Open Problems in Dynamical Astronomy

Galactic Dynamics

Pablo M. Cincotta

Grupo de Caos en Sistemas Hamiltonianos
Facultad de Ciencias Astronómicas y Geofísicas
Universidad Nacional de La Plata &
Instituto de Astrofísica de La Plata, UNLP-CONICET, Argentina

Workshop on Instabilities in Hamiltonian Systems
June 13-17, 2011

This talk is based on:

- Reviews by **C. Efthimiopoulos et al., D. Merritt et al.**;
- Papers in collaboration with **C. Simó**;
- Works together with **S. Ferraz–Mello**;
- Papers/reviews with my co–worker **C. Giordano** and our students;
- All that Carles attempted to teach me, although I am afraid, failed to learn to the extent I should have;
- Chirikov's reviews and papers.

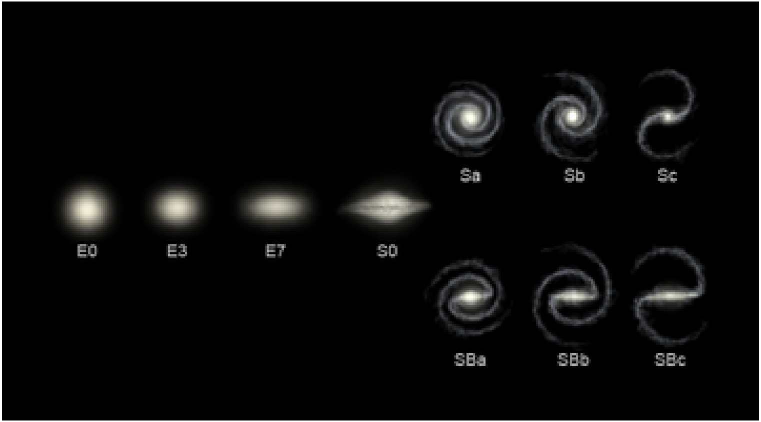
I'm very grateful to all of them for allowing me to borrow material from their papers and lectures as well as for illuminating comments. Thanks to our present and former students, **N. Maffione, M. Mestre, L. Darriba, F. Cachucho da Silva, M. J. Pérez**, for performing some of the figures included in this presentation and useful discussions.

Galaxies

A galaxy is a very complicated system:

- $\sim 10^{11}$ stars whose $\rho(\boldsymbol{x}, t)$ generates its own field;
- rotating pattern, spiral arms, bars,
- gas and dust: SPH codes (Smoothed–particle hydrodynamics),
- star formation,
- supernova explosions,
- chemical evolution,
- interaction with other galaxies and/or globular cluster system;
- a super–massive black hole lies in the center?
- are they in steady state?. Probably not $\cdots$
- dark matter? Alternative theories $\cdots$

Hubble Sequence

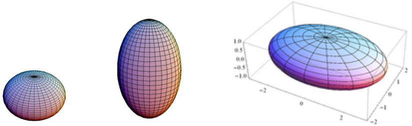


Some pictures of real galaxies

Open Problems· · ·

In Galactic Dynamics the very nature of problems does not permit a coarse classification in open and close. Almost all practically interesting problems are still *largely open*. The main obstruction to closing problems is the lack of sufficient observational data, which, in many cases, is due to our fundamental inability to obtain such data. For instance:

- What is the shape of an observed elliptical galaxy, axisymmetric (oblate or prolate) or triaxial?



- Can the central part of the velocity dispersion curve be justified on the basis of the luminous matter distribution alone, or we need to consider a central black hole and of what mass?
- Is the distribution function of the galaxy best fitted by a two–integral or three–integral model?
- How do we understand the observed correlations between the value of the central mass and the shape - velocity dispersion of a galaxy? Are there observational traces of central black hole - driven secular evolution?

Idealization: a galaxy as a N – particle Hamiltonian dynamical system

Collisionless approximation (Binney & Tremaine)

Let $f^{(N)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N, \boldsymbol{p}_1, \cdots, \boldsymbol{p}_N, t)$ be the N–particle probability density or distribution function (DF) on 2N–D phase space Γ, so it satisfies Liouville theorem:

$$\frac{\mathrm{d}f^{(N)}}{\mathrm{d}t} = 0.$$

Let's define the 1 – particle DF:

$$f^{(1)}(\boldsymbol{x}_1, \boldsymbol{p}_1, t) = \int f^{(N)} \mathrm{d}^3 x_2 \cdots \mathrm{d}^3 x_N \mathrm{d}^3 p_2 \cdots \mathrm{d}^3 p_N,$$

Assuming that:

