Full torsion in the planetary N-body problem

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FIELDS

Perturbation theory for the $(1+n)$–BP

$1+n$ gravitational point masses with $m_0 = 1$ ("Sun") and, for $i \geq 1$, $m_i \ll 1$ (n planets)

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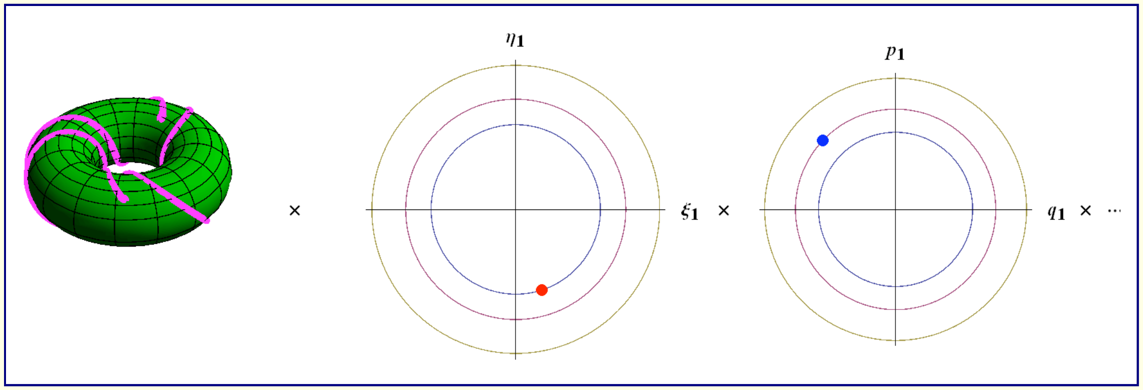

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Brief history and references (Lagrangian tori)

[A63] V.I. Arnold. *Small denominators and problems of stability of motion in classical and celestial mechanics.* Russian Math. Surveys, 18(6):85–191, 1963

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[CP11a] L. C. and G. Pinzari. *The planetary n-body problem: Symplectic foliation, reductions and invariant tori.* Invent. Math., 1–77, 2011

- direct, "second order" KAM (full non–degeneracies and Kolmogorov's normal forms)

Brief history and references (elliptic tori)

[F02] J. Féjoz. *Quasiperiodic motions in the planar three-body problem.* J. Differential Equations, 183(2):303–341, 2002

☛ planar, $n = 2$;

[BCV03] L. Biasco, L. C., and E. Valdinoci. *Elliptic two-dimensional invariant tori for the planetary three-body problem.* Arch. Rational Mech. Anal., 170:91–135, 2003. **and** 180: 507–509, 2006.

☛ spatial, $n = 2$;

[BCV06] L. Biasco, L. C., and E. Valdinoci. *n-dimensional elliptic invariant tori for the planar $(n + 1)$-body problem.* SIAM Journal on Mathematical Analysis, 37(5):1560–1588, 2006.

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[CP11b] L. C. and G. Pinzari. *Planetary Birkhoff normal forms.* Preprint, 2011

☛ aymptotic stability of secular actions (eccentricities and mutual inclinations) in non–resonant phase sets

Degeneracies

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Classical setting (used by Arnold, Nehoroševev, Robutel, Niederman, Herman, Féjoz):
Poincaré symplectic variables (regularization of Delaunay action-angle variables around zero eccentricities and inclinations):

$\big((\Lambda,\lambda),(\eta,\xi),(p,q)\big) \in \mathcal{M}^{6n}_{\epsilon} := \{0 < a_i < \delta\ a_{i+1}\} \times \mathbb{T}^n \times B^{4n}_{\epsilon}$,

$\Big(\Lambda_i = M_i\sqrt{\bar m_i a_i}; \delta < 1$ fixed $\Big)$ $\qquad d\Lambda \wedge d\lambda + d\eta \wedge d\xi + dp \wedge dq$

$$(\eta_j^2+\xi_j^2)/2 = \Lambda_j - G_j = \Lambda_j\,(1-\sqrt{1-e_j^2}) \simeq \tfrac{\Lambda_j}{2}\, e_j^2 \qquad \Big(G_j = |j^{\rm th}\ \text{ang. mom.}|\Big)$$

$$(p_j^2+q_j^2)/2 = G_j - \Theta_j = G_j(1-\cos\iota_j) \simeq \tfrac{G_j}{2}\,\iota_j^2 \qquad \Big(\Theta_j = \text{vert. comp. } j^{\rm th}\ \text{ang. mom.}\Big)$$

