# Diffusion along mean motion resonances in the restricted three body problem

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- We consider the Restricted Planar Elliptic 3 Body Problem.
- Namely, we study the motion $q(t)$ of a massless body (Asteroid) under the influence of two primaries $q_1(t)$ and $q_2(t)$ of masses $\mu$ and $1-\mu$, which move along ellipses of eccentricity $e_0 > 0$ around their center of mass.
- We consider
  - $\mu = 10^{-3}$ which is a realistic value for the Sun-Jupiter model.
  - $e_0 > 0$ arbitrarily small.

## The 2 Body Problem

- If we omit the influence of Jupiter ($\mu = 0$), the system is reduced to two uncoupled 2 Body Problems (Sun-Jupiter and Sun-Asteroid).
- The motion of the Asteriod is given by Kepler Laws.
- First Kepler Law: Orbits of the 2BP are conic sections
- Assume that the Asteroid is moving along an ellipse.
- An ellipse can be given by its semimajor axis $a$ and its eccentricity $0 < e < 1$.
- For the 2BP these parameters are constants of motion.

## The mean motion resonances

- Third Kepler Law: Period of motion of the ellipse is $2\pi a^{3/2}$ where $a$ is the semimajor axis of the ellipse.
- Mean motion resonance is resonance between the period of the Asteroid and the period of Jupiter
- If we normalize the period of Jupiter to $2\pi$ (and its semimajor axis to 1), mean motion resonance appears when $a^{3/2}$ is rational.

- We want to see the influence of Jupiter ($\mu = 10^{-3}$) on the shape of the ellipse when the Asteroid is in mean motion resonance.
- We have focused our study in the mean motion resonance $1 : 7$ (period of the Asteroid is seven times the period of Jupiter).
- We expect that analogous phenomena take place in the other mean motion resonances.

**Theorem** For the Restricted Planar Elliptic 3 Body Problem with mass ratio $\mu = 10^{-3}$ and eccentricity $e_0$ small enough, there exist $T > 0$ and a trajectory whose (osculating) semimajor $a(t)$ and eccentricity $e(t)$ satisfy that

