Small dissipative perturbations of area preserving flows on surfaces.

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joint work with Mark Freidlin and Leonid Koralov

Summary.

Hyperbolic equilibrium point causes instabilities in small perturbations of integrable Hamiltonian systems.

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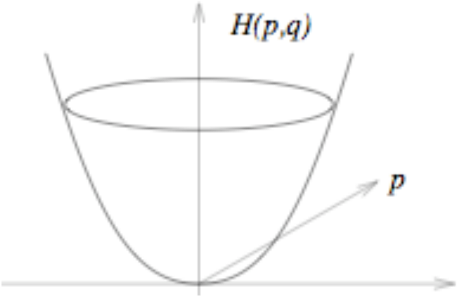
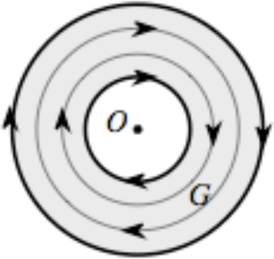
Hyperbolic equilibrium point causes instabilities in small perturbations of integrable Hamiltonian systems.
We illustrate this paradigm for one degree of freedom systems.

One well potential





$$\ddot{x} = -U'(x) - \varepsilon \dot{x}$$

$$E = \frac{(\dot{x})^2}{2} + U, \quad \dot{E} = -\varepsilon (\dot{x})^2.$$

Averaging

$$\ddot{x} = -U'(x) - \varepsilon \dot{x}$$

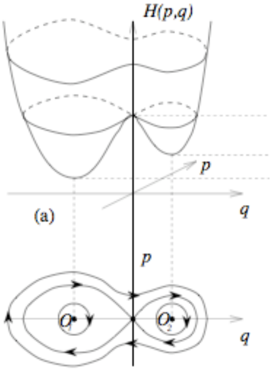
$$\dot{E} = -\varepsilon (\dot{x})^2$$

$$E(T) - E(0) \approx -\varepsilon \oint (\dot{x})^2 dt = -\varepsilon \oint \dot{x} dt = -\varepsilon \mathrm{Area}(\mathrm{Int}(\gamma(E))$$

$E \approx \bar{E}$ where

$$\frac{d\bar{E}}{dt} = -\varepsilon \frac{\mathrm{Area}(\mathrm{Int}(\gamma(\bar{E}))}{T(\bar{E})}.$$

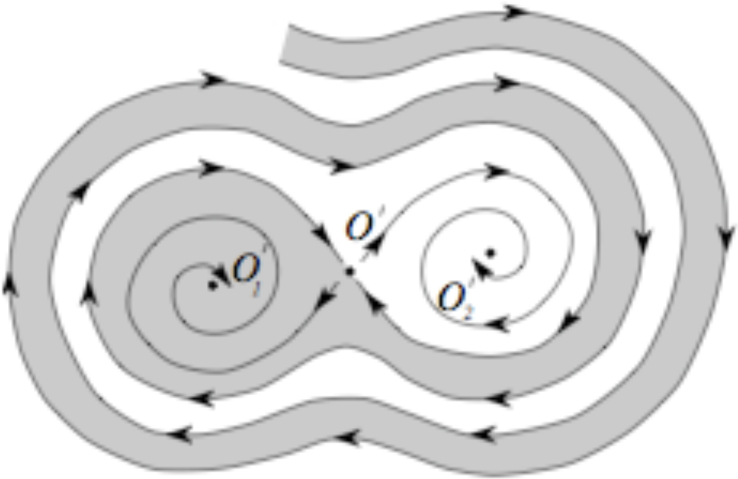
Double well potential



Which equilibrium point the orbit converges to?

Multi well potential

Which equilibrium point the orbit converges to?



$O(\varepsilon)$ changes in initial conditions lead to different answers.
So it makes sense to consider convergence to O_1 (O_2) as random events.

