

Examples of Einstein Manifolds in Odd Dimensions

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Connections in Geometry and Physics
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We construct Einstein metrics on some fiber bundles over a Fano K-E manifold.

- $D^2 \times S^1$ bundle
 - Negative scalar curvature
 - Conformally compact Einstein (CCE)
 - Q -flat conformal infinity
 - Zero scalar curvature
 - Asymptotically a degenerate cone
 - Slow volume growth and quadratic curvature decay
- S^3 bundle
 - Positive scalar curvature
 - Diffeomorphism classification of the total spaces for the base to be $\mathbb{C}P^2$
 - Moduli space of Einstein structures with more than one component

Principal Bundle

- (V^n, h) : a Fano K-E manifold of $\dim_{\mathbb{C}} = n$.
 - $c_1(V) = p \cdot \eta$: $p \in \mathbb{Z}_+$, and η is a generator of $H^2(V; \mathbb{Z})$.
 - $\text{Ric}(h) = ph$, and the Kähler form is $\omega = 2\pi[\eta] \in H^2(V; \mathbb{R})$.
- P_{q_1, q_2} : an $S^1 \times S^1$ bundle over V classified by $(q_1 \cdot \eta, q_2 \cdot \eta)$.
 - $P_{q_1, q_2} \cong \Delta^*(P_{q_1} \times P_{q_2})$: $\Delta : V \rightarrow V \times V$ is the diagonal map, and P_{q_i} is an S^1 bundle over V with Euler class $q_i \cdot \eta$.

$$\begin{array}{ccc} P_{q_1, q_2} & \rightarrow & P_{q_1} \times P_{q_2} \\ \downarrow & & \downarrow \\ V & \xrightarrow{\Delta} & V \times V \end{array}$$

Associated Bundle

- The effective action of $S^1 \times S^1$ on \mathbb{C}^2

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$$

induces two fiber bundles associated with P_{q_1, q_2}

- $\text{Sol}_{q_1, q_2} = P_{q_1, q_2} \times_{S^1 \times S^1} (D^2 \times S^1 \subset \mathbb{C}^2)$
- $\text{Sph}_{q_1, q_2} = P_{q_1, q_2} \times_{S^1 \times S^1} (S^3 \subset \mathbb{C}^2)$

- Open dense subspace

- $\text{Sol}_{q_1, q_2}^0 = P_{q_1, q_2} \times_{S^1 \times S^1} (D^2 \times S^1 \setminus \{0\} \times S^1)$
- $\text{Sph}_{q_1, q_2}^0 = P_{q_1, q_2} \times_{S^1 \times S^1} (S^3 \setminus \{0\} \times S^1, S^1 \times \{0\})$
- $\text{Sol}_{q_1, q_2}^0 \simeq \text{Sph}_{q_1, q_2}^0 \simeq \text{Int} \times P_{q_1, q_2}$

- Ansatz 1: Introduce the following metric on $\text{Int} \times P_{q_1, q_2}$

$$g = \alpha(s)^{-1} ds^2 + \sum_{i,j=1}^2 B_{ij}(s) \theta^i \otimes \theta^j + \beta(s) h,$$

$\alpha = \det B$, and θ^i is a connection on P_{q_i} with curvature $d\theta^i = q_i \omega$.

- The Einstein equation $\text{Ric}(g) = \lambda g$ reduces to a system of ODEs

$$n\alpha\left(-\frac{\beta''}{\beta} + \frac{1}{2}\left(\frac{\beta'}{\beta}\right)^2\right) - n\frac{\alpha'}{2}\frac{\beta'}{\beta} - \frac{\alpha''}{2} + \frac{\det(B')}{2} = \lambda,$$

$$\frac{\alpha}{2}\left(-n\frac{\beta'}{\beta}\Phi - \Phi'\right) - \frac{\alpha'}{2}\Phi + \frac{n}{2\beta^2}U^T Q = \lambda \text{Id}_2,$$

$$\frac{\alpha}{2}\left(-\frac{\beta''}{\beta} - (n-1)\left(\frac{\beta'}{\beta}\right)^2\right) - \frac{\alpha'}{2}\frac{\beta'}{\beta} + \frac{p}{\beta} - \frac{\Delta}{2\beta^2} = \lambda,$$

$\Phi = B'B^{-1}$, $Q = (q_1, q_2)$, $U = QB$, and $\Delta = QBQ^T$.