$$a(t) \sim 7^{2/3} \quad \text{for all } t \in [0, T]$$

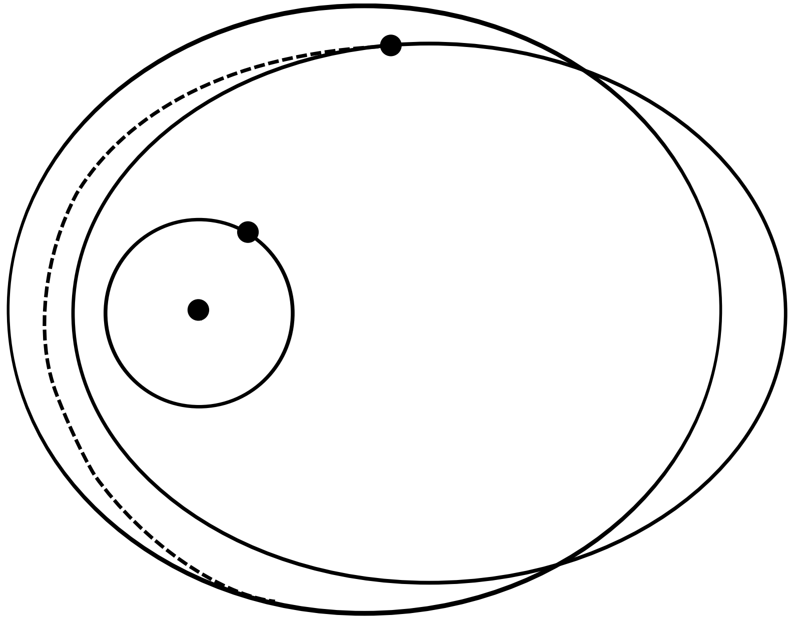
and

$$e(0) < 0.48 \quad \text{and} \quad e(T) > 0.66.$$

Namely,

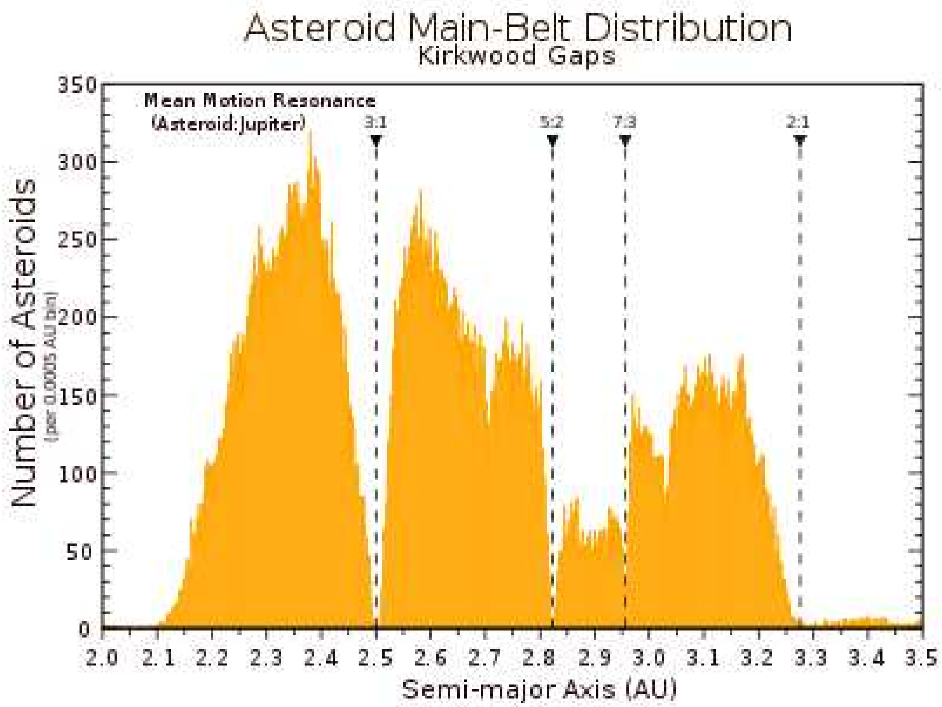
- The Asteroid keeps the semimajor axis almost constant and thus it remains in mean motion resonance.
- It drifts along the resonance, undergoing considerable changes in the eccentricity and thus in the shape of the ellipse.

- Schematically, we obtain orbits:



- Even though we have focused on the resonance $1:7$, we expect the same to happen in the other mean motion resonances.

## The Kirkwood gaps




- The Asteroid Belt is the region of the Solar System located roughly between the orbits of the planets Mars and Jupiter.

- At mean motion resonances of small order $3:1$, $2:1$, $5:2$, $7:3$, there are visible gaps in the distribution of the Asteroids, called Kirkwood gaps.

- This diffusing mechanism could give a justification of its existence.
- The eccentricity of Jupiter is $e_0 \sim 1/20$ and we need $e_0$ arbitrarily small.
- Another mechanism of instability in the $3:1$ Kirkwood gap based on Adiabatic chaos can be seen in Neishtadt-Sidorenko (2004).

## General comments on the proof

- The proof relies on geometric methods commonly used in the study of Arnol'd diffusion.
- This problem does not have big gaps.
- Some parts rely on high-accuracy numerical computations.
- We expect that these parts can be turned into a Computer Assisted Proof.

## Sketch of the proof

Main steps:

- Step 1: Consider the Action-Angle coordinates for the 2BP in elliptic regime (Delaunay coordinates)
- Step 2: Geometrical features of the Circular Problem ($\mu = 10^{-3}$ and $e_0 = 0$).
- Step 3: Study of the Elliptic Problem ($e_0 > 0$) as a perturbation of the Circular One.

## The two body problem

- When $\mu = 0$, the Hamiltonian becomes

$$H(q,p) = \frac{\|p\|^2}{2} - \frac{1}{\|q\|}.$$

- The Delaunay coordinates are the Action-Angle coordinates for the 2BP in elliptic regime:
  - $\ell$ is the mean anomaly.
  - $L$ is the square of the semimajor axis.
  - $\widetilde{g}$ is the argument of the perihelion.
  - $G$ is the angular momentum.

