Conformal Geometry and Metrics of Holonomy Split G_2

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Connections in Geometry and Physics Fields Institute May 14, 2011

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Example: Hol $(M, g) \subset U(n/2)$ if and only if g is Kähler.



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For many, but not all, groups on the list, examples were known of (M,g) with that holonomy.

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Manifolds of holonomy G_2^c arise in M-theory as an analogue of Calabi-Yau manifolds.



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From now on, $G_2 = G_2^s$.

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or as a nonholonomic constraint in a classical mechanical system.

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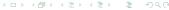
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So any diffeomorphism preserving \mathcal{D} also preserves the (2,3) conformal structure on $S^2 \times S^3$! True locally too.

Nurowski's Conformal Structures

Theorem. (Nurowski, 2005) Any $\mathcal{D} \subset TM^5$ generic. There is a conformal class [g] on M of signature (2,3) associated to \mathcal{D} .

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Approximately 70 terms. Very nasty.



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If n is even, there is a formal obstruction at order n/2.

Put these together:

$$\mathcal{D} \subset \mathit{TM}^5 \stackrel{\mathit{Nurowski}}{\to} (\mathit{M},[\mathit{g}]) \stackrel{\mathit{Ambientmetric}}{\to} (\widetilde{\mathcal{G}}^7,\widetilde{\mathit{g}})$$

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- If one of a_3 , a_4 , a_5 , $a_6 \neq 0$, then $\mathsf{Hol}(\widetilde{\mathcal{G}},\widetilde{\mathsf{g}}) = \mathcal{G}_2$.



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But what about \widetilde{g} for other \mathcal{D} ?

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Say that \mathcal{A} is 3-nondegenerate at p if the only vector $X \in \mathcal{D}_p$ such that $\mathcal{A}(X,Y,Y,Y)=0$ for all $Y \in \mathcal{D}_p$ is X=0.

Conditions for $Hol(\widetilde{\mathcal{G}}, \widetilde{g}) = G_2$

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Theorem. Given (M, \mathcal{D}) real analytic. If there are $p, q \in M$ so that L_p is injective and A_q is 3-nondegenerate, then \widetilde{g} has holonomy $= G_2$.

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Reinterpret $\varphi|_{\mathcal{G}}$ in terms of the tractor bundle of (M,[g]).

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Conclusion. $\varphi|_{\mathcal{G}}$ can be viewed as a section of $\Lambda^3\mathcal{T}^*$: a tractor 3-form.

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Immediately conclude $\operatorname{Hol}(\widetilde{\mathcal{G}},\widetilde{g})\subset \mathit{G}_{2}.$



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$$\mathcal{H}_+ = \{|x|^2 - |y|^2 = 1\}$$
 and $\mathcal{H}_- = \{|x|^2 - |y|^2 = -1\}.$

Then $g_+ = \widetilde{g}|_{\mathcal{TH}_+}$ has signature (p, q+1) and curvature +1.

Likewise $g_-=\widetilde{g}|_{\mathcal{TH}_-}$ has signature (p+1,q) and curvature -1.

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Analogous construction for general (M, [g]):

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Constructing \widetilde{g} is equivalent to constructing g_{\pm} .



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So we obtain new examples of metrics of these types parametrized by $\mathcal{D}\subset TM^5$.