Exact Solution

- Ansatz 2: $\beta = \kappa s$
- $\alpha = -\frac{2\lambda}{n+1}s^2 + \frac{2p}{\kappa(n+1)}s + c_1s^{1-n} + c_2s^{-n}$
- $\Delta = \frac{2p\kappa}{n+1}s + \kappa^2c_2s^{-n}$
- $U_1 = \frac{2\kappa(pw_1 + npq_2\kappa^n c_1 + q_2\lambda\kappa^{n+1}c_2(n+1))}{(n+1)(q_1w_1 + q_2w_2)}s + \frac{\kappa^2c_2(w_1 - q_2\kappa^n c_1)}{q_1w_1 + q_2w_2}s^{-n}$
- $U_2 = \frac{2\kappa(pw_2 - npq_1\kappa^n c_1 - q_1\lambda\kappa^{n+1}c_2(n+1))}{(n+1)(q_1w_1 + q_2w_2)}s + \frac{\kappa^2c_2(w_2 + q_1\kappa^n c_1)}{q_1w_1 + q_2w_2}s^{-n}$
- $b_{11} = \frac{U_1^2 + q_2^2\alpha}{\Delta}, b_{12} = \frac{U_1U_2 - q_1q_2\alpha}{\Delta}, b_{22} = \frac{U_2^2 + q_1^2\alpha}{\Delta}$
- Parameters: $\kappa, c_1, c_2, w_1, w_2$.
- Constraint: $(q_1w_1 + q_2w_2)^2 = 2(n+1)\kappa^{2n+3}c_2(p c_1 + c_2\lambda\kappa) > 0$.

Sol_{q_1, q_2} ($q_1^2 + q_2^2 > 0$) admits a two-parameter family of Einstein metrics with negative scalar curvature.

- Smooth collapse at $t = t_L$: $b_{11}(t_L) = 0$ and $\frac{d}{dt}|_{t=t_L} \sqrt{b_{11}(t)} = 1$.
- Geodesic distance: $t = \int \frac{ds}{\sqrt{\alpha(s)}} = O(\ln s)$
- Conformally compact:
 - Defining function: $\rho = s^{-\frac{1}{2}}$.
 - Geodesic defining function: $d \ln \sigma = \frac{ds}{\sqrt{\alpha(s)}}$
 - The conformal infinity ($\{\sigma = 0\}$, $[\sigma^2 g|_{\{\sigma=0\}}]$) has zero Q-curvature.
 - $\int_{\sigma=0} Q = 0$: (Graham-Zworski) The asymptotic expansion
 $\text{Vol}_g(\{\sigma > \epsilon\}) = c_0 \epsilon^{-(2+2n)} + (\text{even powers}) + c_{2n} \epsilon^{-2} + O(1)$.
 - $Q \equiv \text{const}$: (Fefferman-Graham) The unique solution to
 $\Delta_g u = 2n + 2$ of the form $u = \ln \sigma + A(\sigma) + B(\sigma) \sigma^{2n+2} \ln \sigma$.

Sol_{q_1, q_2} ($q_1^2 + q_2^2 > 0$) admits a one-parameter family of Ricci flat metrics.

- Smooth collapse at $t = t_L$: $b_{11}(t_L) = 0$ and $\frac{d}{dt}|_{t=t_L} \sqrt{b_{11}(t)} = 1$.
- Geodesics distance: $t = \int \frac{ds}{\sqrt{\alpha(s)}} = O(s^{\frac{1}{2}})$.
- Asymptotic cone: $g_c = dt^2 + t^2 (\sum_{i,j=1}^2 \hat{U}_i \hat{U}_j \theta^i \otimes \theta^j + \hat{\kappa} h)$
- Slower-than-Euclidean volume growth:
 $\text{Vol}_g = \int \beta^n ds = O(s^{n+1}) = O(t^{2n+2})$.
- Sectional curvatures decay like t^{-2} .

Sph_{q_1, q_2} ($|q_1| > |q_2| > 0$) admits an Einstein metric with positive scalar curvature.

