A geometric mechanism of diffusion in a priori unstable Hamiltonian systems

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Amadeu Delshams[1]
(Rafael de la Llave[2], Tere M. Seara[1])

[1]Universitat Politècnica de Catalunya

[2]University of Texas at Austin

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Related collaborators: Elisabet Canalias, Marian Gidea, Gemma Huguet, Vadim Kaloshin, Josep Masdemont, Pau Roldán, . . .

Instability for a priori unstable Hamiltonian systems

We consider a periodic in time perturbation of n pendula and a d-dimensional rotor described by the non-autonomous Hamiltonian,

$$H(p,q,I,\varphi,t,\varepsilon) = P(p,q) + h(I) + \varepsilon Q(p,q,I,\varphi,t,\varepsilon), \tag{1}$$

with

$$P(p,q) = \sum_{j=1}^{n} P_j(p_j,q_j), \quad P_j(p_j,q_j) = \pm\left(\frac{1}{2}p_j^2 + V_j(q_j)\right),$$

where $I \in \mathcal{I} \subset \mathbb{R}^d, \varphi \in \mathbb{T}^d$, $\mathcal{I}$ an open set, $p,q \in \mathbb{R}^n$, $t \in \mathbb{T}^1$, and $P_j(p_j,q_j)$ is a *pendulum* for the saddle variables p_j, q_j.

For $\epsilon = 0$, the d-dimensional action I remains constant.

Main question: What happens to $I(t)$ for small $\epsilon \neq 0$? Is there global instability? i.e., $I(t) - I(0) = \mathcal{O}(1)$, or even does $I(t)$ perform rather arbitrary excursions in $\mathcal{I}$?

Elementary and regularity assumptions

- **H1** We will assume that the functions h, V_j, Q are C^r in their corresponding domains with $r \geq r_0$ sufficiently large.
- **H2** We will assume that the potentials V_j have non-degenerate local maxima, say at $q_j = 0$, each of which gives rise to a homoclinic orbit $(p_j^*(t), q_j^*(t))$ of the pendulum P_j:

$$\begin{aligned} &\frac{d}{dt} q_j^*(t) = p_j^*(t); \quad \frac{d}{dt} p_j^*(t) = -V_j'(q_j^*(t)); \\ &\lim_{t \to \pm\infty} (p_j^*(t), q_j^*(t)) = (0, 0). \end{aligned} \tag{2}$$

- **H3** The mapping $I \to \omega(I) := \frac{\partial}{\partial I} h(I)$ is a local diffeomorphism from $\mathcal{I}$ to its image.

Simplifying assumption

We will furthermore assume the simplifying hypothesis:

- **H4** The function Q in (1) is a trigonometric polynomial on (φ, t):

$$Q(p,q,I,\varphi,t,\varepsilon) = \sum_{(k,l)\in\mathcal{N}_Q} Q_{k,l}(p,q,I,\varepsilon)\exp(2\pi\ i(k\cdot\varphi + lt)) \quad (3)$$

with $\mathcal{N}_Q \subset \mathbb{Z}^d \times \mathbb{Z}$ a finite set, with $Q_{k,l} \neq 0$ in $\mathcal{I} \times \mathcal{U}$, if $(k,l) \in \mathcal{N}_Q$.

Remark

For the case $d = 1$, $n = 1$, Hypothesis **H4** *appeared in [D-Llave-Seara06], and was eliminated in [D-Huguet09], [Gidea-Llave06], under generic assumptions. Similar improvements are clearly possible in this higher dimensional case.*

Melnikov potential

$$L(\tau, I, \varphi, s) = -\int_{-\infty}^{\infty} \Big[Q(p^*(\tau+\sigma), q^*(\tau+\sigma)I, \varphi + \sigma\omega(I), s+\sigma) - Q(0, 0, I, \varphi + \sigma\omega(I)\sigma, s+\sigma) \Big] \, d\sigma \tag{4}$$

where $\tau = (\tau_1, \cdots, \tau_n)$, $p^*(\tau+\sigma) = (p_1^*(\tau_1+\sigma), \ldots, p_n^*(\tau_n+\sigma))$, $q^*(\tau+\sigma) = (q_1^*(\tau_1+\sigma), \ldots, q_n^*(\tau_n+\sigma))$.

- **H5** Assume that the system of equations

$$\frac{\partial}{\partial \tau} L(\tau, I, \varphi, s) = 0 \tag{5}$$

admits a non degenerate solution. That is, there exists $(\tau_0, I_0, \varphi_0, s_0)$ such that:

$$\frac{\partial}{\partial \tau} L(\tau_0, I_0, \varphi_0, s_0) = 0, \quad \det\left(\frac{\partial^2}{\partial \tau^2} L(\tau_0, I_0, \varphi_0, s_0) \right) \neq 0. \tag{6}$$

Poincaré reduced function

By the implicit function theorem, we can find a branch of solutions $\tau = \tau^*(I, \varphi, s)$ in a domain $I \in \mathcal{I}^*, \varphi \in \mathcal{G}^*, s \in [a, b]$, such that the point $(\tau^*(I, \varphi, s), I, \varphi, s)$ also verifies (6).
Define the Poincaré reduced function

$$\mathcal{L}^*(I, \theta) = L(\tau^*(I, \theta, 0), I, \theta, 0) \tag{7}$$

which satisfies

$$L(\tau^*(I, \varphi, s), I, \varphi, s) = \mathcal{L}^*(I, \varphi - \omega(I)s)$$

- **H6** Assume that the function $\mathcal{L}^*(I, \varphi - \omega(I)s)$ satisfies some non-degeneracy conditions, stated later, in the domain $\mathcal{I}^* \times \mathcal{G}^* \times [a, b]$.