- $|(\boldsymbol{x}_i, \boldsymbol{p}_i)|^k f^{(N)} \to 0$ as $|(\boldsymbol{x}_i, \boldsymbol{p}_i)| \to \infty \quad \forall k,\, \forall i = 1, \cdots, N,$
- $f^{(N)}$ is symmetric in $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N; \boldsymbol{p}_1, \cdots, \boldsymbol{p}_N$,
- The 2–particle DF: $f^{(2)}(\boldsymbol{x}_1, \boldsymbol{p}_1, \boldsymbol{x}_2, \boldsymbol{p}_2, t) =$ $f^{(1)}(\boldsymbol{x}_1, \boldsymbol{p}_1, t)\, f^{(1)}(\boldsymbol{x}_2, \boldsymbol{p}_2, t) + g(\boldsymbol{x}_1, \boldsymbol{p}_1, \boldsymbol{x}_2, \boldsymbol{p}_2, t)$,
- the 2 – particle correlation function $g \approx 0$,
- $N \gg 1$,
- and defining $f(\boldsymbol{x}, \boldsymbol{v}, t) \equiv \mathrm{N} f^{(1)}(\boldsymbol{x}_1, \boldsymbol{p}_1/m, t)$
- $(\boldsymbol{x}, \boldsymbol{v}) \equiv (\boldsymbol{x}_1, \boldsymbol{p}_1/m)$

we arrive to the so-called collisionless Boltzmann (or Vlasov) eq.

$$\frac{\partial f}{\partial t} + [f, H] = 0, \quad H(\boldsymbol{p}, \boldsymbol{x}) = \frac{\boldsymbol{p}^2}{2} + \Phi(\boldsymbol{x}, t), \qquad \boldsymbol{p} \equiv \boldsymbol{v},$$

being $\Phi(\boldsymbol{x}, t)$ a smooth potential generated by the star distribution.
The Liouville theorem in the 6–D phase space, μ.

Thus we need to solve:

$$\frac{\partial f}{\partial t} + \nabla_{\boldsymbol{x}} f \cdot \boldsymbol{v} - \nabla_{\boldsymbol{x}} \Phi \cdot \nabla_{\boldsymbol{v}} f = 0$$

$$\nabla^2 \Phi = 4\pi G m \int f(\boldsymbol{x}, \boldsymbol{v}, t) \mathrm{d}^3 v$$

Steady state solutions

Jeans Theorem: *The DF of a steady–state galaxy in which almost all orbits are regular with incommensurable frequencies may presumed to be a function only of three independent isolating integrals (Lynden–Bell).*

An *isolating integral* $I(\boldsymbol{x}(t), \boldsymbol{v}(t)) = c$ defines a manifold in μ of dimension lower than $\dim(\mu)$. In Jeans Theorem, the three isolating integrals are, for instance, the three global actions of $H(\boldsymbol{p}, \boldsymbol{x})$.

If the Hamiltonian of a collisionless stellar system in steady-state equilibrium is Arnol'd–Liouville integrable, the distribution function has a constant value at all the points of an invariant torus of the system (Efthymiopoulos et al.).

THE ASTRONOMICAL JOURNAL VOLUME 69, NUMBER 1 FEBRUARY 1964

The Applicability of the Third Integral Of Motion: Some Numerical Experiments

MICHEL HÉNON* AND CARL HEILES

Princeton University Observatory, Princeton, New Jersey

(Received 7 August 1963)

The problem of the existence of a third isolating integral of motion in an axisymmetric potential is investigated by numerical experiments. It is found that the third integral exists for only a limited range of initial conditions.

1. INTRODUCTION

THERE has recently been a renewal of interest in the question of the existence of the third integral of galactic motion (Contopoulos 1957, 1958, 1960, 1963; Barbanis 1962; van de Hulst 1962, 1963; Ollongren 1962). A thorough review of the problem can be found in Ollongren's work, and we summarize it briefly here. We suppose that the gravitational potential of a galaxy is time-independent and has an axis of symmetry. In a system of cylindrical coordinates R, θ, z, this potential is then a given function $U_g(R,z)$. We are interested in the motion of a star in such a

In the present case, two isolating integrals are known:

$$I_1 = U_g(R,z) + \tfrac{1}{2}(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2), \tag{2}$$

$$I_2 = R^2\dot{\theta}. \tag{3}$$

They are the total energy and the angular momentum per unit mass of the star around the z axis. It can be shown that two of the other integrals, for example I_4 and I_5, are generally nonisolating. The problem is then: what is the nature of the last integral, I_3?

For many years, it was assumed that I_3 is nonisolating (see, for example, Jeans 1915, 1919; Lindblad

But galaxies should present a divided phase space.
So the implementation of Jeans theorem in realistic stellar systems is problematic:

- How to incorporate approximate integrals in the arguments of the distribution function when the system is close to integrability?;
- Resonances and resonance intersections;
- One or two integrals do not exist for the chaotic domains, which co–exist with the regular ones within any energy surface.