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$$\boxed{\mathcal{H} = h(\Lambda) + \mu f(\Lambda,\eta,\xi,\mathrm{p},\mathrm{q})}$$

proper degeneracy

$$\begin{cases} m_0 = 1, \quad m_i = \mu\bar m_i = O(\mu) \ll 1 \\ h(\Lambda) := h_{\text{Kepler}}(\Lambda) := \displaystyle\sum_{i=1}^n \frac{\kappa_i}{2\Lambda_i^2} \\ \langle f\rangle_\lambda = c_0(\Lambda) + \displaystyle\sum_{j=1}^n \Omega_j^{\text{pl}}\,\frac{\xi_j^2+\eta_j^2}{2} + \Omega_j^{\text{vr}}\,\frac{\mathrm{q}_j^2+\mathrm{p}_j^2}{2} + O(4) \quad \text{(up to rotation)} \end{cases}$$

Arnold's properly degenerate KAM theory

The general proper degeneracy problem was overcome by Arnold [A63] with a general ("second order") KAM theory for (real-analytic) Hamiltonians of the form $h(\Lambda) + \mu f(\Lambda, \lambda, y, x)$ with $\langle f \rangle_\lambda = c_0(\Lambda) + \sum \Omega_j \frac{y_j^2 + x_j^2}{2} + o(|(y, x)|^2)$ provided:

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(i) $\Lambda \to h'(\Lambda)$ is a diffeo ✓

(ii) the "secular" frequencies Ω_j (1st order Birkhoff invariants) are non–resonant up to order 4 so that $\langle f \rangle \to c_0 + \Omega \cdot r + \frac{1}{2}\tau \, r \cdot r + o(|r|^2)$ with $r_j = \frac{y_j^2 + x_j^2}{2}$

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⇨ for μ small enough, can find positive measure Kolmogorov's set (Lagrangian, Diophantine invariant tori with non-degenerate normal form)

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M. Herman computed $\det \tau_2$ $(n = 2)$ in the asymptotic $a_1 \to 0, a_2 = 1$ and found 0 and writes

"*J'ignore si* $\det \tau_2$ *est identiquement nulle!*"

[H?] M.R. Herman. *Torsion du problème planètaire*, edited by J. Fejóz in 2009. Electronic Archives Michel Herman on Yoccoz' web page

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But because of rotational and Herman resonances, Rüssmann's condition is violated in the planetary problem.

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- use the fact that *if* $\{H, F\} = 0$ *and* $\mathcal{T}$ *is an invariant, Lagrangian and transitive torus for* H *then it is invariant also for* F to conclude that the KAM invariant tori for $\mathcal{H}_\delta$ are also invariant tori for $\mathcal{H}$ (since $\{\mathcal{H}, \mathcal{H}_\delta\} = 0$). ■

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(ii) pass to their action-angle version (which will play the rôle of Delaunay variables) $(L, \Gamma, \Psi, \ell, \gamma, \psi) \in \mathbb{R}_+^{3n} \times \mathbb{T}^{3n}$ where: (L_i, ℓ_i) and $\Gamma_i = G_i$ are Delaunay ($L_i = \mu_i \sqrt{M_i a_i}$, ℓ_i=mean anomaly, $\Gamma_i = |C^{(i)}|$).

$$\boxed{\Psi_i = \Big| \sum_{j=1}^{i+1} C^{(j)} \Big| \quad (\text{for } 1 \le i \le n-1) \ , \quad \Psi_n := C_3 \qquad (\text{note} : \Psi_{n-1} = |C|)}$$

(Recall: Delaunay actions are $(\Lambda, \Gamma, \Theta)$ with $\Theta_i = C_3^{(i)}$)