- One can define the eccentricity of the Asteroid using these coordinates as

$$e = \sqrt{1 - \frac{G^2}{L^2}}.$$

- In these coordinates the Hamiltonian of the 2BP become

$$H_{2BP}(\ell, L, \widetilde{g}, G) = -\frac{1}{2L^2}.$$

- If we apply the change to the RPE3BP we obtain

$$\begin{aligned} H(\ell, L, \widetilde{g}, G) = & -\frac{1}{2L^2} + \mu\Delta H_{\mathrm{circ}}(\ell, L, \widetilde{g} - t, G) \\ & + \mu e_0 \Delta H_{\mathrm{ell}}(\ell, L, \widetilde{g} - t, G, t) \end{aligned}$$

- The circular perturbating term only depends on $t$ through $\widetilde{g} - t$.

## Rotating Delaunay coordinates

- One can define a new system of coordinates with $g = \widetilde{g} - t$: Rotating Delaunay coordinates.
- New Hamiltonian

$$
\begin{aligned}
H(\ell, L, g, G, t) = & -\frac{1}{2L^2} - G + \mu \Delta H_{\mathrm{circ}}(\ell, L, g, G) \\
& + \mu e_0 \Delta H_{\mathrm{ell}}(\ell, L, g, G, t)
\end{aligned}
$$

- Since $\Delta H_{\mathrm{circ}}$ is independent of $t$, when $e_0 = 0$ the system has 2 degrees of freedom and the energy is preserved.
- This corresponds to the preservation of the Jacobi constant.
- We will look for diffusing orbits in this Hamiltonian when $e_0 > 0$.

## The mean motion resonance in Rotating Delaunay coordinates

- When $\mu = 0$ we have the Hamiltonian

$$H(\ell, L, g, G, ) = -\frac{1}{2L^2} - G$$

- Frequencies:

$$\dot{\ell} = \frac{1}{L^3} \quad \text{and } \dot{g} = -1$$

- The mean motion resonance $1 : 7$ corresponds to take $L = 7^{1/3}$ so that

$$\dot{\ell} = \frac{1}{7} \quad \text{and } \dot{g} = -1$$

- We want to obtain diffusing orbits along the mean motion resonance.
- Namely we will keep $L \sim 7^{1/3}$ (which implies keeping the semimajor axis almost constant)
- Since

$$e = \sqrt{1 - \frac{G^2}{L^2}},$$

big changes in $G$ are equivalent to big changes in $e$.

## The extended system

- In fact, we will consider the full 3 dof freedom system introducing the variable $I$ conjugate of $t$

$$\begin{aligned} H(\ell, L, g, G, t, I) = &- \frac{1}{2L^2} - G + \mu \Delta H_{\mathrm{circ}}(\ell, L, g, G) \\ &+ \mu e_0 \Delta H_{\mathrm{ell}}(\ell, L, g, G, t) + I \end{aligned}$$

- We restrict ourselves at the energy level $H = 0$.
- Since the perturbating terms are small and $L$ almost constant, to obtain an orbit with big changes in $G$ is equivalent to obtain an orbit with big changes in $I$.
- We look for orbits with big changes in $I$.

## The circular problem

- For a moment let us forget about $I$ and $t$.
- The Hamiltonian for the circular problem is

$$H(\ell, L, g, G) = -\frac{1}{2L^2} - G + \mu \Delta H_{\rm circ}(\ell, L, g, G)$$

- The energy is conserved.
- It has two degrees of freedom so it is impossible to obtain diffusion.
- We study the mean motion resonance numerically.
- We take advantage of the fact that this system is reversible with respect to the involution

$$R(\ell, L, g, G) = (-\ell, L, -g, G).$$

**Theorem** Consider the Hamiltonian

$$H(\ell, L, g, G) = -\frac{1}{2L^2} - G + \mu \Delta H_{\rm circ}(\ell, L, g, G)$$