Non-degeneracy assumptions

- **H7** Assume that the perturbation Q satisfies some non-degeneracy conditions, stated later, in the connected domain $\mathcal{I}^* \times \mathcal{G}^* \times [a, b]$.
- **H8**
  Consider the set $\mathcal{N}^{[\leq 2]} = \mathcal{N}_1 \cup \mathcal{N}_2 \subset \mathbb{Z}^{d+1}$ where $\mathcal{N}_1$ is the support of the Fourier series of $Q(I, \varphi, p, q, t, 0)$, $\mathcal{N}_2 = (\mathcal{N}_1 + \mathcal{N}_1) \cup \bar{\mathcal{N}}$, and $\bar{\mathcal{N}}$ is the support of the Fourier series of $\frac{\partial Q}{\partial \varepsilon}(I, \varphi, p, q, t, 0)$.
  Assume that for any for any $(k, l) \in \mathcal{N}^{[\leq 2]}$ the set

$$\{I \in \mathcal{I}, Dh(I)k + l = 0, \quad k^\top D^2 h(I) k = 0\} \tag{8}$$

  is a codimension 2 set in $\mathcal{I}$.

Remark

If $\tilde{h}(I_0, I) = I_0 + h(I)$ is a quasi convex function the set (8) is an empty set. Therefore Hypothesis **H8** *is true for any perturbation in this case.*

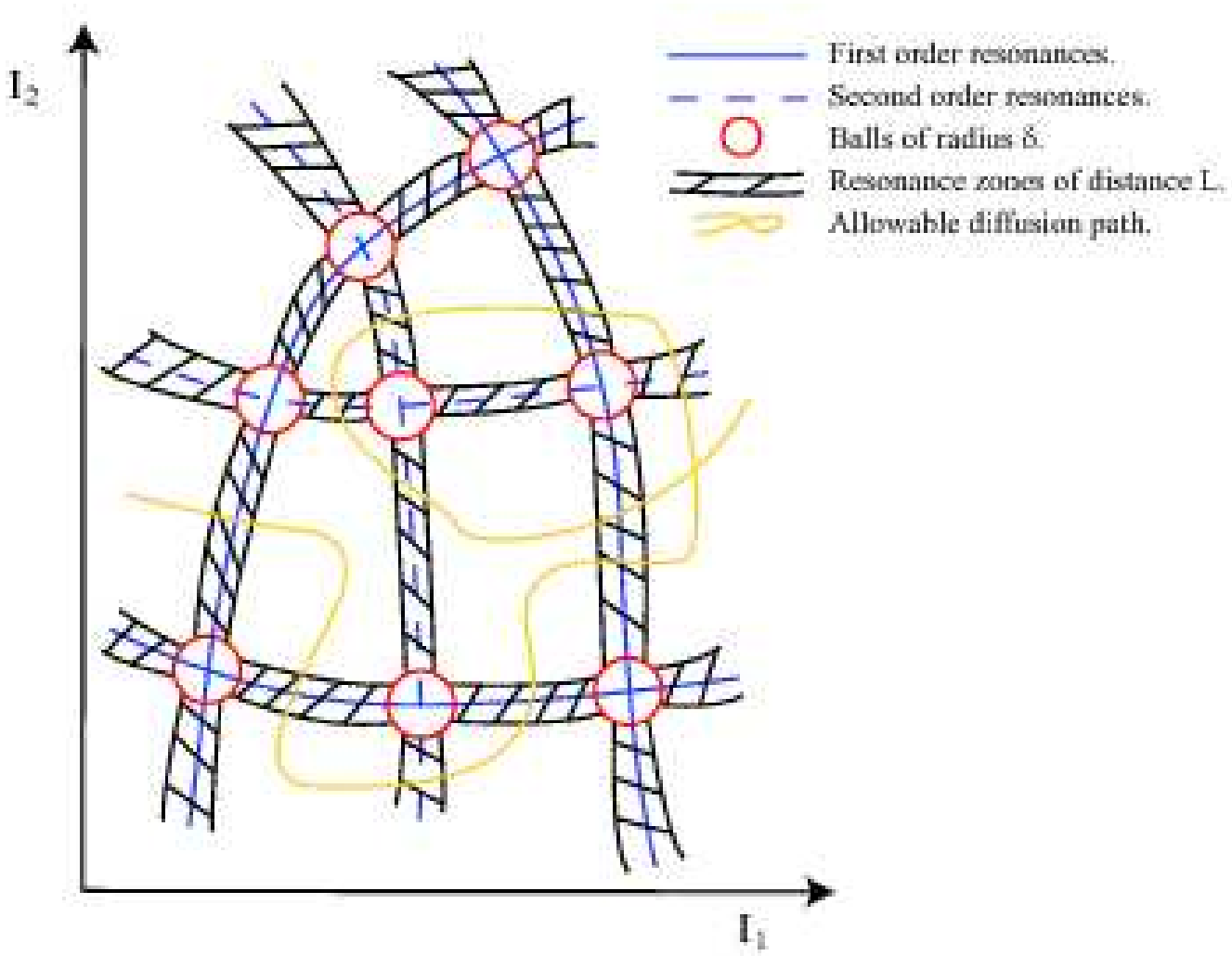
Main Result

Theorem

Let H be a Hamiltonian of the form (1) *satisfying the elementary assumptions* **H1, H2**, *the regularity assumption* **H3**, *the simplifying assumption* **H4** *and the nondegeneracy asumptions* **H5, H6, H7, H8**. *Then, for every $\delta > 0$, there exists $\varepsilon_0 > 0$, such that for every $0 < |\varepsilon| < \varepsilon_0$, given $I_\pm \in \mathcal{I}^*$,there exists a solution $\tilde{x}(t) = (p(t), q(t), I(t), \varphi(t))$ of* (1) *and $T > 0$, such that*

$$|I(0) - I_-| \leq C\delta \quad \text{and} \quad |I(T) - I_+| \leq C\delta \tag{9}$$

- One can forget about δ and prescribe arbitrary paths on a set $\mathcal{I}^*$. This set $\mathcal{I}^*$ is described precisely in the course of the proof, and is determined by the non-degeneracy assumptions **H5-H8**. The main idea is that $\mathcal{I}^*$ is obtained from the domain of definition, just eliminating some sets of codimension 2, like double resonances, from the open set described in **H5** (where the intersection of stable and unstable manifolds of a normally hyperbolic invariant manifold is transversal).
- All the conditions **H5-H8** are generic: C^2 open and hold except in sets of infinite codimension. The only non-generic hypothesis is the assumption **H4**, maintained here to simplify the exposition.
- Codimension 2 objects do not separate the regions and can be contoured so that they do not obstruct the change along the paths.