Generalization of D. Merritt to non–integrable potentials:
The phase space density of a stationary stellar system must be constant within every well-connected region.

The definition of "well-connected" is "...one that cannot be decomposed into two finite regions such that all trajectories lie on either one or the other (what the mathematicians call *metric transitivity*)" *(Merritt).* **Let us discuss this point later.**

Focus our attention first in fully–integrable galactic potentials, like for instance, the Perfect Ellipsoid, which is represented by a stratified density function:

$$\rho(x, y, z) = \frac{\rho_0}{\left(1 + m^2(x, y, z)\right)^2},$$

where ρ_0 *represents the central density and*

$$m^2(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \quad a \geq b \geq c \geq 0,$$

is constant on an ellipsoidal shell.

Following Chandrasekhar, the the Perfect Ellipsoid in ellipsoidal coordinates (λ, μ, ν) has the Stäckel form:

$$V(\lambda, \mu, \nu) = -\frac{1}{4h_\lambda^2}\frac{G(\lambda)}{(\lambda+\beta)} - \frac{1}{4h_\mu^2}\frac{G(\mu)}{(\mu+\beta)} - \frac{1}{4h_\nu^2}\frac{G(\nu)}{(\nu+\beta)},$$

where

$\beta = -b^2$; and h_λ^2, h_μ^2, h_ν^2

are the metric coefficients of the ellipsoidal coordinates and $G(\tau)$ is given in terms of an elliptical function of the third kind (de Zeeuw).

The Stäckel model has three explicit global analytic integrals, namely, the integrals I_2 and I_3 besides the total energy H.

The integrals can be considered as generalizations of the angular momentum integrals that exist in axisymmetric and spherical potentials, but also as generalizations of the energy integrals always present in separable potentials in Cartesian coordinates. The integrals I_2 and I_3 are, in fact, linear combinations of other integrals J and K:

$$I_2 = \frac{\alpha^2 H + \alpha J + K}{\alpha - \gamma}, \qquad I_3 = \frac{\gamma^2 H + \gamma J + K}{\gamma - \alpha},$$

where $\alpha = -a^2, \beta = -b^2, \gamma = -c^2$ and the energy H, J and K are functions of the ellipsoidal coordinates and conjugate momenta (de Zeeuw).

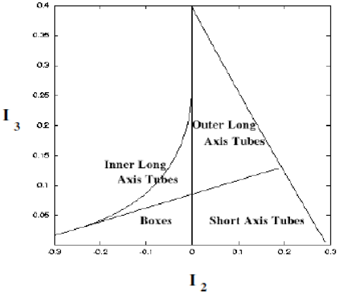


Fig. 1. Orbital Classification for the Perfect Ellipsoid in the $I_2 I_3$-plane for a fixed value of the energy $E = -0.3$, de Zweeu (1985). Curve drawn in this plane are separatrices of different families of orbits.

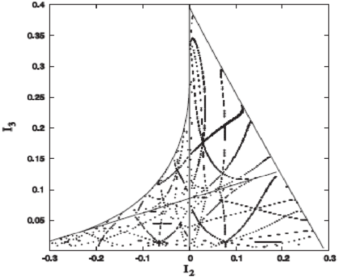
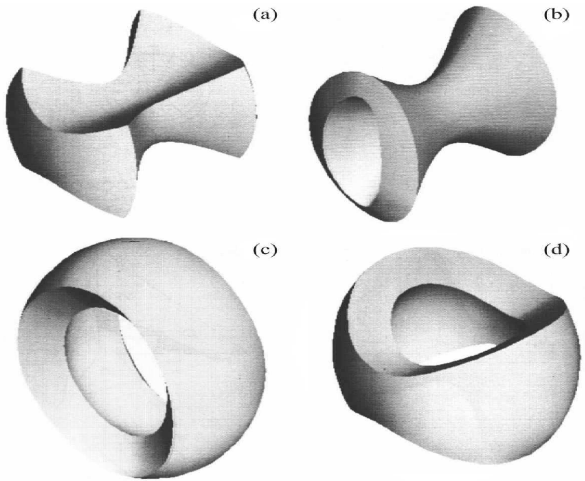


Fig. 2. Resonance structure of the Perfect Ellipsoid onto the $I_2 I_3$-plane, for a particular value of the energy $E = -0.3$. The curves shown in this figure are the intersection of the energy surface and several resonant surfaces calculated from (17) for different resonant vectors $\boldsymbol{m}$ satisfying $|m| = |m_1| + |m_2| + |m_3| < 8$ (see text).