Then, at each energy level $H \in [H_-, H_+] = [-1.81, -1.56]$,

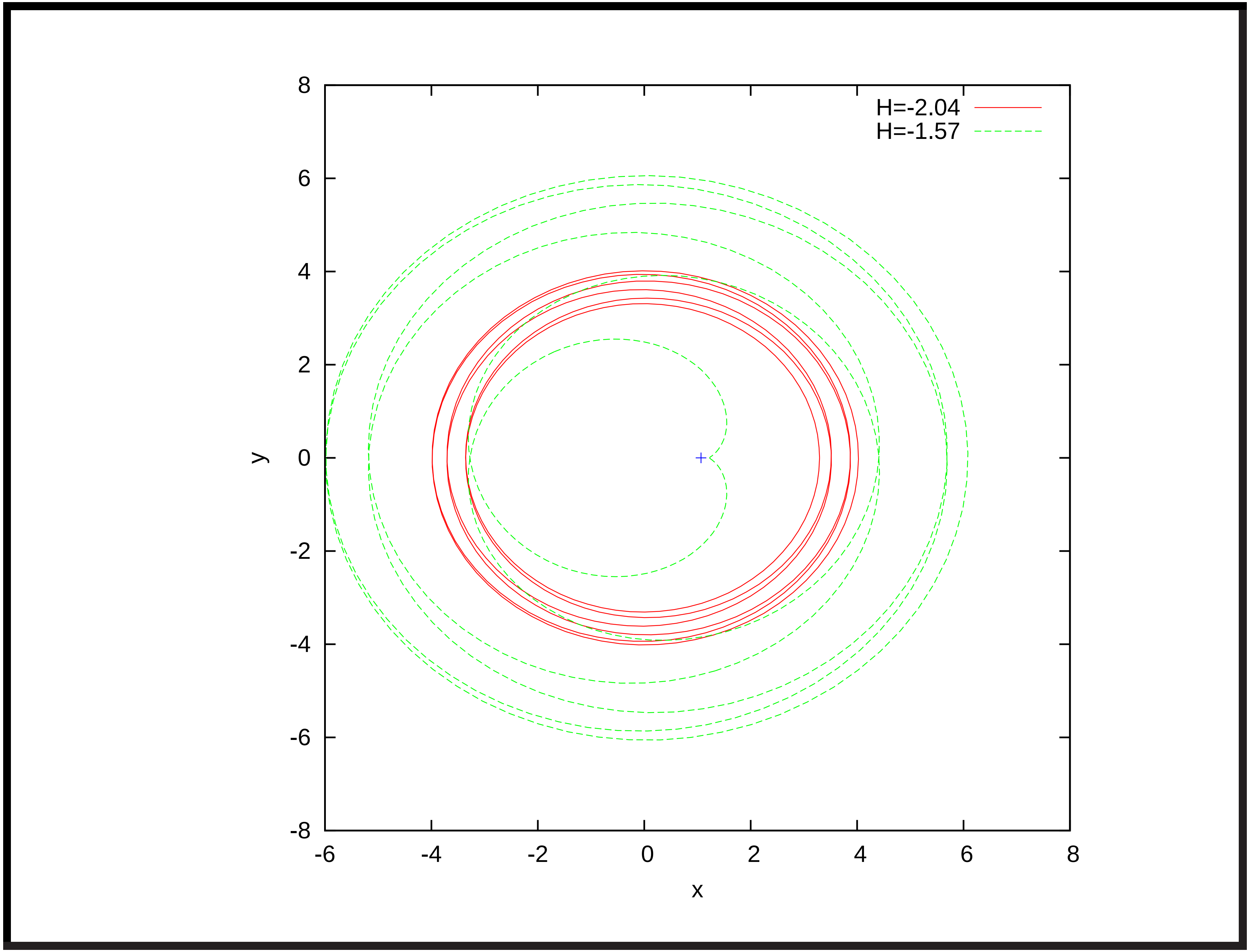
- There exists a hyperbolic periodic $\gamma_H$ orbit whose period satisfies

$$|T - 14\pi| < 60\mu = 60 \cdot 10^{-3}$$

- $\gamma_H$ has two branches of stable and unstable invariant manifolds.
- At each energy level either one set of branches of $\gamma_H$ or the other intersect transversally at the symmetry axis.