Ways to define the probability of an event

1. **Random initial condition regularization** (Arnold): Take initial conditions uniformly distributed on $B(x_0, \delta)$. Compute $\lim_{\varepsilon\to 0} \mathbb{P}_{\varepsilon,\delta}(O_j)$ and then take $\delta \to 0$.
2. **Small noise regularization** (Freidlin): Consider

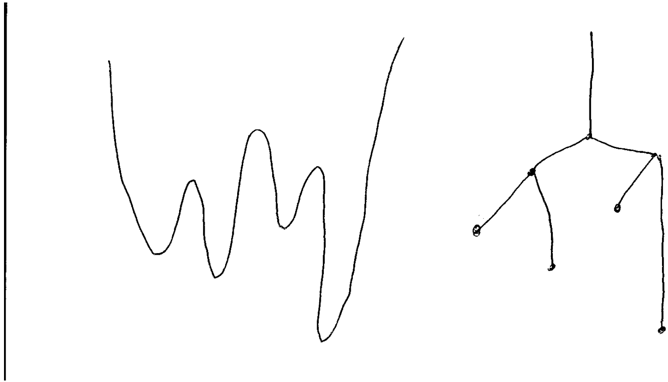
$$\dot{z} = \nabla^{\perp} H(z) + \varepsilon b(z) + \delta\sqrt{\varepsilon}\dot{w}$$

   Compute $\lim_{\varepsilon\to 0} \mathbb{P}_{\varepsilon,\delta}(O_j)$ and then take $\delta \to 0$.

In both definitions the results should **not** depend on the choice of the Riemann metric.
Theorem (Neishtadt, Brin-Freidlin)

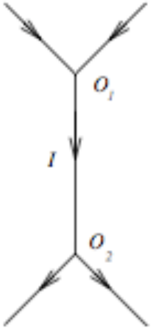
$$\mathbb{P}(O_1) = \frac{\mathrm{Area}(\mathrm{Int}(\Omega_1))}{\mathrm{Area}(\mathrm{Int}(\Omega_1)) + \mathrm{Area}(\mathrm{Int}(\Omega_2))}.$$

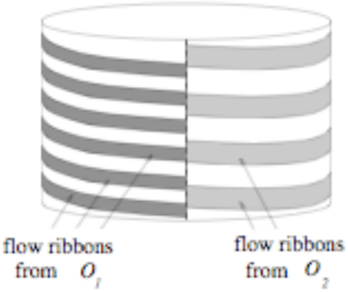
Multiple separatrix passages



$$\dot{z} = \nabla^{\perp} H(z) + \varepsilon b(z)$$

Question. Are multiple separatrix passages independent?





Answer: (Brin-Freidlin)

- YES for small noise regularization
- SOMETIMES for initial condition regularization

Restatement.

Consider the equation

$$\dot{z} = \nabla^{\perp} H(z) + \varepsilon b(z)$$

on a plane or a compact surface.

Theorem. (Brin-Fredlin) Take $\tau = t/\varepsilon$. Then the motion of the slow component converges (after the small noise regularization) to the Markov process such that

- The motion along the edges is deterministic and given by the averaging principle
- The the process comes to a vertex it immediately moves to the next edge.
- The next edge is chosen with probability proportional to separatrix integrals.

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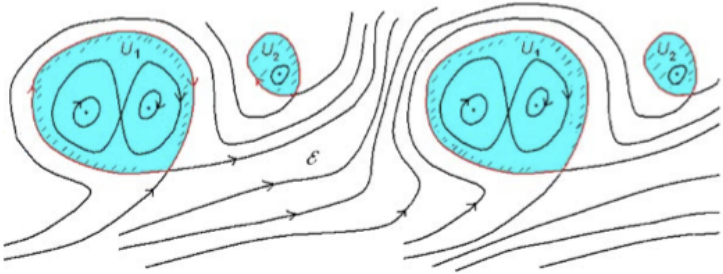
Question 1. What is the limiting process for random initial condition regularization?

Question 2. (Khanin, 1993) What if we consider perturbations of area preserving (non Hamiltonian) flows on surfaces?

Flows on surfaces.

Assume that

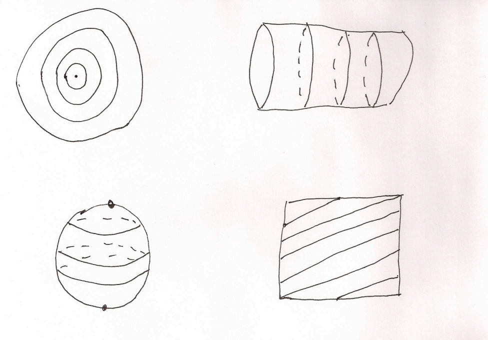
- Equilibrium points are non-degenerate;
- No saddle connections