Figure: Structure of action or integral space in a completely integrable elliptic potential of a galaxy model (perfect ellipsoid) for a given H, in the plane I_2, I_3, after de Zeeuw. Main resonance structure of the system, after J. Pérez et al.

Typical regular orbital structure of elliptical galaxies



(a) Box, (b) ILAT, (c) OLAT, (d) SAT. The four types of (regular) orbits in 3D space (for perfect ellipsoid). These orbits are very good guides for the form of regular orbits that exist in most galactic models (after Statler 1987).

Near–integrable elliptical potentials

If we add a "small" ($\epsilon \ll 1$), non–integrable smooth perturbation to the Perfect Ellipsoid, the structure of phase space should look like:

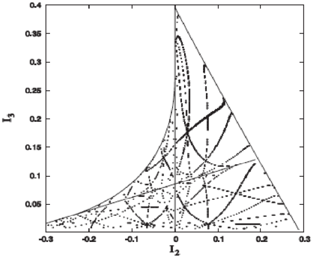
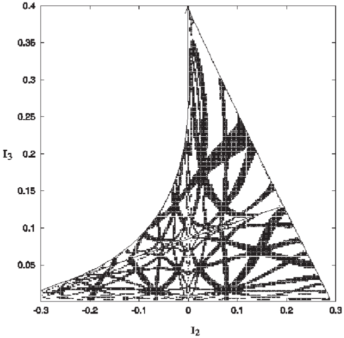


Fig. 2. Resonance structure of the Perfect Ellipsoid onto the $I_2 I_3$-plane, for a particular value of the energy $E = -0.3$. The curves shown in this figure are the intersection of the energy surface and several resonant surfaces calculated from (17) for different resonant vectors $\boldsymbol{m}$ satisfying $|m| = |m_1| + |m_2| + |m_3| < 8$ (see text).



A more realistic non–integrable triaxial potential, generated by a N–body simulation, could be:

$$\Phi(x, y, z) = -f_0(r) - f_x(r) \cdot (x^2 - y^2) - f_z(r) \cdot (z^2 - y^2),$$

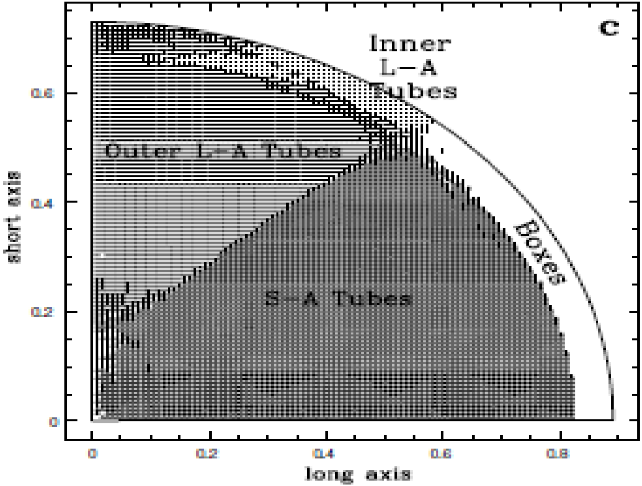
where

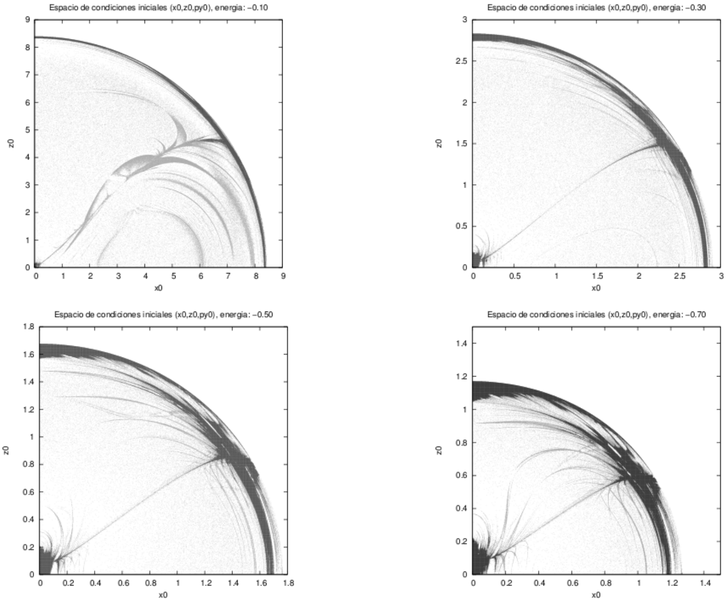
$$f_s(r) = \frac{C_s}{\left[p^{k_s}(r) + q_s^{k_s}\right]^{l_s/k_s}}, \qquad s = 0, x, z,$$