## Remarks on the theorem

- $H \to H_-$ implies that $e$ decreases (the orbit becomes more circular).
- In this regime the periodic orbit becomes weakly hyperbolic and the angle between the invariant manifolds decreases exponentially.
- It becomes harder to detect.
- When $H \to H_+$ the periodic orbit approaches the invariant manifolds of the point Lagrangian Equilibrium Point $L_2$.
- The period of the periodic orbit explodes and thus we move away from the resonance.


H=-2.04
H=-1.57
8
6
4
2
0
-2
-4
-6
-8
y
-6
-4
-2
0
2
4
6
8
x

## A priori chaotic versus a priori stable

- This theorem gives us at every energy level a periodic orbit which has a transversal homoclinic orbit for $\mu = 10^{-3}$.
- We will use this hyperbolic structure to obtain diffusing orbits when $e_0 > 0$.
- This type of systems are usually called *a priori chaotic*, since for the unperturbed problem ($e_0 = 0$) they present chaotic motion at each energy level (but not global instabilities).
- It presents similar features to the so-called Mather Problem: existence of orbits whose energy grows arbitrarily in geodesic flows with a periodic potential.

- When $\mu \to 0$, the system becomes nearly completely integrable: it is an *a priori stable* system.
- The splitting angle at the homoclinic points is exponentially small with respect to $\mu$ and therefore it is very difficult to prove the transversality of the invariant manifolds.

## The extended circular problem

- Hamiltonian for the extended circular problem

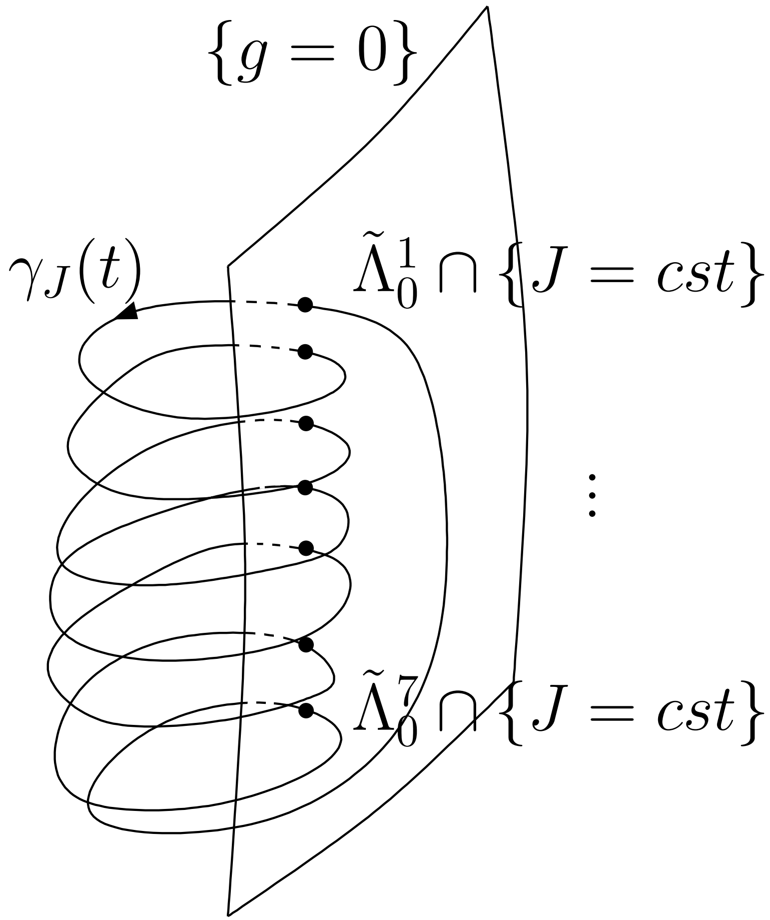
$$H(\ell, L, g, G, t, I) = -\frac{1}{2L^2} - G + \mu \Delta H_{\mathrm{circ}}(\ell, L, g, G) + I$$

  at the energy level $H = 0$.

- Now the dynamics is restricted to planes $I =$ constant.
- If we take into account the variable $t$, the periodic orbits of the circular problem are now 2 dimensional tori
- The union of the periodic orbits form a 3 dimensional Normally Hyperbolic Invariant Manifold $\Lambda_0$.
- We want to define inner and outer dynamics associated to it.