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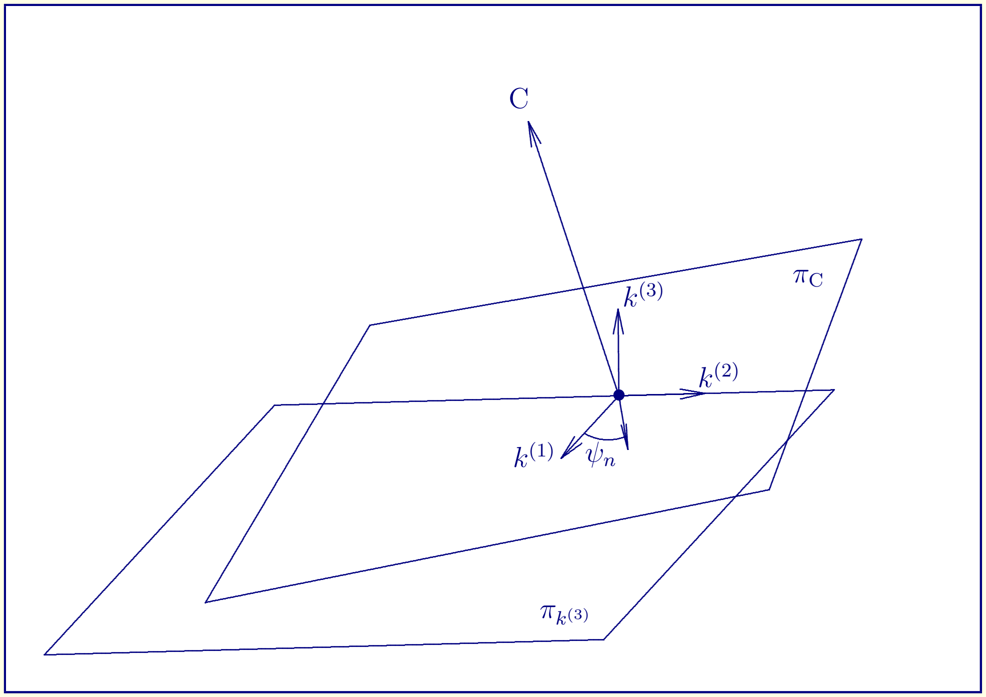
(Recall: Delaunay actions are $(\Lambda, \Gamma, \Theta)$ with $\Theta_i = C_3^{(i)}$)

(iii) regularize around co-circular and co-planar motions

$\longrightarrow (\Lambda, \lambda, \eta, \xi, p, q) \in \mathcal{M}_\epsilon^{6n} := \{0 < a_i < \delta\ a_{i+1}\} \times \mathbb{T}^n \times B_\epsilon^{4n}$

☛ Key point is that $\begin{cases} p_n = \sqrt{2(|C| - C_3)}\ \cos\psi_n \\ q_n = -\sqrt{2(|C| - C_3)}\sin\psi_n \end{cases}$ with

ψ_n = total angular momentum node



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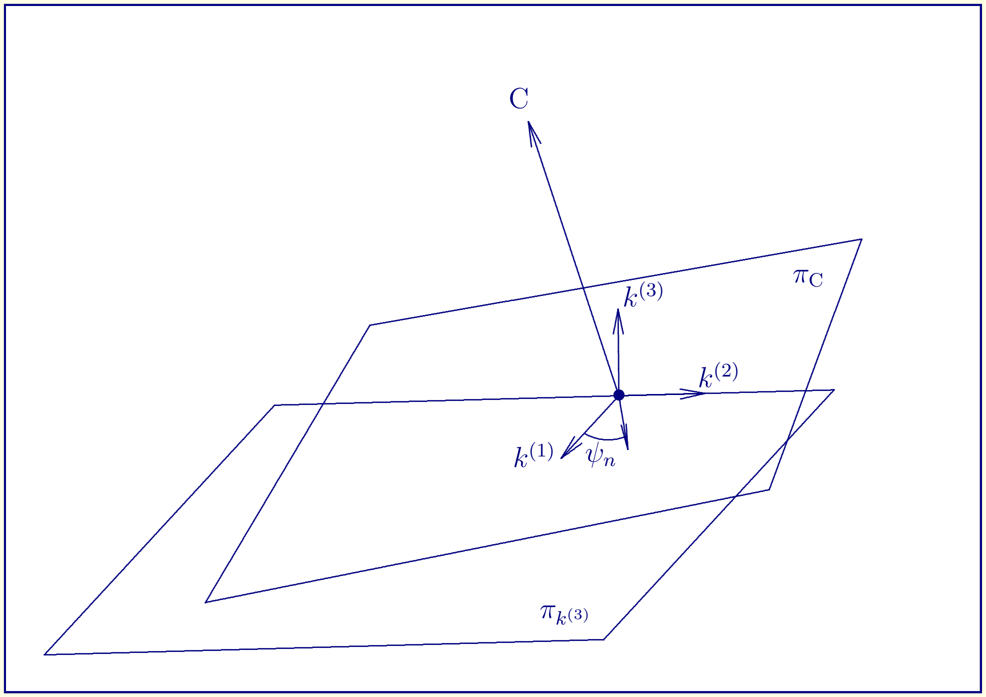