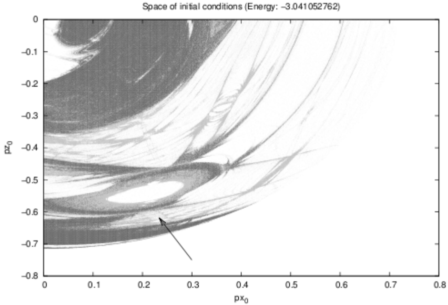
$$p_s^2(r) = r^2 + \epsilon^2 \quad \text{if} \quad s = 0, \qquad p_s^2(r) = r^2 + 2 \cdot \epsilon^2 \quad \text{if} \quad s = x, z,$$

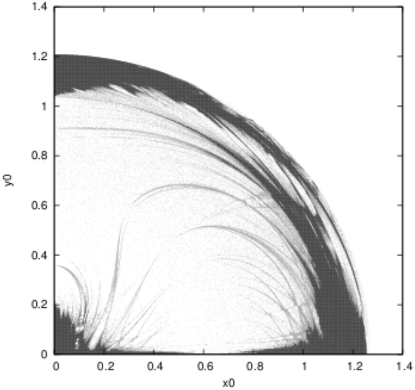
C_s, k_s, q_s, $l_s, \epsilon = 0.01$ are constants, fixed by a quadrupolar interpolation of the N–body simulation.

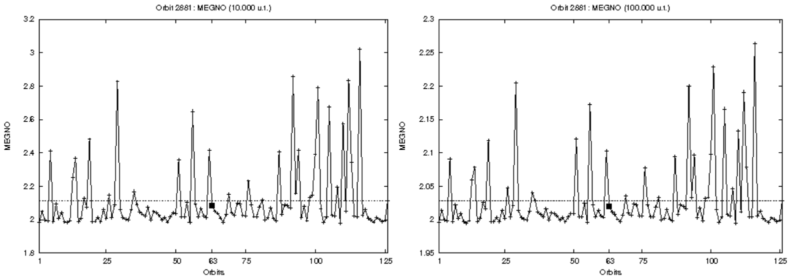
Usual planes to display dynamics in general 3D elliptical galactic potentials (Papaphilippou and Laskar, Schwarzschild):




Espacio de condiciones iniciales (x0,z0,py0), energia: −0.10
z0
x0
Espacio de condiciones iniciales (x0,z0,py0), energia: −0.30
z0
x0
Espacio de condiciones iniciales (x0,z0,py0), energia: −0.50
z0
x0
Espacio de condiciones iniciales (x0,z0,py0), energia: −0.70
z0
x0


Space of initial conditions (Energy: –3.041052762)
0
–0.1
–0.2
–0.3
–0.4
–0.5
–0.6
–0.7
–0.8
pz_0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
px_0


1.4
1.2
1
0.8
0.6
0.4
0.2
0
y0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
x0



MEGNO values for $T = 10^4$ and $T = 10^5$ for 125 orbits in a domain $\sigma = 10^{-7}$ around the pointed orbit. $T_c(E) \sim 1$. (Maffione et. al.).

In a recent paper it was shown the strong relationship between the MEGNO and the FLI (Mestre et al.).

clearly this a typical sticky orbit.

Some Theoretical Considerations

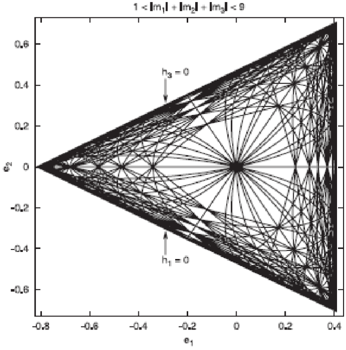
- Well–known mechanisms that lead to transition from regularity to gross chaos (or gross instabilities) are overlap of resonances (heteroclinic intersections) and Arnol'd diffusion–like processes (we will discuss this later).
- "Classical" Arnol'd diffusion, in our opinion, does not play any role in galactic dynamics (even in asteroidal dynamics), since in general "the perturbation" is not small enough, and even if it might work, its time–scale is quite large for any real system. And we do not think that it could describe global instabilities.
- Though one could get accurate values of any indicator of the stability of the motion, they only provide the local rate of exponential divergence.

- Chaos should be understood as large variation of the unperturbed integrals (diffusion). Unfortunately, as far as we know, it does not exist yet any theory that could describe global diffusion (instability) in phase space.
- A given orbit in a chaotic component could have for instance two positive and large values of the LCN, but this does not mean that the unperturbed integrals would change too much.
- What is actually relevant is the extent of the domain over which the unperturbed integrals would change and, physically, the time–scale over which this diffusion may occur.