Then $\omega(z)$ is

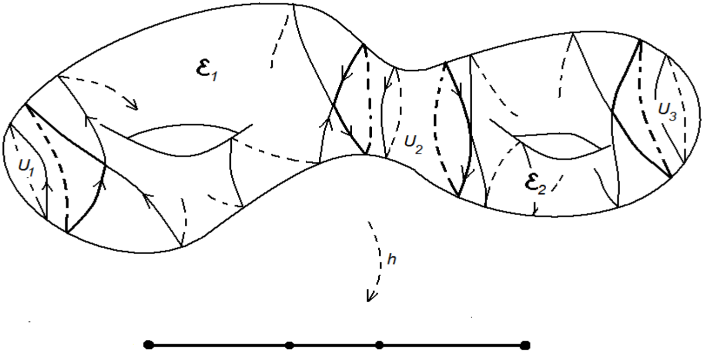
- equilibrium point or
- periodic orbit or
- suspension flow over an interval exchange transformation

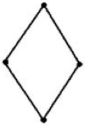
Flows on surfaces.

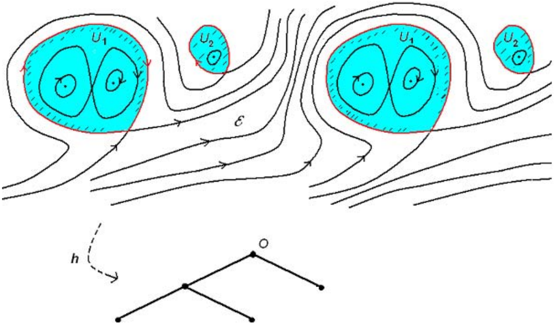


Periodic orbits can be divided into finitely many components where each component is

- **Disc** or
- **Cylinder** or
- Sphere or
- Torus


$\mathcal{E}_1$
U_1
U_2
U_3
$\mathcal{E}_2$
h





U_1
U_2
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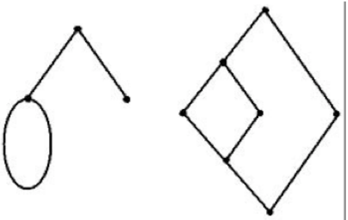

Main result.

$$\dot{z} = v + \varepsilon b.$$

$$r_k = \int_{\Omega_k} \langle b, \nabla H \rangle dt \text{ where } v = \nabla^{\perp} H \text{ near } \Omega_k, \quad H = 0 \text{ on } \Omega_k.$$

Theorem. Take $\tau = t/\varepsilon$. Then the motion of the slow component converges (after the small noise regularization) to the Markov process such that

- The motion along the edges is deterministic and given by the averaging principle
- The the process comes to a vertex it
  - leaves it immediately if the vertex corresponds to a saddle point
  - **Stays for a random time having exponential distribution with parameter $\lambda(E) = \sum_k \frac{r_k}{\mathrm{Area}(E)}$ if the vertex corresponds to a positive measure component E**
- The next edge is chosen with probability proportional r_k.

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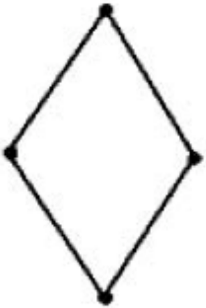
Question 1. What about random initial condition regularization?

Intermittency.



In particular small dissipative perturbations of area preserving flows can lead to an intermittent behavior if the corresponding graph has cycles.

Small random perturbations of area preserving flows

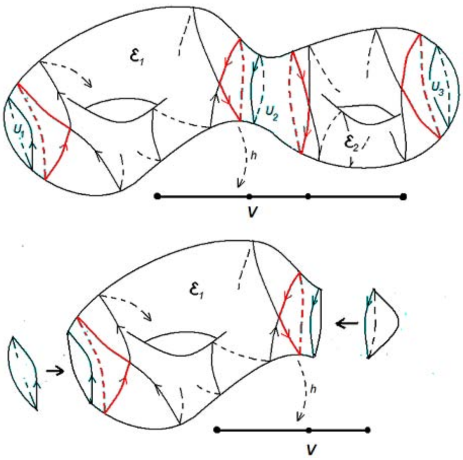
Our main result follows from

Theorem. Consider a Markov process with generator

$$L_\varepsilon = \frac{1}{\varepsilon}\langle v, \nabla\rangle + L$$

where L is a generator of a non-degenerate diffusion when as $\varepsilon \to 0$ the motion of the slow component converges to a Markov process on the graph with explicit generator.