I_2
I_1
First order resonances.
Second order resonances.
Balls of radius δ.
Resonance zones of distance L.
Allowable diffusion path.

- It is customary to refer to models of the form (1) as a-priori-unstable models [Chierchia-Gallavotti94]. This distinction makes sense for analytic models depending only on one parameter. The results we will present could be applied just as well for μV_i instead of V_i in (1) and $0 < \mu \ll 1$, but require to choose ε very small (even exponentially small) with respect to μ. In particular, one could use this method to produce systems that present instability but which are as close to integrable as desired. This procedure was pioneered in [Arnold64].
- Hamiltonian (1) can be considered as a simplified model of what happens in a neighborhood of a resonance of multiplicity n in a near integrable Hamiltonian. The averaging method shows that near a resonance, one can reduce the system to a Hamiltonian of the form

$$h(I) + \sum_{i=1}^{n} \frac{p_i^2}{2} + \varepsilon V(q_1, \ldots, q_n, I) + O(\varepsilon^2). \tag{10}$$

The assumption that the averaged system is given by uncoupled pendula is not general, but is made often [Holmes-Marsden82,Haller].

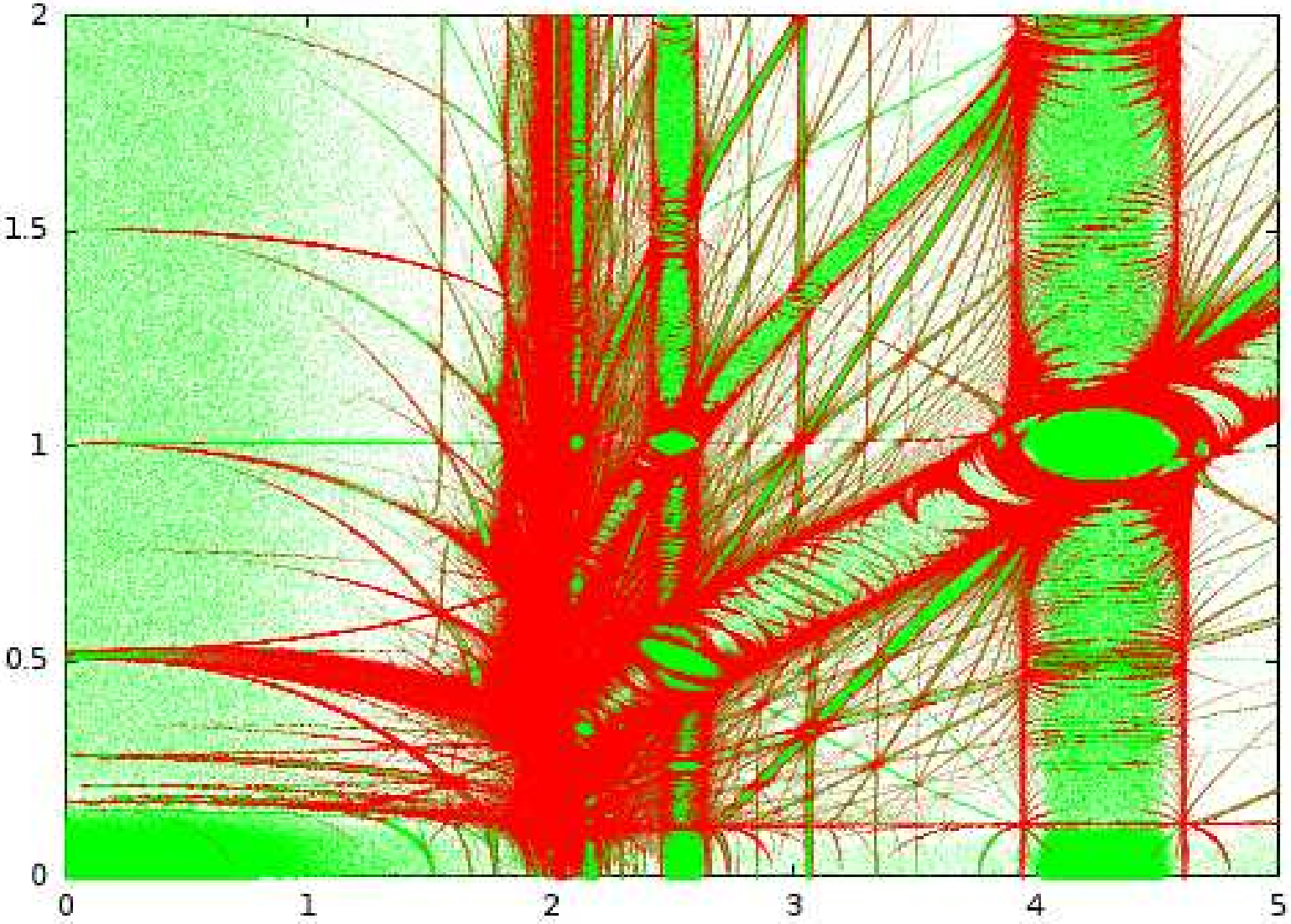
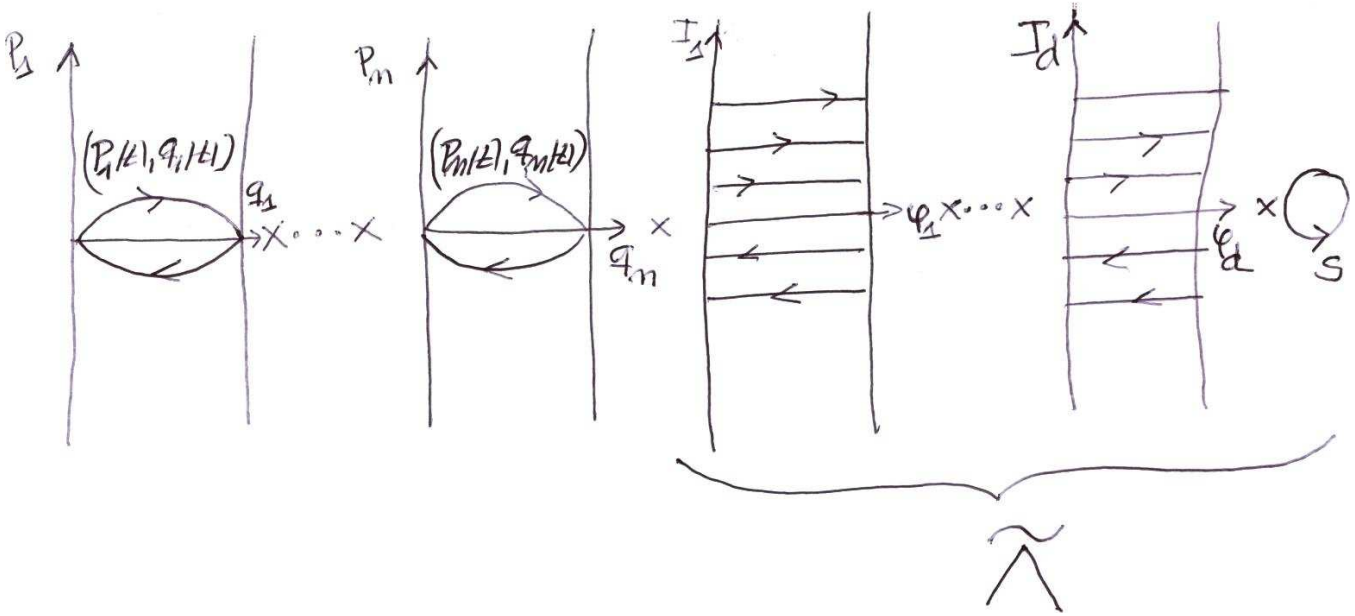