- For galaxies, time–scales less than the Hubble time $T_H \sim 10^3 T_c$, seem to be too short in order to any diffusion may operate. Perhaps in some cases, overlap of resonances, whose rate is $\sim$ some power of the perturbation parameter could lead to some diffusion.
- As far as we know, Arnol'd diffusion, or to be precise, Arnol'd mechanism, only states that two point of the phase space separated by a distance of $\boldsymbol{O}(1)$ could be connected. This result does not imply any global instability and requires exponentially large times. Thus, we believe that Arnol'd diffusion does not play any role in galactic dynamics.
- Arnol'd diffusion–like processes perhaps also could not operate in galactic dynamics, as we shall see next.
- Let's consider a rather toy model:

$$H(\boldsymbol{p}, \boldsymbol{q}) = \frac{\boldsymbol{p}^2}{2} + \frac{1}{4}(x^4 + y^4 + z^4) + \epsilon x^2(y + z).$$

It is easy to write it as $H_0(\boldsymbol{I}) + \epsilon V(\boldsymbol{I}, \boldsymbol{\theta})$.
Now $(I_1, I_2, I_3) \to (h_x, h_y, h_z) \to (e_1, e_2, e_3)$ where e_3- axis is normal to the energy surface.



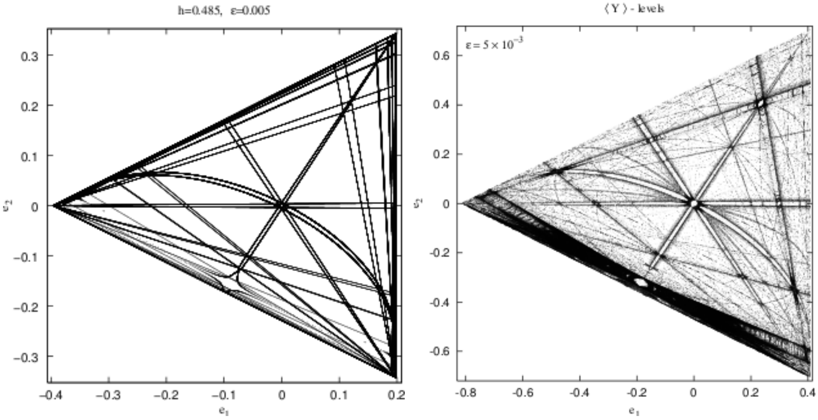


Figure: Main resonances of the systems: computed analytically up to $O(\epsilon^2)$ and numerically by means of the MEGNO (Mestre et al., Simó).

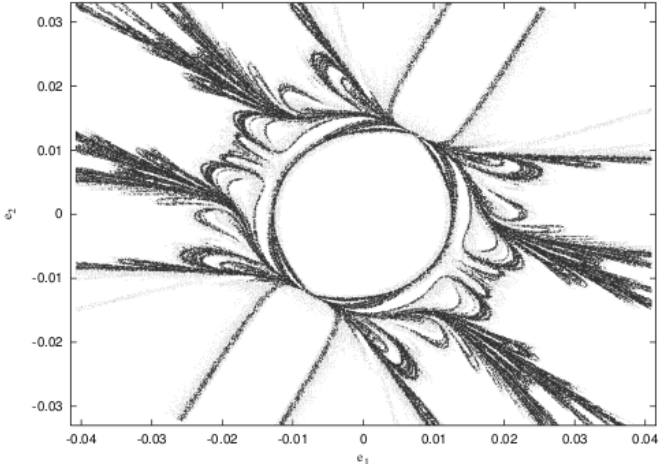


Figure: Blow-up around the origin in the contour plot displaying the $\overline{Y}$–levels on the energy surface for $\epsilon = 5 \times 10^{-3}$ (Simó).

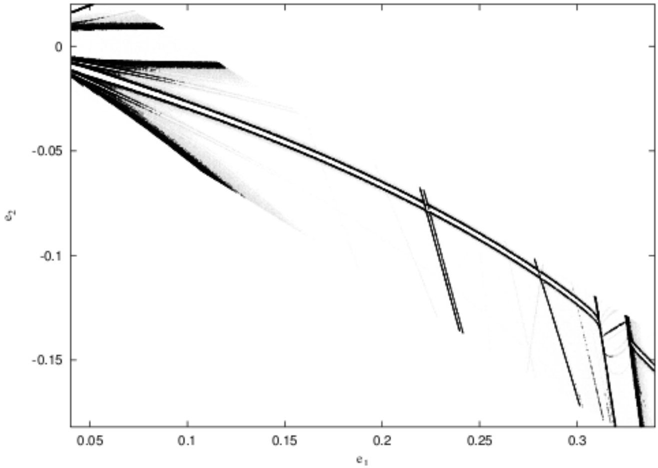


Figure: Zoom along a thin resonance channel (Simó).