Localization



We may assume that our graph is star-shaped.

Diffusions with boundary conditions: Brownian motion.

$S_{n+1} - S_n = \pm 1$ with equal probabilities.
$\frac{S_{Nt}}{\sqrt{N}} \Rightarrow$ Brownian motion.
Density of the limiting process satisfies heat equation.
Weak (martingale) formulation:

$$\mathbb{E}(u(w(T)) - u(W(0))) = \mathbb{E}\left(\frac{1}{2}\int_0^T \Delta u(W(s))ds\right)$$

for smooth test functions u.

Diffusions with boundary conditions: skew Brownian motion.

$S_{n+1} - S_n = \pm 1$ with equal probabilities except if $S_n = 0$ then it moves right with probability p and left with probability q.
$\frac{S_{Nt}}{\sqrt{N}} \Rightarrow$ skew Brownian motion.
Martingale formulation:

$$\mathbb{E}(u(w(T)) - u(W(0))) = \mathbb{E}\left(\frac{1}{2}\int_0^T \Delta u(W(s))ds\right)$$

if $pu'_+(0) = qu'_-(0)$.

Diffusions with boundary conditions: slowly reflecting Brownian motion.

$S_{n+1} - S_n = \pm 1$ with equal probabilities except if $S_n = 0$ then it moves right with probability $\frac{p}{\sqrt{N}}$ and stays at 0 with probability $1 - \frac{p}{\sqrt{N}}$.
$\frac{S_{Nt}}{\sqrt{N}} \Rightarrow$ skew Brownian motion.
Martingale formulation:

$$\mathbb{E}(u(w(T)) - u(W(0))) = \mathbb{E}\left(\frac{1}{2}\int_0^T \Delta u(W(s))ds\right)$$

if $pu'(0) = \frac{1}{2}\Delta u(0)$.

Diffusions with boundary conditions: general case.

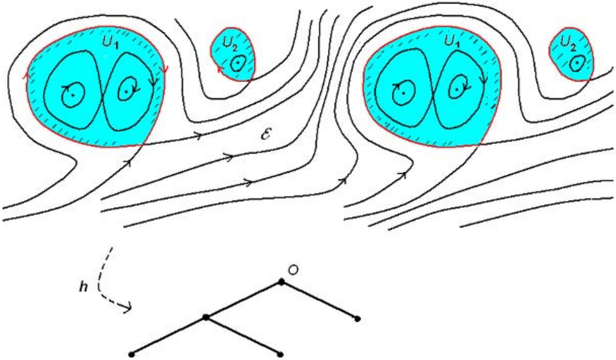
Martingale formulation:

$$\mathbb{E}(u(w(T)) - u(W(0))) = \mathbb{E}\left(\frac{1}{2}\int_0^T (Lu)(W(s))ds\right)$$

if $\sum_j p_j u_j'(0) = a(Lu)(0)$.
$a = a(c)$ where the invariant measure satisfies

$$d\mu = \rho dx + c\delta_0.$$

Key ingredient



We need to show that the limiting process is Markov that is for $x \in E$ and $\delta > 0$

$$\mathbb{P}_x(\tau_{\Omega_j} > 0) \to 0 \text{ as } \varepsilon \to 0.$$

Key ingredient

Berestycki–Hamel–Nadirashvili (2005):

$$L_\varepsilon = \frac{1}{\varepsilon}\langle v, \nabla\rangle + L$$

where L is a non-degenerate diffusion on a manifold M with non empty boundary. Assume that Lebesgue measure is invariant. Then

$$\mathbb{P}(\tau_{\partial M} > \delta) \to 0 \text{ as } \varepsilon \to 0$$

iff v has no $\mathbf{H^1_0}$-eigenfunctions.
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This result was improved in Constantin–Kiselev–Ryzhik–Zlatos (2008) and Zlatos (2010)
In our case the absence of $\mathbf{H}_0^1$-eigenfunctions follows from Katok (1973) whereas the absence of $\mathbf{L}^2$-eigenfunctions is only known for almost all rotation numbers Khanin–Sinai (1992) and Ulcigrai (2010) and is open in general.

Open question