Figure: Regions of regular motion (grey) and chaotic motion (black) in the (p, I) plane, from *Global dynamics and fast indicators*, C. Simó

Sketch of the proof

- **Part I**: Existence of a normally hyperbolic invariant manifold with associated stable and unstable manifolds.
- **Part II**: Outer dynamics.
- **Part III**: Inner dynamics.
- **Part IV**: Combination of both dynamics.

$\epsilon = 0$



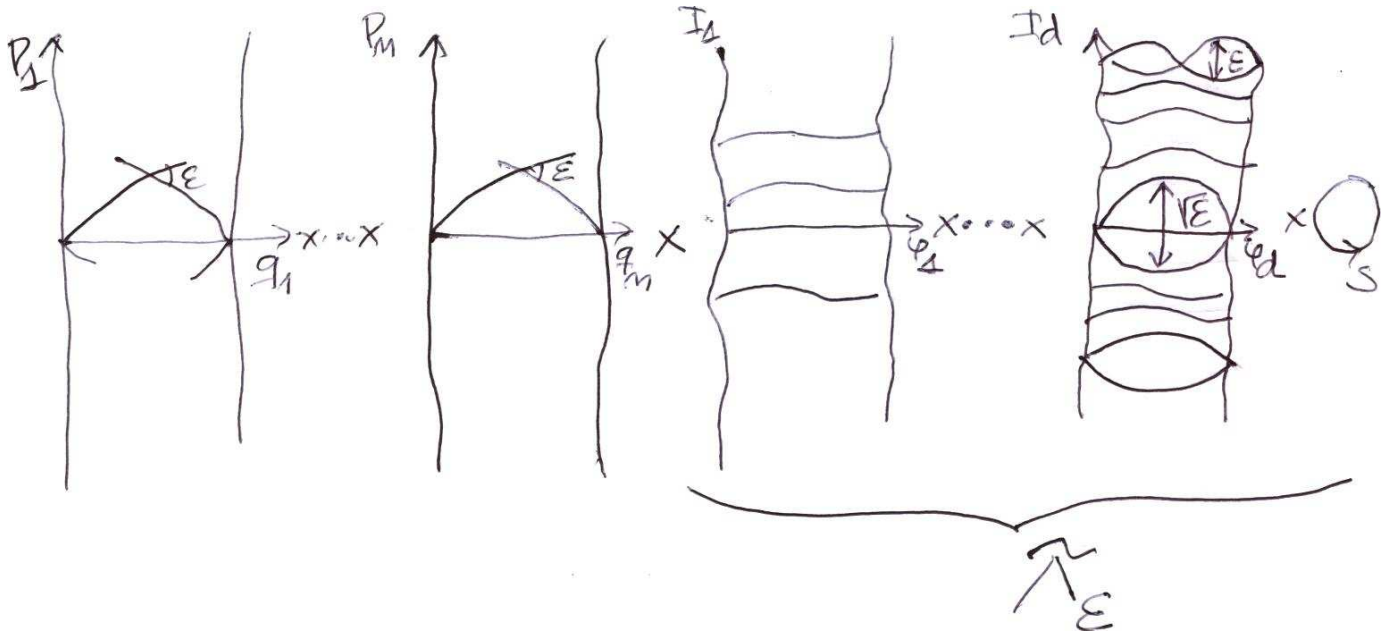
- Normally hyperbolic invariant manifold $(2d+1)$-dimensional:

$$\tilde{\Lambda} = \{(0, 0, I, \varphi, s) : (I, \varphi, s) \in \mathbb{R} \times \mathbb{T}^2\}$$