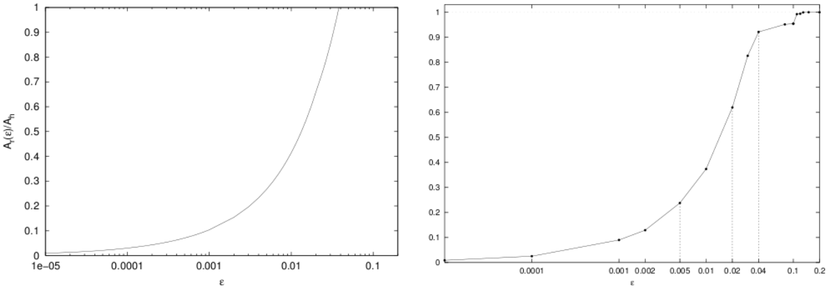


Figure: Theoretical estimation of fraction of chaotic motion A_r/A_T (on the left) and fraction of chaotic motion (on the right) both vs. ϵ, in logarithmic scale (Mestre et al., Simó).

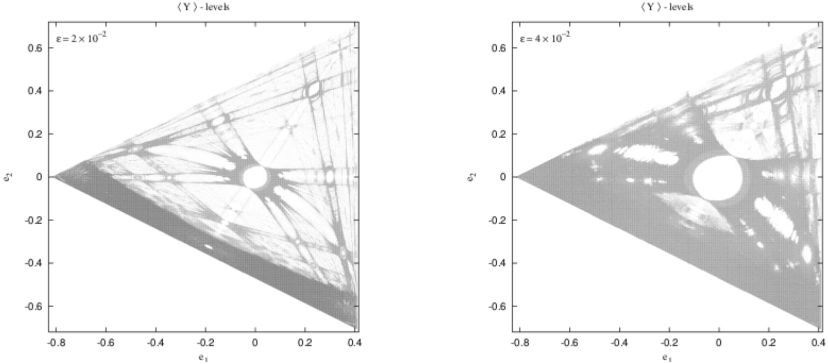


Figure: Phase space structure of the system for two particular values of ϵ: $\sim$ moderate perturbation, $\sim$ large perturbation (Giordano et al.)

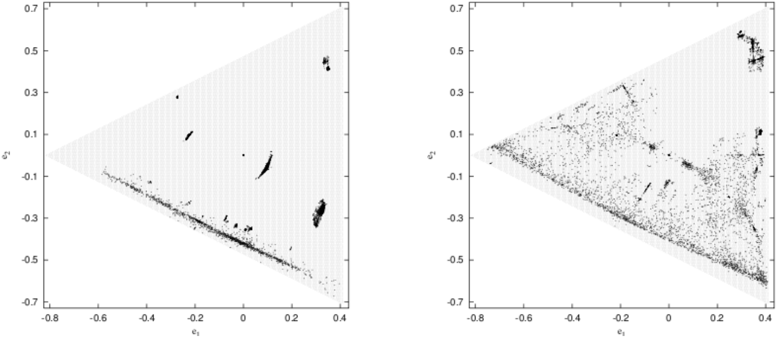


Figure: Diffusion on the energy surface at moderate–to–high perturbations after $3 \times 10^6 T_c$ for 8 orbits with highest MEGNO values (except the origin) for: [left] $\epsilon = 0.02$, and [right] $\epsilon = 0.04$.

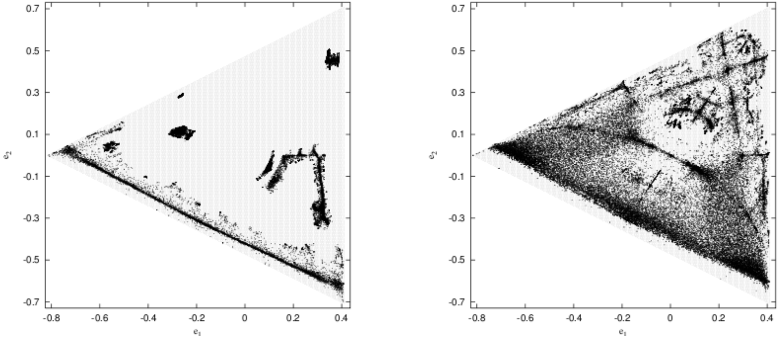


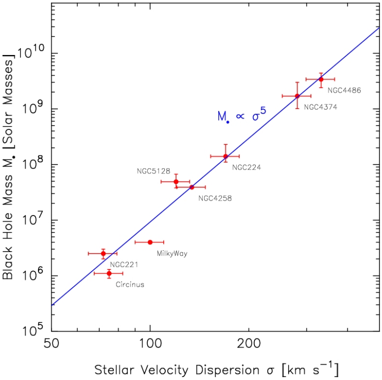
Figure: Diffusion on the energy surface at moderate–to–high perturbations after $3 \times 10^8 T_c$ of for the same orbits for: [left] $\epsilon = 0.02$, and [right] $\epsilon = 0.04$.

