The cylindrical contact homology of universally tight sutured solid tori

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Outline

- Introduction
- Cylindrical Contact Homology
- Sutures and Contact Homology
- Results

Motivation

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- In 2010, Colin, Ghiggini, Honda and Hutchings have generalized contact homology to sutured manifolds. This is possible by imposing certain convexity condition on a contact form.
- The goal is to do some computations of the sutured contact homology which do not follow from the computations in the closed case.

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- The Reeb vector field R_α that is associated to a contact form α is characterized by equations $\alpha(R_\alpha) = 1$ and $d\alpha(R_\alpha, \cdot) = 0$.
- A Reeb orbit is a closed orbit of the Reeb flow, i.e., a smooth map $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ for some $T > 0$ such that $\dot{\gamma}(t) = R_\alpha(\gamma(t))$.

Consider Reeb orbit γ passing through a point $x \in M$. The linearization of the Reeb flow on the contact planes along γ with respect to some fixed symplectic trivialization determines a linearized return map $P_\gamma : \xi_x \to \xi_x$. This linear map is symplectic and it does not depend on x (up to conjugation).

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Assume that all Reeb orbits, including multiply covered ones, are non-degenerate.

Definition

A Reeb orbit γ is called elliptic or positive (respectively negative) hyperbolic if the eigenvalues of P_γ are on the unit circle or the positive (resp. negative) real line respectively.

Conley-Zehnder index

Definition

If γ is elliptic, then there is an irrational number $\phi \in \mathbb{R}$ such that P_γ is conjugate in $SL_2(\mathbb{R})$ to a rotation by angle $2\pi\phi$, and

$$\mu_\tau(\gamma^k) = 2\lfloor k\phi \rfloor + 1,$$

where $2\pi\phi$ is the total rotation angle with respect to the fixed trivialization τ of the linearized flow around the orbit.

If γ is positive (respectively negative) hyperbolic, then there is an even (respectively odd) integer r such that the linearized flow around the orbit rotates the eigenspaces of P_γ by angle πr with respect to τ, and

$$\mu_\tau(\gamma^k) = kr.$$

Cylindrical Contact Homology Module

There are 2 ways the Conley-Zehnder index of γ^m can behave :

- the parity of $\mu_\tau(\gamma^m)$ is the same for all $m \geq 1$.
- the parity for the even multiples $\mu_\tau(\gamma^{2k})$, $k \geq 1$, disagrees with the parity for the odd multiples $\mu_\tau(\gamma^{2k-1})$, $k \geq 1$.

In the second case, the even multiples γ^{2k}, $k \geq 1$, are called bad orbits. An orbit that is not bad is called good.

Definition

Cylindrical contact homology module $C_*(M, \alpha)$ is a $\mathbb{Q}$-module freely generated by all good Reeb orbits.

Almost complex structure

Consider $(\mathbb{R} \times M, d(e^s \alpha))$, where s is a coordinate on $\mathbb{R}$.
An almost complex structure J on $\mathbb{R} \times M$ is called admissible if

- J is $\mathbb{R}$-invariant;
- $J(\xi) = \xi$;
- $d\alpha(v, Jv) > 0$ for nonzero $v \in \xi$;
- $J(\partial_s)$ is a positive multiple of R_α.

Holomorphic cylinders

Fix an admissible almost complex structure J on $\mathbb{R} \times M$. Given two good Reeb orbits γ_- and γ_+ we denote by $M^J(\gamma_-, \gamma_+)$ the moduli space of maps (u, j), where

(1) j is a complex structure on $\dot{S}^2 := S^2 \setminus \{0, \infty\}$ and $u : (\dot{S}^2, j) \to (\mathbb{R} \times M, J)$ is J-holomorphic, i.e., u satisfies $du \circ j = J \circ du$;

(2) u is asymptotically cylindrical over γ_- at the negative end of $\mathbb{R} \times M$ at the puncture $0 \in S^2$; and u is asymptotically cylindrical over γ_+ at the positive end of $\mathbb{R} \times M$ at the puncture $\infty \in S^2$;

(3) $(u, j) \sim (v, j')$ if there is a diffeomorphism $f : \dot{S}^2 \to \dot{S}^2$ such that $v \circ f = u$, $f_* j = j'$ and f fixes all punctures.

Definition

Define the cylindrical contact homology differential

$$\partial\gamma_+ := \sum_{\gamma_-} n_{\gamma_+,\gamma_-} \cdot \gamma_-,$$

where n_{γ_+,γ_-} is the signed weighted count of points in $\mathcal{M}^J(\gamma_+,\gamma_-)/\mathbb{R}$ with $dim\, M^J(\gamma_+,\gamma_-)/\mathbb{R} = 0$.

If all good Reeb orbits are non-contractible, then cylindrical contact homology is defined.