- Invariant manifolds $(2d+n+1)$-dimensional:

$$W^s\tilde{\Lambda} = W^u\tilde{\Lambda} = \{(p^*(\tau), q^*(\tau), I, \varphi, s) : \tau \in \mathbb{R}^n, I \in \mathcal{I}, (\varphi, s) \in \mathbb{T}^{(d+1)}\}$$

$0 < \epsilon \ll 1$



- $\widetilde{\Lambda}$ persists to $\widetilde{\Lambda}_\epsilon$, which is ϵ-close to $\widetilde{\Lambda}$.
- $W^s\widetilde{\Lambda}_\epsilon$ and $W^u\widetilde{\Lambda}_\epsilon$ are ϵ-close to the unperturbed ones.
- Using the Melnikov potential $L(\tau, I, \varphi, s)$, one has to check that $W^s\widetilde{\Lambda}_\epsilon \pitchfork W^u\widetilde{\Lambda}_\epsilon$ along a homoclinic manifold $\widetilde{\Gamma}_\epsilon$.

- Like in [D-Llave-Seara06], for any $(I, \varphi, s) \in \mathcal{I} \times \mathbb{T}^{(d+1)}$ and for any non-degenerate critical point $\tau^* = \tau^*(I, \varphi, s)$ of

$$\tau \in \mathbb{R}^n \mapsto L(\tau, I, \varphi, s) \in \mathbb{R} \tag{11}$$

there exists a locally unique point z,

$$z = z(I, \varphi, s; \epsilon) = (p^*(\tau^*) + \mathcal{O}(\epsilon), q^*(\tau^*) + \mathcal{O}(\epsilon), I, \varphi, s), \tag{12}$$

such that $z \in W^s(\tilde{\Lambda}_\epsilon) \pitchfork W^u(\tilde{\Lambda}_\epsilon)$.

- By hypothesis **H5** there exists an open set $(I, \varphi, s) \in \mathcal{I} * \times \mathcal{G}^* \times [a, b]$, such that the function (11) has non-degenerate critical points at $\tau = \tau^*(I, \varphi, s)$.

Scattering map (outer map)

Ingredients:

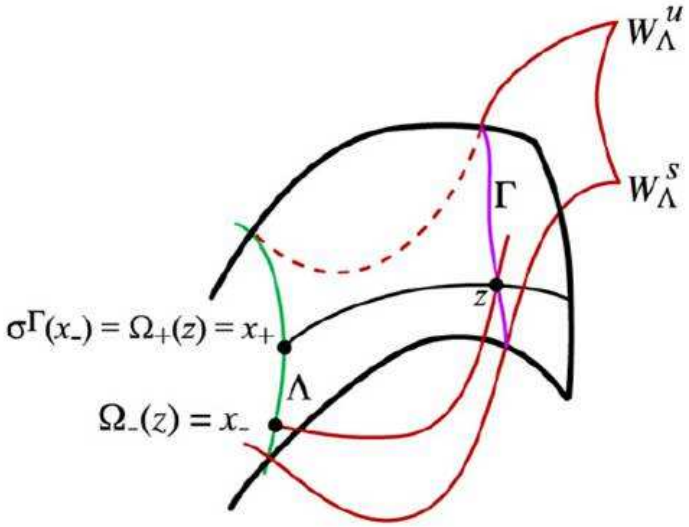
- Consider the foliations $\mathcal{F}_{s,u}$:

$$W^{s,u}_{\widetilde{\Lambda}_\epsilon} = \cup_{x\in\widetilde{\Lambda}_\epsilon} W^{s,u}_x$$

- Define the wave operators Ω_+, Ω_-:

$$\begin{array}{rccc} \Omega_\pm : & W^{s,u}_{\widetilde{\Lambda}_\epsilon} & \longrightarrow & \widetilde{\Lambda}_\epsilon \\ & x & \longmapsto & \Omega_\pm(x) \end{array}$$

defined by $x \in W^{s,u}_{\Omega_\pm(x)}$.



- Ω_- is a diffeomorphism from $\widetilde{\Gamma}_\epsilon$ to $H^{\widetilde{\Gamma}_\epsilon}_- \equiv \Omega_-(\widetilde{\Gamma}_\epsilon)$.

Define

$$S^{\widetilde{\Gamma}}_\epsilon = \Omega^+ \circ (\Omega^{\widetilde{\Gamma}_\epsilon}_-)^{-1}$$

- Scattering map (outer map):

$$\begin{array}{rccc} S_\epsilon : & H_-^{\widetilde{\Gamma}_\epsilon} \subset \widetilde{\Lambda}_\epsilon & \longrightarrow & H_+^{\widetilde{\Gamma}_\epsilon} \subset \widetilde{\Lambda}_\epsilon \\ & x_- & \longmapsto & x_+ \end{array}$$

defined by $x_+ = S_\epsilon(x_-) \Leftrightarrow \exists\, z \in \widetilde{\Gamma}_\epsilon$, such that

$$\mathrm{dist}(\Phi_t(z), \Phi_t(x_\pm)) \to 0 \qquad \text{for} \quad t \to \pm\infty$$

- S_ϵ is exact symplectic [D-Llave-Seara08].

