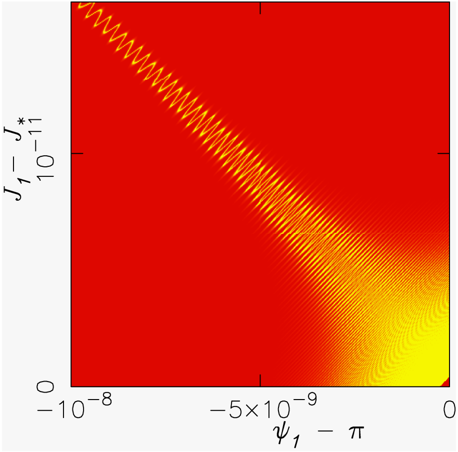
We find correlations between the amplitude of the lobes of unstable manifold of the resonant hyperbolic torus and the oscillations of the diffusion coefficient



Unstable manifold for $\varepsilon = 10^{-6}$ and $a = 10^{-6}$

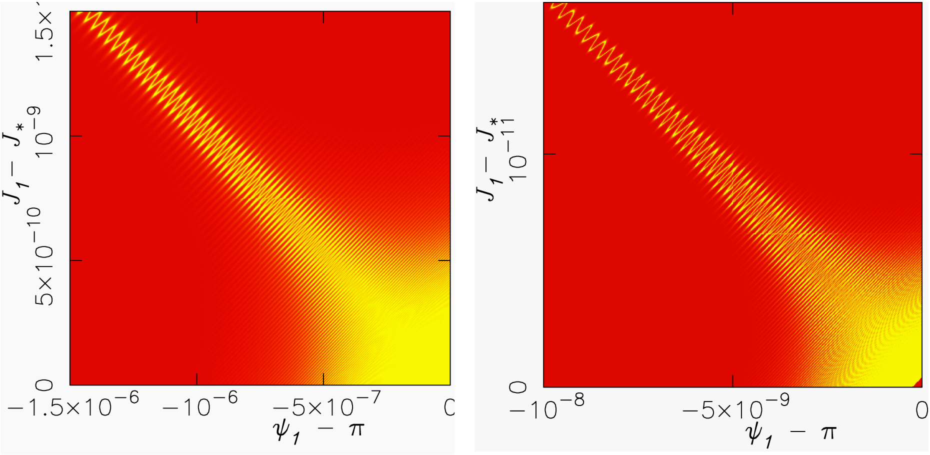


Phase portrait



Unstable manifold of the hyperbolic torus (GLF, 2011)

Comparison of the unstable manifolds

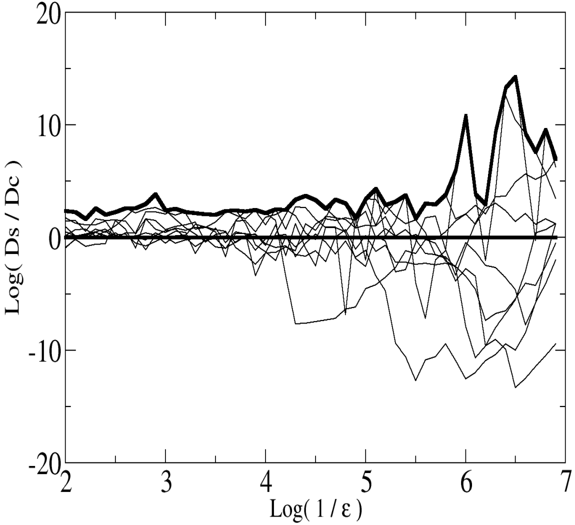


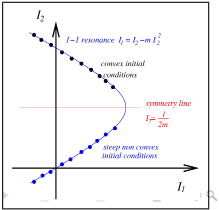
$\varepsilon = 4\ 10^{-7}$, $\log D \sim -27.9$

$\varepsilon = 10^{-6}$, $\log D \sim -36.2$

Guzzo, Lega, Froeschlé, to appear on Chaos, 2011.

Ratio D_s/D_c





Guzzo, Lega, Froeschlé, to appear on Chaos, 2011.