- To this end we consider a Poincaré map.
- We fix the Poincaré section $\{g = 0\}$ and the map

$$\mathcal{P}_0 : \{g = 0\} \longrightarrow \{g = 0\}$$




- The Poincaré map $\mathcal{P}_0$ has a 2 dimensional NHIM $\widetilde{\Lambda}_0$
- It has seven connected components

$$\widetilde{\Lambda}_0 = \cup_{j=0}^{6} \widetilde{\Lambda}_0^j$$

- In fact, $\mathcal{P}_0(\widetilde{\Lambda}_0^j) = \widetilde{\Lambda}_0^{j+1}$.
- They are invariant by $\mathcal{P}_0^7$.
- $\mathcal{P}_0^7$ has seven NHIMs:

$$\widetilde{\Lambda}_0^j \ , j = 0, \dots, 6.$$

- $(I, t)$ are global coordinates for each of these connected components.
- We can use them to define the inner and outer dynamics.
- One could also have used the Poincaré map associated to $\{t = 0\}$ and use coordinates $(G, g)$.
- The advantage of using $(I, t)$ is that $I$ is constant for the circular problem and therefore it will be easier to study the influence of the elliptic perturbation in order to prove diffusion.

## Inner and outer dynamics of the circular problem

- We chose one of the cylinders: $\widetilde{\Lambda}_0^3$.
- Recall that it is invariant by $\mathcal{P}_0^7$.
- By $\mathcal{P}_0^7$ it has heteroclinic connections with $\widetilde{\Lambda}_0^2$ and $\widetilde{\Lambda}_0^4$.
- We choose it because the heteroclinic connections between $\widetilde{\Lambda}_0^3$ and $\widetilde{\Lambda}_0^4$ intersect transversally at the symmetry axis and thus are easier to compute.
- We want to define:
  - Inner dynamics
  - Outer dynamics

## Inner map of the circular problem

- It is given by the Poincaré map $\mathcal{P}_0^7$ restricted to $\widetilde{\Lambda}_0^3$.
- Since $I$ is constant, it is integrable.
- Then, it is of the form

$$\mathcal{F}_0^{\mathrm{in}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \mathcal{T}_0(I) \end{pmatrix}.$$

- $14\pi + \mathcal{T}_0(I)$ is the period of the periodic orbit we have obtained for the circular problem.
- It can be checked (numerically) that it is twist.

## Outer map of the circular problem

- At each energy level either $W^u(\widetilde{\Lambda}_0^3) \pitchfork W^s(\widetilde{\Lambda}_0^4)$ or $W^u(\widetilde{\Lambda}_0^4) \pitchfork W^s(\widetilde{\Lambda}_0^3)$.
- Associated to the transversal homoclinic points we can define a scattering map, which we call forward or backward.

$$\mathcal{S}_0^{\mathrm{f}} : \widetilde{\Lambda}_0^3 \to \widetilde{\Lambda}_0^4$$

$$\mathcal{S}_0^{\mathrm{b}} : \widetilde{\Lambda}_0^4 \to \widetilde{\Lambda}_0^3$$

- $\mathcal{S}_0^*(x_-) = x_+$ if the homoclinic orbit tends to $x_-$ in the past and to $x_+$ in the future.

- We want to define outer maps from $\widetilde{\Lambda}_0^3$ to itself $\widetilde{\Lambda}_0^3$.
- We compose the scattering map with the Poincaré map

$$\mathcal{F}_0^{\mathrm{out,f}} = \mathcal{P}_0^6 \circ \mathcal{S}_0^{\mathrm{f}}$$

$$\mathcal{F}_0^{\mathrm{out,b}} = \mathcal{S}^{\mathrm{b}} \circ \mathcal{P}_0$$

- As $I$ is a first integral, the outer maps (wherever they are defined) are of the form

$$\mathcal{F}_0^{\mathrm{in},*} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \omega^*(I) \end{pmatrix} \quad * = \mathrm{f}, \mathrm{b}$$

- We compute $\omega^*(I)$ numerically.