- Perturbative formula for the Hamiltonian $\mathcal{S}_\epsilon$ generating the deformation of the scattering map S_ϵ:

$$\mathcal{S}_\epsilon(I,\varphi,s) = -\mathcal{L}^*(I,\varphi - s\omega(I)) + \mathcal{O}(\epsilon). \tag{13}$$

where the reduced Poincaré function $\mathcal{L}^*(I,\theta)$ satisfies

$$L(\tau^*(I,\varphi,s),I,\varphi,s) = \mathcal{L}^*(I,\varphi - s\omega(I)). \tag{14}$$

- Up to first order in ϵ, S_ϵ is the ϵ-time flow of $-\mathcal{L}^*(I,\theta)$, where $\theta = \varphi - s\omega(I)$:

$$S_\epsilon(I,\varphi,s) = (I,\varphi,s) + \epsilon \mathrm{J}\nabla\left(\mathcal{L}^*(I,\varphi - s\omega(I))\right) + \mathcal{O}(\epsilon^2), \tag{15}$$

- The scattering map can jump distances of $\mathcal{O}(\epsilon)$ along the trajectories of the Hamiltonian $\mathcal{L}^*(I,\theta)$.

- We need now to study the inner dynamics in $\widetilde{\Lambda}_\epsilon$ and more precisely its invariant tori $\mathcal{T}$ to construct a transition chain along $\widetilde{\Lambda}_\epsilon$, i.e., a sequence of whiskered tori $\{\mathcal{T}_1\}_{i=1}^N$ such that

$$W^u_{\mathcal{T}_i} \pitchfork W^s_{\mathcal{T}_{i+1}}$$

- Standard shadowing methods [Fontich-Martin00] provide orbits connecting arbitrary small neighborhoods of $\mathcal{T}_1$ and $\mathcal{T}_N$.
- The key point is to use the property

$$S_\epsilon(\mathcal{T}_i) \pitchfork_{\widetilde{\Lambda}_\epsilon} \mathcal{T}_{i+1} \Rightarrow W^u_{\mathcal{T}_i} \pitchfork W^s_{\mathcal{T}_{i+1}}$$

to choose convenient transition chains.

Inner Dynamics

The flow of Hamiltonian (1) restricted to the NHIM $\tilde{\Lambda}_\epsilon$ can be parameterized on $\tilde{\Lambda}_0 = \tilde{\Lambda}$ and is generated by a $\mathcal{C}^{r-1}$ time dependent Hamiltonian vector field with Hamiltonian of the form

$$K_\varepsilon(I, \varphi, s) = h(I) + \sum_{i=1}^{N} \varepsilon^i K^i(I, \varphi, s) + O_{C^{r-N-2}}(\varepsilon^{N+1}), \qquad (16)$$

where each of the terms K^i is a trigonometric polynomial in the φ, s variables.
Moreover, K^i is an algebraic expression in terms of $\nabla^\ell Q(p = 0, q = 0, I, \varphi, s; \varepsilon = 0)$, for $\ell = 0, \ldots, i-1$. In particular, $K^1(I, \varphi, s) = Q(0, 0, I, \varphi, s, 0)$.
Standard averaging far from resonances and close to single secular resonances provide adequate approximations for KAM tori.

Non resonant KAM tori

- For $\epsilon > 0$, KAM theorem ensures the preservation, with some deformation, of the invariant tori with frequency vector $(\omega(I), 1)$ satisfying Diophantine conditions.
- The invariant tori with frequency vector $(\omega(I), 1)$ satisfying Diophantine conditions fill a Cantorian set of relative measure $1 - \mathcal{O}(\sqrt{\epsilon})$, called the *non-resonant region*.
- Since these invariant tori are just deformed by the perturbation, they are called *primary tori*. Moreover, they are given by the level sets of a d-dimensional vector function $F = (F_1, \ldots, F_d)$ of the form

$$F(I, \varphi, s) = I + \mathcal{O}(\epsilon).$$

Primary and secondary tori in the resonant region

- Resonant tori (corresponding to *resonances*, i.e., values of I such that $\omega(I)\cdot k + l = 0$, for some $(k,l)\in\mathbb{Z}^{d+1}$) are typically destroyed by the perturbation, creating gaps in the foliation of the persistent primary tori of size up to $\mathcal{O}(\sqrt{\epsilon})$ centered around resonances. Other invariant objects are created inside these gaps, like *secondary tori*, which are (d+1)-dimensional invariant KAM tori contractible to d-dimensional invariant tori.
- Given any $(k,l)\in\mathbb{Z}^{d+1}$, $k\neq 0$, $\gcd(k,l)=1$, determining a single resonant region around $\omega(I)\cdot k + l = 0$, for simplicity of notation, assume $k_d\neq 0$ and write $k=(\hat{k},k_d)$ with $\hat{k}\in\mathbb{Z}^{d-1}$.
- By averaging theory, the invariant tori in this resonant region can be approximated by the level sets of a vector function $F=(F_1,\ldots,F_d)=(\hat{F},F_d)$ where $\hat{F}=\hat{J}-\frac{l_d}{k_d}\hat{I}$, and $F_d=K_\varepsilon+\frac{l_d}{k_d}I$. Once fixed the value of $\hat{F}$, thanks to hypothesis **H4**, F_d is the Hamiltonian of a pendulum.

There are two types of resonant regions depending whether the size of the gaps created by the single resonances is bigger or smaller than the size ϵ of the heteroclinic jumps of the Scattering map S:

Big gaps region It corresponds to those secular resonances centered around $\omega(I) \cdot k + l = 0$. For this region we have the *large gap problem*. In these regions, we will include primary as well as secondary KAM tori.

Small gaps region Centered around $\omega(I) \cdot k + l = 0$ for non-secular resonances, the gaps are of size strictly smaller than ϵ in terms of the I variable, so that it is possible to connect two primary tori on both sides of the gap. This case does not present the large gap problem and can be treated analogously as in the non-resonant region.