Definition

For a Reeb orbit γ we define symplectic action

$$\mathcal{A}(\gamma) := \int_{\gamma} \alpha.$$

Symplectic action and contact energy

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- **Fact**
  $E(u) \geq 0$ and $E(u) = 0$ if and only if $\gamma_+ = \gamma_-$, and in this case the image of u is $\mathbb{R} \times \gamma_-$.

Definition

A sutured manifold (M, Γ) consists of the following data:

(1) M is a compact, oriented 3-manifold with corners;
(2) Γ is a union of disjoint annuli on ∂M;
(3) $\partial\Gamma$ is a set of corners;
(4) $\partial M \setminus (\Gamma \setminus \partial\Gamma) = R_+(\Gamma) \bigsqcup R_-(\Gamma)$, where R_+ and R_- are surfaces with boundary;
(5) orientation of R_+ agrees with the boundary orientation; orientation of R_- is opposite to the boundary orientation.

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Definition

A contact form α (Reeb vector field R_α) is adapted to (M, Γ) if:

(1) R_α is transverse to $R_+(\Gamma)$ and exits M;
(2) R_α is transverse to $R_-(\Gamma)$ and enters M;
(3) R_α is tangent to Γ and on each component $A = S^1 \times [0, 1]$ is of the form $\frac{\partial}{\partial t}$, where t is a coordinate of $[0, 1]$.
(4) technical condition.

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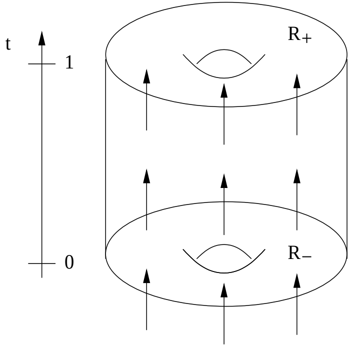

Figure: Product sutured contact manifold $(\Sigma \times [0,1], \Gamma = \partial\Sigma \times [0,1], dt + \beta)$. Here $(\Sigma, d\beta)$ is a Liouville manifold.

Theorem(Colin, Ghiggini, Honda, Hutchings)

Let (M, Γ, ξ) be a sutured contact 3-manifold with an adapted contact form α such that all Reeb orbits are non-degenerate and non-contractible. Then the contact homology $HC^{cyl}(M, \Gamma, \xi)$ is defined and is independent of the choice of contact 1-form α with $ker(\alpha) = \xi$, adapted almost complex structure J, and abstract perturbation.

Results

Theorem(G)

Let $(S^1 \times D^2, \Gamma)$ be a sutured manifold, where Γ is a set of $2n$ parallel closed curves of slope $k \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$. Then there is a contact form α which makes $(S^1 \times D^2, \Gamma, \alpha)$ a sutured contact manifold, $HC^{cyl}(S^1 \times D^2, \Gamma, \alpha)$ is defined, is independent of the contact form α for $\xi = ker\, \alpha$ and the almost complex structure J and

$$HC_h^{cyl}(S^1 \times D^2, \Gamma, \xi) \simeq \begin{cases} \mathbb{Q}, & \text{for } k \nmid h > 0; \\ \mathbb{Q}^{n-1}, & \text{for } k \mid h > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Here h is a homological grading.

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- adapted to $(D^2 \times [0,1], S^1 \times [0,1])$;
- adapted to the gluing map ψ, roughly speaking $\psi : D^2 \times \{1\} \to D^2 \times \{0\}$ given by $R_{-1/k} \circ \varphi_{X_H}$, where $R_{-1/k}$ is a $-1/k$-rotation around the origin and φ_{X_H} is a time-1 map of the Hamiltonian flow given by H, i.e. $\alpha = dt + \beta$ with $d\beta > 0$ on D^2 near $D^2 \times \{0\}$ and $\alpha = dt + \phi^* \beta$ near $D^2 \times \{1\}$ and R_α collinear to ∂t.

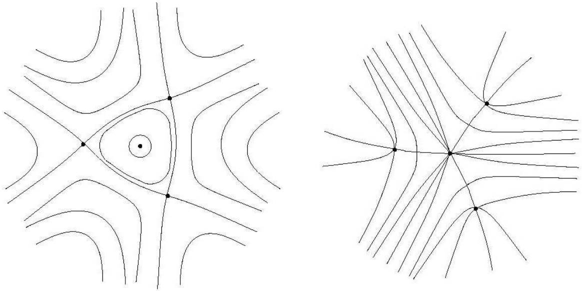

Figure: Level sets of H (left) and characteristic foliation of β (right) when $n = 1$, $|k| = 3$.

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5. 3 and 4 imply that $\partial \gamma_i^s = 0$ and $\partial \gamma^t = 0$ for $k \nmid t$.
6. We prove that ∂ does not vanish only on γ^t for $k \mid t$. This completes the proof.

Any Questions?