## Conclusion

- We have inner and (two) outer dynamics associated to $\Lambda_0^3$ for the circular problem.
- They are given by

$$\mathcal{F}_0^{\mathrm{in}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \mathcal{T}_0(I) \end{pmatrix}$$

and

$$\mathcal{F}_0^{\mathrm{out},*} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I \\ t + \omega^*(I) \end{pmatrix} \quad * = \mathrm{f}, \mathrm{b}$$

- They are all integrable.

## The elliptic problem

- We study the elliptic problem ($e_0 > 0$) as a perturbation of the circular one ($e_0 = 0$).
- For $e_0$ small enough
  - The NHIM $\widetilde{\Lambda}_0^j$ are preserved, slightly deformed, as $\widetilde{\Lambda}_{e_0}^j$.
  - Roughly speaking, for each $I$, $W^u(\widetilde{\Lambda}_0^3) \pitchfork W^s(\widetilde{\Lambda}_0^4)$ or $W^u(\widetilde{\Lambda}_0^4) \pitchfork W^s(\widetilde{\Lambda}_0^3)$ are transversal.
  - We can associate inner and outer dynamics to $\widetilde{\Lambda}_0^3$: $\mathcal{F}_{e_0}^{\mathrm{in}}$, $\mathcal{F}_{e_0}^{\mathrm{out,f}}$ and $\mathcal{F}_{e_0}^{\mathrm{out,b}}$ as in the circular problem.
  - We study them perturbatively.

## A particular feature of the elliptic Hamiltonian

- Hamiltonian

$$H(\ell, L, g, G, t, I) = -\frac{1}{2L^2} - G + \mu\Delta H_{\mathrm{circ}}(\ell, L, g, G) + \mu e_0 \Delta H_{\mathrm{ell}}(\ell, L, g, G, t) + I$$

- We can expand

$$\Delta H_{\mathrm{ell}} = \Delta H^1_{\mathrm{ell}} + e_0 \Delta H^2_{\mathrm{ell}} + \mathcal{O}\left(e_0^2\right)$$

- $\Delta H^1_{\mathrm{ell}}$ only has the $t$-harmonics $\{\pm 1\}$.
- $\Delta H^2_{\mathrm{ell}}$ only has the $t$-harmonics $\{0, \pm 1, \pm 2\}$.

## The perturbed inner map

- Is of the form

$$\mathcal{F}_{e_0}^{\mathrm{in}} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 A_1(I,t) + e_0^2 A_2(I,t) + \mathcal{O}\left(e_0^3\right) \\ t + \mathcal{T}_0(I) + e_0 \mathcal{T}_1(I,t) + e_0^2 \mathcal{T}_2(I,t) + \mathcal{O}\left(e_0^3\right) \end{pmatrix}.$$

- $A_1$ and $\mathcal{T}_1$ only have $t$-harmonics $\{\pm 1\}$.
- $A_2$ and $\mathcal{T}_2$ only have $t$-harmonics $\{0, \pm 1, \pm 2\}$.
- As we will see later, we only need to know explicitly

$$A_1(I,t) = A_1^+(I)e^{it} + A_1^-(I)e^{-it}.$$

- We compute $A_1^{\pm}(I)$ numerically.

## The perturbed outer maps

- Are of the form

$$\mathcal{F}_{e_0}^{\mathrm{out},*} : \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 B_1^*(I,t) + \mathcal{O}\left(e_0^2\right) \\ t + \omega^*(I) + \mathcal{O}(e_0) \end{pmatrix} \quad * = \mathrm{f}, \mathrm{b}.$$

- $B_1^*$ are computed (numerically) through Melnikov integrals.
- $B_1^*$ only has $t$-harmonics $\{\pm 1\}$.

- We have defined inner and outer dynamics.
- We want to combine them to obtain a transition chain of tori.
- Namely, we want to obtain a sequence of invariant tori of the inner map $\{\mathbb{T}_j\}_{j=1\ldots N}$ such that

$$W^u\left(\mathbb{T}_j\right) \pitchfork W^s\left(\mathbb{T}_{j+1}\right)$$

- It is equivalent to see that for each tori $\mathbb{T}_j$ either

$$\mathcal{F}_{e_0}^{\mathrm{out,f}}(\mathbb{T}_j) \cap \mathbb{T}_{j+1} \neq \emptyset \quad \text{or } \mathcal{F}_{e_0}^{\mathrm{out,b}}(\mathbb{T}_j) \cap \mathbb{T}_{j+1} \neq \emptyset$$

- To obtain it we perform two steps of averaging to the inner map.