Flat tori region The union of the small gaps region and the non-resonant region. The dominant term in ϵ of the invariant tori is given there in first order by the function

$$F^*(I, \varphi) = I. \tag{17}$$

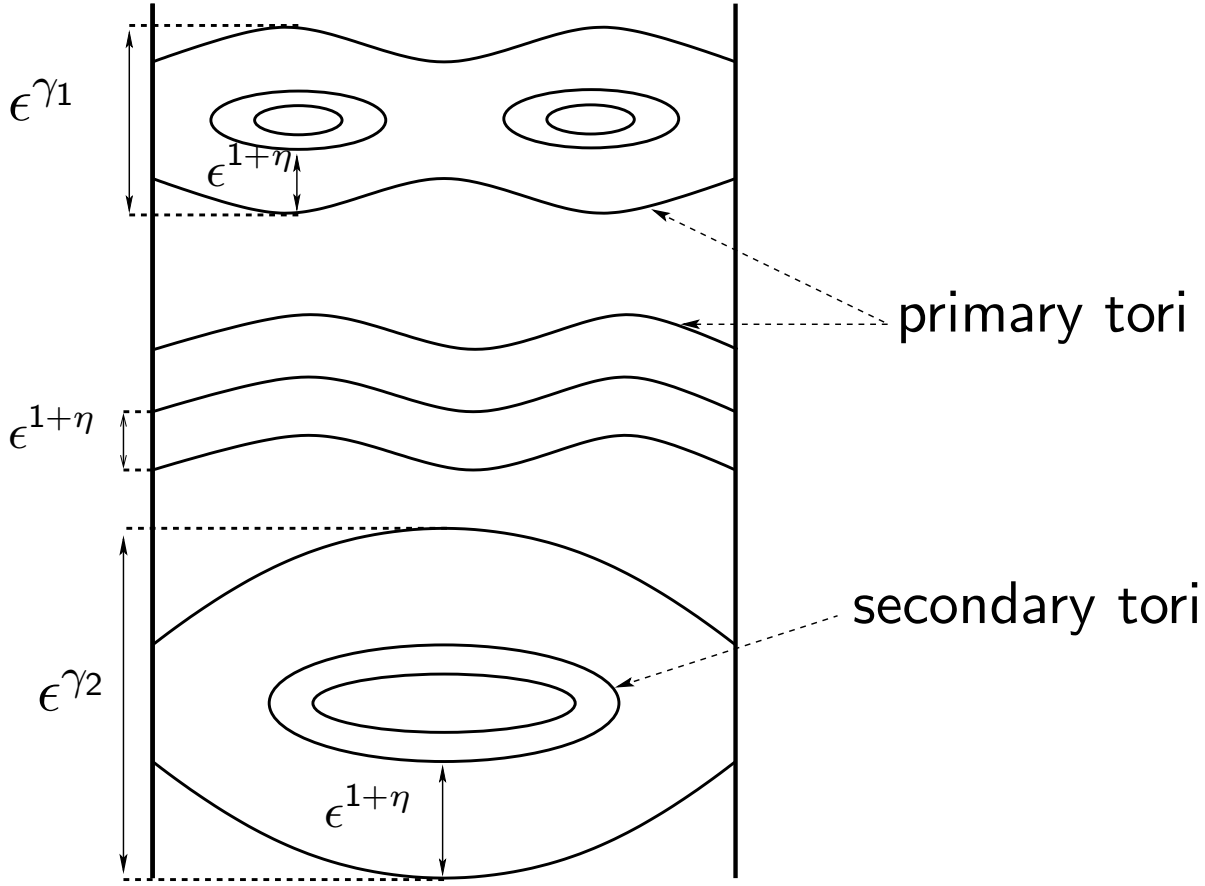
Which of them survive and at what distance when we add the perturbation term?

KAM theorem (Quantitative version)

- Flat tori region. We can apply KAM Theorem straightforwardly. Tori at a distance $\epsilon^{1+\eta}$, with $0 < \eta \ll 1$ (after m averaging steps).
- Big gaps region. The integrable system is a (d-1)-dimensional rotor plus a pendulum. We need to write the Hamiltonian in action-angle variables. Tori at a distance $\epsilon^{1+\eta}$, with $0 < \eta \ll 1$ (after m' averaging steps).

For $I \in \mathcal{I}$, there exists in $\widetilde{\Lambda}_\epsilon$ a discrete sequence of invariant tori $\mathcal{T}_i$ (some primary and some secondary) which are $\epsilon^{1+\eta}$-closely spaced, with $0 < \eta \ll 1$. They are given by the leaves $L_E^{F^*}$ of a foliation $\mathcal{F}_{F^*}$, with F^* close to F.