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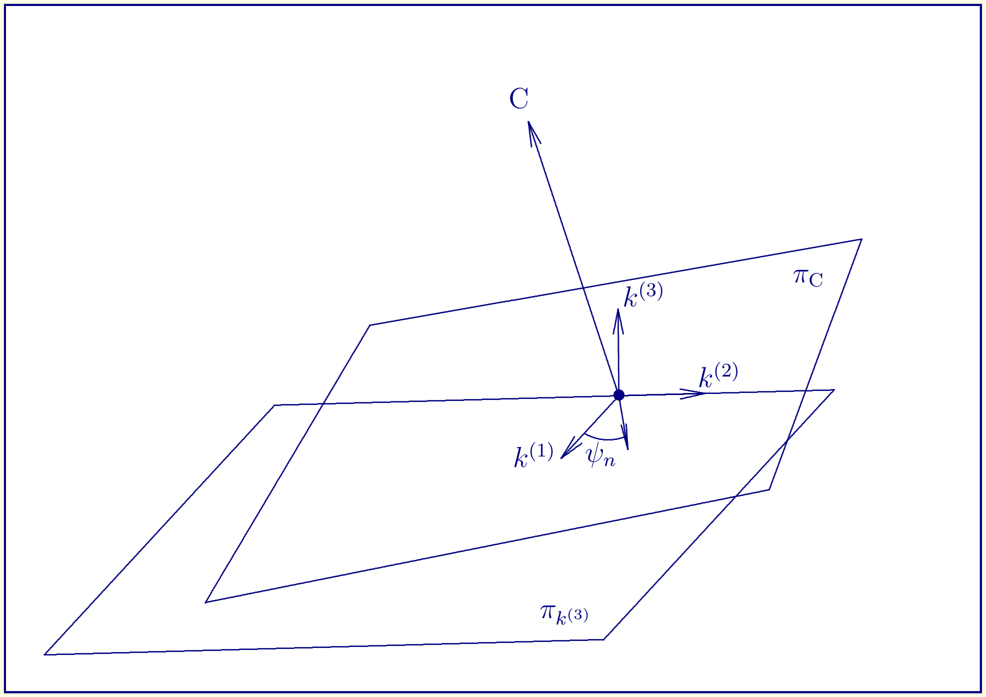

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⇨ the phase space $\mathcal{M}^{6n}_{\epsilon}$ is foliated by symplectic invariant manifolds $\mathcal{M}^{6n-2}_{\epsilon;p_n,q_n}$ with symplectic variables $(\Lambda, \lambda, \eta, \xi, \bar{p}, \bar{q})$, $\bar{p} = (p_1, ..., p_{n-1})$, $\bar{q} = (q_1, ..., q_{n-1})$

Relation between RPS and Poincaré variables:

$\phi : (\Lambda, \lambda, z) \to (\Lambda, \lambda, \mathrm{z})$, $\quad z := (\eta, \xi, p, q), \qquad \mathrm{z} = (\upeta, \upxi, \mathrm{p}, \mathrm{q})$

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- more in general ϕ has the form

$$\lambda = \lambda + \varphi(\Lambda, z) \ , \qquad \mathrm{z} = Z(\Lambda, z) \quad \text{with}$$

  - $\varphi(\Lambda, 0) = 0$
  - $z \to Z(\Lambda, z)$ is 1:1, symplectic and $\exists\, U = U(\Lambda) \in \mathrm{SO}(n)$:

$$\begin{cases} \pi_{\eta} Z = \eta + O(|z|^3) \\ \pi_{\xi} Z = \xi + O(|z|^3) \end{cases} \qquad \begin{cases} \pi_{\mathrm{p}} Z = Up + O(|z|^3) \\ \pi_{\mathrm{q}} Z = Uq + O(|z|^3) \end{cases}$$