The 6D case: the double resonances

We consider the 6D symplectic map:

$$
\begin{aligned}
\varphi_1' &= \varphi_1 + I_1 \\
\varphi_2' &= \varphi_2 - I_2 + m\, I_2^2 \\
\varphi_3' &= \varphi_3 + I_3 \\
I_j' &= I_j - \varepsilon \frac{\partial f}{\partial \varphi_j}(\varphi') \quad , \quad j = 1, 2, 3
\end{aligned}
\qquad (1)
$$

with:

$$
f = \cos(\varphi_1) + \frac{1}{2}\cos(\varphi_1 + \varphi_2) + \frac{a}{\cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) + c}
$$

The 6D case: the double resonances

The frequencies are:

$$\omega = (I_1, -I_2 + m I_2^2, I_3) = \nabla h(I)$$

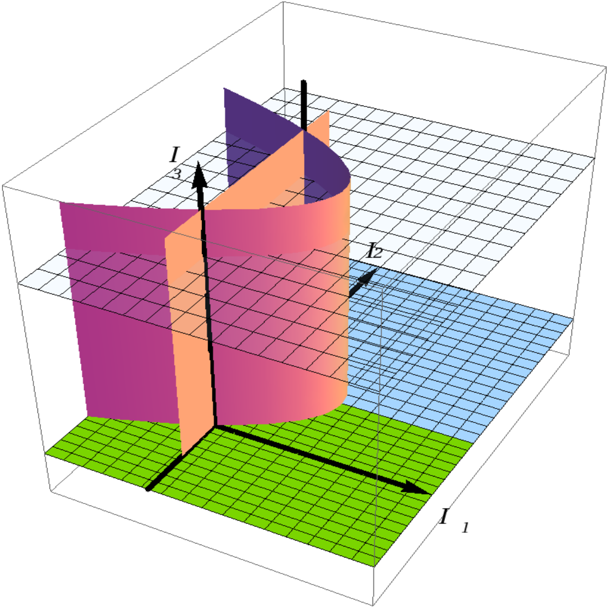
with:

$$h = \frac{I_1^2}{2} - \frac{I_2^2}{2} + m\,\frac{I_2^3}{3} + \frac{I_3^2}{2}$$

The map is designed to study the diffusion in the 1-0-0 and 1-1-0 resonance:

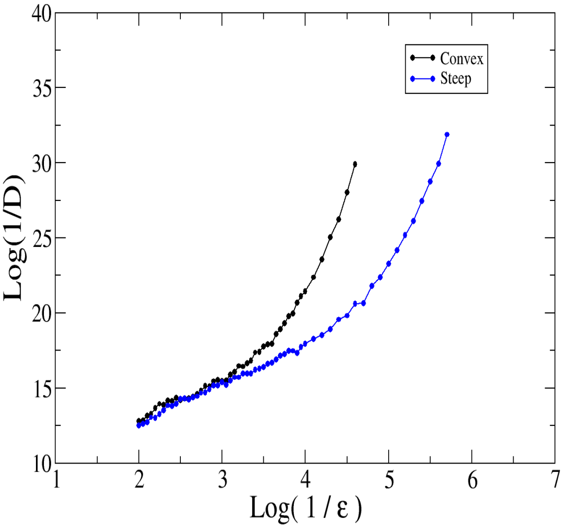
$$f = \cos(\varphi_1) + \frac{1}{2}\cos(\varphi_1 + \varphi_2) + \frac{a}{\cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) + c}$$

The 6D case: the double resonances



We measure the diffusion of the action I_3.

Diffusion coefficients



Guzzo, Lega, Froeschlé, to appear on Chaos, 2011.

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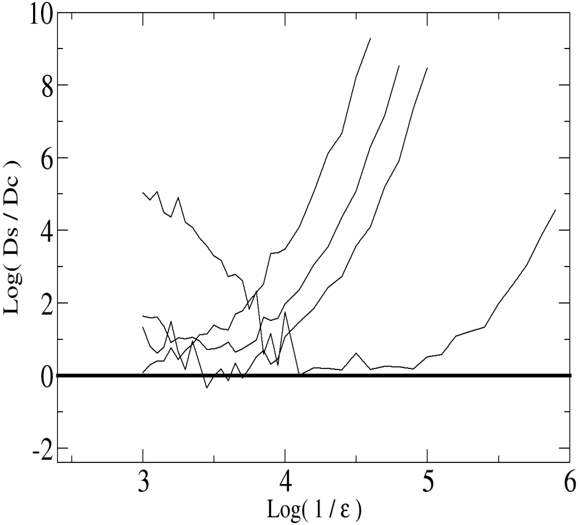

Guzzo, Lega, Froeschlé, to appear on Chaos, 2011.

Conclusions

For both convex and non–convex initial conditions the numerically measured diffusion decreases faster than a power law, but with important differences:

- In single resonances: the diffusion curve in the non-convex case is characterized by large oscillations: this behaviour is confirmed also by computations done with other perturbations and maps; the agreement of the numerical computations with the conjecture is not sharp and it is found after a sup over different initial conditions.
- In double resonances: the agreement of the numerical computations with the conjecture is sharp.