Some statements in the astrophysical literature

- Observations with HST revealed the presence of very high stellar densities at the centers of early-type galaxies, suggesting a power law ($r^{-\gamma}$) to fit them. The evidence for large central masses was also reforced from high-resolution kinematical studies of nuclear stars and gas, which disclosed the presence of compact dark objects with masses in the rage of $10^{6.5} - 10^{9.5} M_{\odot}$, presumably super–massive black holes. These observational results have produced a substantial change in the classic ideas on dynamics in triaxial galaxies.
- Results obtained from numerical simulations show that the addition of a central mass to an integrable triaxial potential has deep effects on its dynamics, at least for the boxlike orbits which mainly cover the central region of triaxial galaxies.

- Black holes and central density cusps scatter these particular orbits during each close passage giving rise to chaos in the system. The sensitivity of boxlike orbits to deflexions also drives to a rounder central distribution of mass. This slow evolution towards axisymmetry suggests that stationary triaxial configurations could not exist for a central density cusp.
- For such large value of M_{bh}, the box–orbit phase space is almost completely stochastic and diffusive processes could take place in very short timescales.
- This result turned out to be substantially attractive because this critical black hole mass was close to the observed one and also close to the mass which induced a sudden evolution towards the axisymmetry in N–body simulations.

- From recent works it has been known that the mass of black holes in galaxies from the black hole demographic relationships are $0.1 - 0.2\%$ of the ellipsoid mass in which they are embedded.



- Merritt and Fridman, arrive to similar conclusions analyzing two triaxial power law models $r^{-\gamma}$: the steep ($\gamma = 2$) and the weak ($\gamma = 1$) cusp. They find, in agreement with Gerhard and Binney, and Schwarzschild, that triaxial galaxies with such huge concentration of mass would evolve towards a central axisymmetry, as box orbits loose their distinguishability.
- For these models, in which a large fraction of phase space is dominated by a chaotic dynamics, the construction of self-consistent solutions requires the inclusion of stochastic orbits besides the regular ones. A system thus built evolves, mainly close to its center, as stochastic orbits mix through phase space.
- Though it is possible to build this kind of solutions for a weak cusp model, this is not the case for a strongly concentrated model. This would imply that triaxility is not consistent with a high central density.

Final Remarks

- The question on nature's ability to build stationary non–axisymetric stellar systems is still open.
- Merritt's generalization of Jeans Theorem rests under a very strong assumption: in a 3D systems with divided phase space, a completely connected chaotic component must exist.
- It seems that this could happen only when the chaotic component has a large measure and "$t \to \infty$", which, from a physical point of view, it would not be possible in galactic systems, where the chaotic component has a small measure and $t \lesssim T_H$.

- An important fact to be stated is that when chaos sets up, the unperturbed global integrals (or actions) have a discontinuous dependence on phase space variables. Indeed, close to resonant tori, despite the existence of three local integrals, the unperturbed orbital structure is not preserved and the topology of the phase space changes. Moreover, on the stochastic layer at least one integral does not exist.
- Close to strong non–resonant tori, the local integrals are just corrections of order ϵ of the unperturbed global integrals. On the other hand, when the system is close to a elliptic resonant tori, new local integrals appear: the pendulum Hamiltonian H_r and linear combinations, K_2, K_3, of the unperturbed actions at the resonant point.

- It is not possible to assume that the DF, in the whole regular component, has the form $f(H, I_2, I_3)$. This could be true only for strong non–resonant tori, but since resonances are dense in phase space, the DF should be locally defined as: $f_n(H, I_2, I_3) + \epsilon g(H, I_2, I_3)$ in a neighborhood of non–resonant tori and $f_r(H_r, K_2, K_3)$ in a vicinity of an elliptical resonant tori.
- Nothing could be said about the dependence of f in the chaotic domain. Since there is no theoretical support to argue that the whole chaotic region is fully connected. Clearly, a notorious discontinuous dependence of f on the integrals is expected.
- The introduction of a black hole ("singularity") at the origin, changes the approach to the problem $\cdots$

- Regarding diffusion, how to measure it? It seems natural to be related with the variance of the integrals.
- In which way the existence of barriers and "accelerators" of diffusion should be included in the coefficient?
- How to know/predict the diffusion routes?
- *...the global instability properties of near integrable Hamiltonian systems, thirty years after the pioneering work of V.I. Arnold, are far from well–understood. It could almost be said that little progress has been made, and new ideas are definitely called for.* (Lochak 1999).