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Proof $\mathcal{H}\circ\phi = h(\Lambda) + \mu\bar{f}(\Lambda,\lambda,\eta,\xi,\bar{p},\bar{q})$, $\quad (\eta,\xi,\bar{p},\bar{q}) =: \bar{z} \in \bar{B}^{4n-2}_{\epsilon} \subseteq \mathbb{R}^{4n-2}$

and, $\bar{f}_{\rm sec} := \langle \bar{f} \rangle_\lambda$ can be put in Birkhoff normal form up to order 4:

$\boxed{\bar{f}^4_{\rm sec} = c_0(\Lambda) + \bar{\Omega}\cdot r + \frac{1}{2}\bar{\tau}\ r\cdot r + O(|r|^3)}$ $\quad r \in \mathbb{R}^{2n-1}_+$ $\ (r_j = |\bar{z}_j|^2/2)$ and $\boxed{\det\bar{\tau} \neq 0}$

in the well spaced regime ($a_1 \ll a_2 \ll \cdots \ll a_n$) and hence (analytic continuation) in an open dense subset of $\{0 < a_i < \delta\ a_{i+1}\}$. ■

Theorem: *on $\mathcal{M}^{6n-2}_{\epsilon;p_n,q_n}$ the torsion does not vanish (for an open dense set of a_j's).*

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and, $\bar{f}_{\rm sec} := \langle \bar{f} \rangle_{\lambda}$ can be put in Birkhoff normal form up to order 4:

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in the well spaced regime $(a_1 \ll a_2 \ll \cdots \ll a_n)$ and hence (analytic continuation) in an open dense subset of $\{0 < a_i < \delta\ a_{i+1}\}$. ■

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- Herman resonance is the only exact resonance and $\{\bar{f}_{\rm sec}, G\} = 0$ ⇨

to construct the Birkhoff normal form is enough $\Omega\cdot k \neq 0$, $\forall$ integer k such that $\sum k_j \neq 0$. Thus the planetary BNF exists at any order (and in Poincaré variables is degenerate at any order).

Corollaries

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(1) For $\mu < \epsilon^6 < \epsilon_*^6$, the Kolmogorov's set $\mathcal{K} \subseteq \mathcal{M}^{6n-2}_{\epsilon;p_n,q_n}$ satisfies

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(2) The tori $\mathcal{T} \in \mathcal{K}$ are <u>Kolmogorov's tori</u> i.e. $\exists$ a symplectic map

$\nu : B^{3n-1} \times \mathbb{T}^{3n-1} \to \mathcal{M}^{6n-2}_{\epsilon;p_n,q_n}$ such that $\boxed{\mathcal{H} \circ \nu = E + \omega \cdot y + Q}$ with

- $Q = O(|y|^2)$
- $\mathcal{T} = \nu\big(0, \mathbb{T}^{3n-1}\big)$
- ω Diophantine , with: $\omega_j = \begin{cases} O(1) & \text{for } 1 \le j \le n \\ O(\mu) & \text{for } n+1 \le j \le 2n-1 \end{cases}$
- $\det \langle Q_{yy}(0, \cdot) \rangle_{\mathbb{T}^{3n-1}} \neq 0$

(3) By Conley-Zehnder's [1983] argument (based on non degenerate Kolmogorov's NF as in (2) + Birkhoff-Lewis):

$$\mathrm{meas}\Big(\mathrm{cls}\{\textit{periodic orbits in } \mathcal{M}^{6n-2}_{\epsilon;p_n,q_n}\}\Big) \geq \mathrm{meas}\,\mathcal{K} \geq \mathrm{const}\,\epsilon^{2(2n-1)}$$

