Conformal Geometry and Metrics of Holonomy Split G_2

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Connections in Geometry and Physics
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Example: $\text{Hol}(M, g) \subset U(n/2)$ if and only if g is Kähler.

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For many, but not all, groups on the list, examples were known of (M,g) with that holonomy.

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Manifolds of holonomy G_2^c arise in M-theory as an analogue of Calabi-Yau manifolds.

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From now on, $G_2 = G_2^s$.

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or as a nonholonomic constraint in a classical mechanical system.

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This gives the model $\mathcal{D} \subset T(S^2 \times S^3)$.

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Now recall $G_2 \subset SO(3, 4)$

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So any diffeomorphism preserving $\mathcal{D}$ also preserves the $(2, 3)$ conformal structure on $S^2 \times S^3$! True locally too.

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Approximately 70 terms. Very nasty.

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Put these together:

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But what about $\widetilde{g}$ for other $\mathcal{D}$?

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So we obtain an infinite-dimensional space of metrics $\widetilde{g}$ of holonomy G_2, parametrized by an almost arbitrary generic 2-plane field $\mathcal{D}$.

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Theorem. Given $(M, \mathcal{D})$ real analytic. If there are p, $q \in M$ so that L_p is injective and $\mathcal{A}_q$ is 3-nondegenerate, then $\widetilde{g}$ has holonomy $= G_2$.

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Reinterpret $\varphi|_{\mathcal{G}}$ in terms of the tractor bundle of $(M, [g])$.

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This had been previously proved by Gover for $r = 1$, different proof.

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Holds for general conformal structures in any signature and dimension and for parallel tractors having arbitrary symmetry.

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Theorem. Let $(M, [g])$ be a conformal manifold, with ambient metric $\widetilde{g}$. Suppose φ is a parallel tractor-tensor of rank $r \in \mathbb{N}$.

- If n is odd, then φ has an ambient extension $\widetilde{\varphi}$ such that $\widetilde{\nabla}\widetilde{\varphi}$ vanishes to infinite order along $\mathcal{G}$.
- If n is even, then φ has an ambient extension such that $\widetilde{\nabla}\widetilde{\varphi}$ vanishes to order $n/2 - 1$.

This had been previously proved by Gover for $r = 1$, different proof.

Immediately conclude $\mathrm{Hol}(\widetilde{\mathcal{G}}, \widetilde{g}) \subset G_2$.

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Constructing $\widetilde{g}$ is equivalent to constructing $g_\pm$.

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So we obtain new examples of metrics of these types parametrized by $\mathcal{D} \subset TM^5$.