- Smooth collapse at $t = t_L$ and $t = t_R$:

- $b_{11}(t_L) = 0$ and $\frac{d}{dt}|_{t=t_L} \sqrt{b_{11}(t)} = 1$.
- $b_{22}(t_R) = 0$ and $\frac{d}{dt}|_{t=t_R} \sqrt{b_{11}(t)} = -1$.

- $V = \mathbb{C}P^1$ (Hashimoto-Sakaguchi-Yasui, 2005)

$$\text{Sph}_{q_1, q_2} \cong \begin{cases} S^2 \times S^3, & q_1 + q_2 \equiv 0 \pmod{2} \\ S^2 \tilde{\times} S^3, & q_1 + q_2 \equiv 1 \pmod{2} \end{cases}$$

- $V = \mathbb{C}P^2$:

- $\pi_1(\text{Sph}_{q_1, q_2}) = 0$
- $H^2(\text{Sph}_{q_1, q_2}; \mathbb{Z}) \cong \mathbb{Z} \cdot \xi$, $H^3(\text{Sph}_{q_1, q_2}; \mathbb{Z}) = 0$, and
 $H^4(\text{Sph}_{q_1, q_2}; \mathbb{Z}) \cong \mathbb{Z}_{|q_1 q_2|} \cdot \xi^2$

- $\text{Bal}_{q_1, q_2} = P_{q_1, q_2} \times_{S^1 \times S^1} (D^4 \subset \mathbb{C})$:
 - $\partial \text{Bal}_{q_1, q_2} = \text{Sph}_{q_1, q_2}$
 - $H^2(\text{Bal}_{q_1, q_2}; \mathbb{Z}) = \mathbb{Z} \cdot \zeta$, $H^2(\text{Sph}_{q_1, q_2}; \mathbb{Z}) = \mathbb{Z} \cdot \xi$, and $\zeta \approx \xi$
 - Spin: $w_2(\text{Bal}_{q_1, q_2}) = 0$ and $w_2(\text{Sph}_{q_1, q_2}) = 0$
 - Non-spin: $w_2(\text{Bal}_{q_1, q_2}) = \zeta \pmod{2}$ and $w_2(\text{Sph}_{q_1, q_2}) = \xi \pmod{2}$
 - $H^4(\text{Bal}_{q_1, q_2}, \text{Sph}_{q_1, q_2}; \mathbb{Q}) \cong H^4(\text{Bal}_{q_1, q_2}; \mathbb{Q}) \cong \mathbb{Q} \cdot \zeta$

- Diffeomorphism invariants $\chi_i \pmod{\mathbb{Z}}$:

- Spin: $\chi_1 = \frac{p_1^2}{896} - \frac{\sigma}{224}$, $\chi_2 = \frac{\zeta^4}{24} - \frac{\zeta^2 p_1}{48}$, $\chi_3 = \frac{2}{3}\zeta^4 - \frac{\zeta^2 p_1}{12}$
- Non-spin: $\chi_1 = \frac{\zeta^4}{384} - \frac{\zeta^2 p_1}{192} + \frac{p_1^2}{896} - \frac{\sigma}{224}$, $\chi_2 = \frac{5}{24}\zeta^4 - \frac{\zeta^2 p_1}{24}$,
 $\chi_3 = \frac{13}{8}\zeta^4 - \frac{\zeta^2 p_1}{8}$

where $p_1 = p_1(\text{Bal}_{q_1, q_2})$ and $\sigma = \text{sign}(\text{Bal}_{q_1, q_2})$.

- (Kreck-Stolz) $\text{Sph}_{q_1, q_2} \simeq \text{Sph}_{q'_1, q'_2} \Leftrightarrow \chi_i(\text{Sph}_{q_1, q_2}) = \chi_i(\text{Sph}_{q'_1, q'_2})$
- Examples:
 - Spin: $\text{Sph}_{1, 21170} \simeq \text{Sph}_{145, 146}$, $\text{Sph}_{1, 37442} \simeq \text{Sph}_{193, 194}$, ...
 - Non-spin: $\text{Sph}_{2, 19012} \simeq \text{Sph}_{194, 196}$, $\text{Sph}_{2, 297220} \simeq \text{Sph}_{770, 772}$, ...