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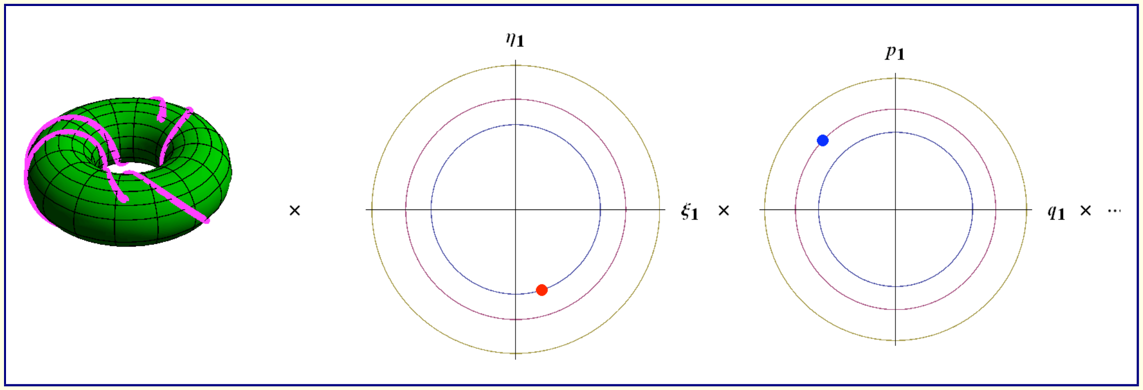
(4) *$\mathcal{K}$ surrounds a family of positive n–dimensional measure of elliptic invariant tori bifurcating from $\bar{z} = 0$.*

(For this result is not needed non-vanishing torsion; enough Melnikov's condition on $\bar{\Omega}$: $\bar{\Omega} \cdot k \neq 0$ for $0 < |k| \leq 2$)



II. Long time action stability

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Theorem (Nehorošev 1977) $\exists\, C, a, b, \alpha, \epsilon_* > 0$: *for any* $0 < \mu < \epsilon^\alpha \leq \epsilon_*^\alpha$ *any motion starting in* $\mathcal{M}_\epsilon^{6n}$ *(with* $a_1 \ll a_2 \ll \cdots \ll a_n$*) satisfies*

$$\begin{cases} |a_j(t) - a_j(0)| < C\mu^b \\ |z(t)| \leq \epsilon \end{cases} \qquad \forall\ |t| < \frac{1}{C\mu} \exp\left(\frac{1}{C\mu^a}\right).$$

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Proof based on Nehorošev's celebrated general result on exponential stability of nearly–integrable, convex (steep) Hamiltonians, regarding the z's as dummy variables and using the conservation of $C_3 = \sum \Lambda_j - \frac{|z|^2}{2}$ to confine the secular variables. ■

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Niederman 1996 improves significantly the constants using Lochack's approach through approximation by periodic motions and Dirichlet simultaneous Diophantine approximation.

¿ Behaviour of the secular actions $r_j = \frac{|z_j|^2}{2}$?

Secular variation of r_j are related to changes in eccentricities and inclinations on the ecliptic $\longrightarrow$ basic question:

Outside $\mathcal{K} \times \{(p_n, q_n)\}$, do the $|z_j|$'s move around B_ϵ^{4n} or do they stay close to their initial values $|z_j(0)|$ (say $\big||z_j(t)| - |z_j(0)|\big| < o(\epsilon)$) for times of the order $\frac{1}{C\mu} \exp\left(\frac{1}{C\epsilon}\right)$ ("exponential stability") or at least for times of order $\frac{1}{\mu\epsilon^k}$ for arbitrary k ("asymptotic stability")?

Asymptotic stability in non-resonant zones

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Theorem ([CP11b]) *Fix* $k \geq 1$. *For* μ *and* ϵ *small enough, there exists a neighborhood* $\mathcal{N}$ *of* $\mathcal{K}$ *such that any motion starting in* $\mathcal{N}$ *satisfies*

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Proof Use: existence of BNF up to any order and “properly–degenerate averaging theory” (analytic part of Nehoroševʼs theorem in the properly–degenerate setting) . ■

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$$\bar{\tau} \ \sim \ \begin{pmatrix} c_2 \begin{pmatrix} \sqrt{17} & 0 & 0 \\ 0 & -\sqrt{17} & 0 \\ 0 & 0 & \frac{4}{17}\frac{\Lambda_1^2}{\Lambda_2^2} \end{pmatrix} & & & \\ & \ddots & & \\ & & c_n \begin{pmatrix} -\frac{m_n}{m_{n-1}} & 0 \\ 0 & 5\,\frac{m_{n-1}}{m_n}\,\frac{a_{n-1}}{a_n} \end{pmatrix} & \\ & & & \ddots \end{pmatrix}$$

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☛ In particular, the integrable truncation $c_0(\Lambda) + \bar\Omega\cdot r + \frac{1}{2}\bar\tau\ r\cdot r$ is <u>non–convex</u>.

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- and, of course , Arnold's diffusion...