Invariant objects in the NHIM $\widetilde{\Lambda}_\epsilon$



- The image under the scattering map S_ϵ of a leaf $L_E^{F^*}$ satisfies

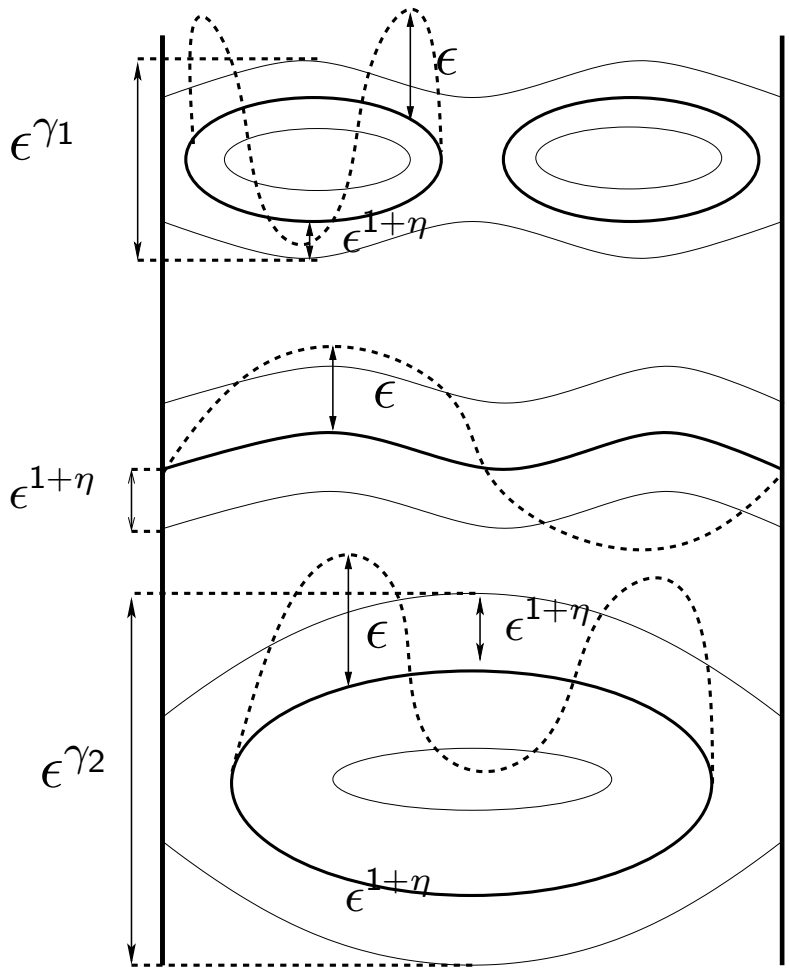
$$F^* \circ S_\epsilon = F^* - \epsilon\{F^*, \mathcal{L}^*\} + h.o.t$$

- For any $j = 1, \ldots, d$, at points

$$\{F_j, \mathcal{L}^*\} < 0 \qquad (18)$$

the scattering map increases the value of F_j by order ϵ.

- The non-degeneracy hypothesis **H6-H7** provide explicit conditions to ensure that $\{\{F_j, \mathcal{L}^*\}, \mathcal{L}^*\} \not\equiv 0$ when $\{F_j, \mathcal{L}^*\} = 0$. For instance, for F_i, $I = 1, \ldots, d-1$, i.e., for $\hat{F}$, they amount to $\det\left(\frac{\partial^2 \mathcal{L}^*}{\partial \hat{\theta}^2}\right) \not\equiv 0$.



The end of the proof

- By the hypotheses of the theorem, for any $I_\pm \in \mathcal{I}$ and for every $\delta > 0$, there exists a path from I_- to I_+ in the set $\mathcal{I}* = \mathcal{I}_\delta$ at a distance δ of the codimension 2 sets where the hypotheses of the Main Theorem are not fulfilled. By the construction presented, there exists $\varepsilon_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$, this path is in an ϵ-neighborhood of (primary and secondary) transition tori $\mathcal{T}_i$ forming a transition chain, so there exists an orbit $\tilde{x}(t) = (p(t), q(t), I(t), \varphi(t))$ of (1) which shadows the transition chain, so that, for some $T > 0$:

$$|I(0) - I_-| \leq C\delta$$
$$|I(T) - I_+| \leq C\delta$$

Illustration of the transport of invariant tori under the scattering map

$$H_\epsilon(p,q,I,\varphi,t) = \pm\left(\frac{p^2}{2}+\cos q-1\right)+\frac{I^\top\cdot I}{2}+\epsilon\cos q\sum_{|k|+|l|=1} a_{kl}\cos(k\cdot\varphi+lt).$$

Red curves: Invariant tori (primary and secondary) around $I=0$

Green curves: Images of these invariant tori under the scattering map.

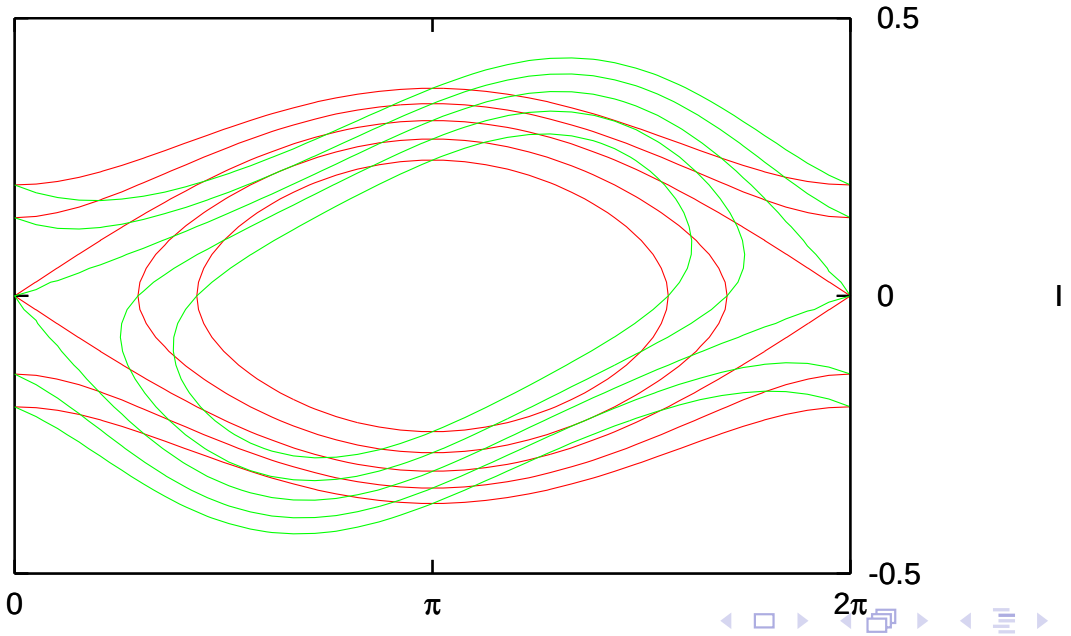


Illustration of how to combine the two dynamics to cross the big gaps region. Invariant tori for the inner dynamics (red curves) and invariant sets for the outer dynamics (blue curves). Inner dynamics is represented by dashed lines whereas outer dynamics is represented by solid lines.