## Two steps of averaging

- Inner map:

$$\mathcal{F}^{\text{in}}_{e_0}: \begin{pmatrix} I \\ t \end{pmatrix} \mapsto \begin{pmatrix} I + e_0 A_1(I,t) + e_0^2 A_2(I,t) + \mathcal{O}\left(e_0^3\right) \\ t + \mathcal{T}_0(I) + e_0 \mathcal{T}_1(I,t) + e_0^2 \mathcal{T}_2(I,t) + \mathcal{O}\left(e_0^3\right) \end{pmatrix}.$$

- The $e_0$ and $e_0^2$ terms of the inner map only have $t$-harmonics $\{0, \pm 1, \pm 2\}$.

- If we perform two steps of averaging, the small divisors that appear are only

$$e^{\pm \mathcal{T}_0(I)} - 1 \quad \text{and } e^{\pm 2\mathcal{T}_0(I)} - 1$$

- The bounds we know for $\mathcal{T}_0(I)$ show that these small divisors never vanish.

## The inner map in the new variables

- Then, we can perform the two steps of averaging globally.
- Inner map in the new variables:

$$\mathcal{F}_{e_0}^{\mathrm{in}} : \begin{pmatrix} \mathcal{I} \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{I} + \mathcal{O}\left(e_0^3\right) \\ \tau + \mathcal{T}_0(\mathcal{I}) + e_0^2 \mathcal{T}_2(\mathcal{I}) + \mathcal{O}\left(e_0^3\right) \end{pmatrix}.$$

- The new inner map is $e_0^3$-close to integrable.
- It is a twist map.
- We can apply KAM Theorem.
- We obtain a sequence of tori $\{\mathbb{T}_j\}_{j=1,\dots,N}$, which are $e_0^{3/2}$-close to each other.

## The outer map in the new variables

- To obtain the transition chain, we consider the outer maps in the new variables:

$$\widetilde{\mathcal{F}}_{e_0}^{\mathrm{out},*} : \begin{pmatrix} \mathcal{I} \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{I} + e_0 \widetilde{B}^* (\mathcal{I}, \tau) + \mathcal{O}\left(e_0^2\right) \\ \tau + \omega^*(\mathcal{I}) + \mathcal{O}(e_0) \end{pmatrix}, \quad * = \mathrm{f}, \mathrm{b}$$

  where

$$\widetilde{B}^* (\mathcal{I}, \tau) = \widetilde{B}^{*,+} (\mathcal{I})\, e^{i\tau} + \widetilde{B}^{*,-} (\mathcal{I})\, e^{-i\tau}$$

  with

$$\widetilde{B}^{*,\pm} (\mathcal{I}) = B^{*,\pm} (\mathcal{I}) - \frac{e^{\pm i\omega^*(\mathcal{I})} - 1}{e^{\pm i\mathcal{T}_0(\mathcal{I})} - 1} A_1^{\pm} (\mathcal{I})\,.$$

- We want the outer map to connect tori which are $e_0^{3/2}$-close.
- It is enough to check that $\widetilde{B}^{*,\pm} (\mathcal{I}) \neq 0$.
- Namely, $\widetilde{B}^{*,\pm}$ is defined through the functions $A_1^{\pm}$, $B^{*,\pm}$, $\mathcal{T}_0$ and $\omega^*$, which have been already computed.

- Since $\widetilde{B}^{*,\pm}(\mathcal{I}) \neq 0$, the jumps of the outer maps are bigger than the distance between tori.
- Namely, there are no big gaps.
- Therefore, $\{\mathbb{T}_j\}_{j=1,\ldots,N}$ is a transition chain.
- Once we have the transition chain, it is enough to use a shadowing method to obtain the true orbit.

## Time of diffusion

- The used methods do not give any estimate on the time of diffusion.
- We expect that it is of the form

$$T \sim \frac{\ln(\mu e_0)}{\mu e_0}.$$