Ergodic properties of some canonical systems driven by thermostats

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Fields Institute

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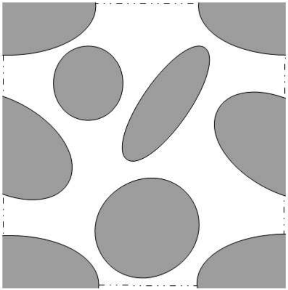
The System



[Lin and Young 2010]

- $\Gamma \subset \mathbb{T}^2$ - a bounded horizon billiard table with convex scatterers.
  - specular particle reflections $v'_\perp = -v_\perp; v'_t = v_t$.
- $D_1, \cdots, D_N$ convex scatterers acting as thermostats:
  - $\beta_1 = \frac{1}{T_1}, \cdots, \beta_N = \frac{1}{T_N}$
  - $v'_\perp = -v_\perp;$
  - v'_t is randomly drawn from $\sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t$
- particles do not interact with each other

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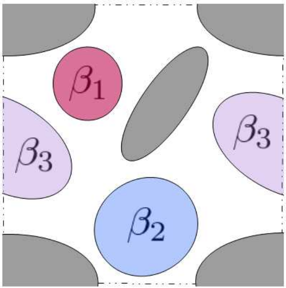

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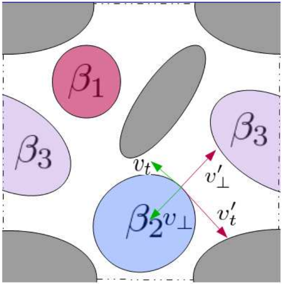

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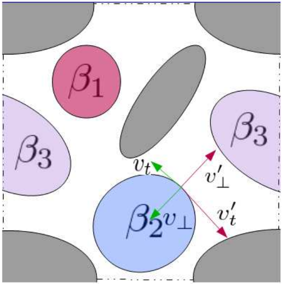

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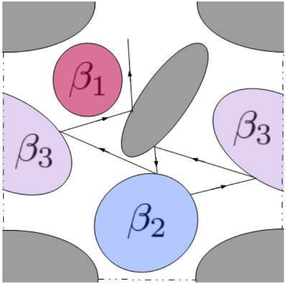

- No particle interactions $\Rightarrow$ can work with one particle
- The Phase Space $\Omega = \{(x, v) : x \in \Gamma, v \in \mathbb{R}^2\}/\sim$
- Markov Process Φ_τ
  - deterministic billiard flow between collisions with thermostats.
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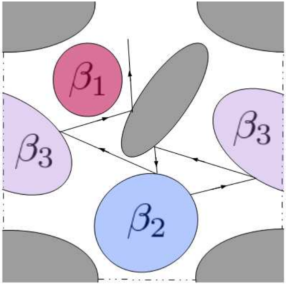

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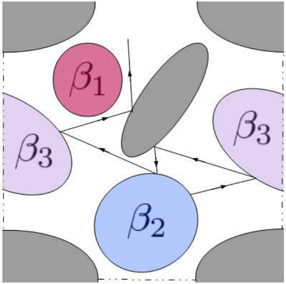

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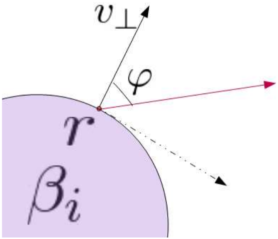

- Choose variables $r \in \partial\Gamma$, $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $v_\perp \in [0, \infty)$.
- Perturbation occurs only in φ variable with density $\rho_{v_\perp}(\varphi) = \sqrt{\frac{\beta}{\pi}} \frac{v_\perp}{\cos^2(\varphi)} e^{-\beta v_\perp^2 \tan^2(\varphi)} d\varphi$.
- Define the Markov Chain Φ on $X = \{(r, v_\perp) : r \in \partial\Gamma, v_\perp \in [0, \infty)\}$ by first drawing φ and then applying the billiard map.
- Denote the transition probability kernel of Φ by $\mathcal{P}$, i.e. $\mathcal{P}((r, v_\perp), A) = P(\Phi_n \in A | \Phi_{n-1} = (r, v_\perp))$

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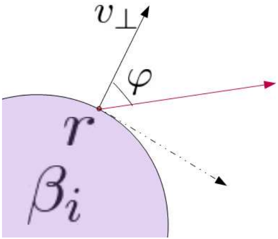

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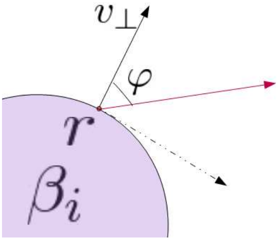

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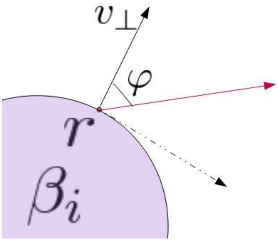

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Invariant Measures

- Existence, uniqueness, absolute continuity w.r.t. Lebesgue measure m, ergodicity, and mixing properties.
- Equilibrium measures: $\beta_1 = \cdots = \beta_N = \beta$
  - for Φ: $d\mu = 2\beta v_\perp e^{-\beta v_\perp^2} dv_\perp dr$
  - for Φ_τ $d\nu = \frac{1}{\pi|D|}\sqrt{\frac{\beta}{\pi}}\frac{v_\perp^2}{\cos^2(\varphi)} e^{-\beta v_\perp^2/\cos^2(\varphi)} dv_\perp dr d\varphi dl$

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There exists unique absolutely continuous (w.r.t. Leb.) and geometrically ergodic invariant measure for the Markov Chain Φ.

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Geometric Ergodicity Theorem

Lemma 1

Φ is irreducible.

Lemma 2

Φ is aperiodic.

Theorem (Meyn and Tweedie Thm 15.01)

If for $C = \{(r, v_\perp) : v_\perp^{\min} \leq v_\perp \leq v_\perp^{\max}\}$ there exist $\gamma < 1$, $b < \infty$, and a function $V \geq 1$ such that $C = \{(r, v_\perp) : V(r, v_\perp) \leq R\}$ and

$$\mathcal{P}V(r, v_\perp) = \int_{-\pi/2}^{\pi/2} f(r', v_\perp')\rho_{v_\perp}(\varphi)d\varphi \leq \gamma V(r, v_\perp) + b\mathbf{1}_C(r, v_\perp), \quad \forall (r, v_\perp) \in X$$

then $\exists!$ a.c. and geometrically ergodic invariant measure for Φ, s.t.

$$\sup_{|f| \leq V} |\mathcal{P}^n f - \nu(f)| \leq RV((r, v_\perp))\rho^n, \quad \rho < 1$$

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Lemma 1

Φ is irreducible in a sense that for any $A \subset X$ with $Leb(A) > 0$, $P_{(r,v_\perp)}(\tau_A < \infty) > 0 \; \forall (r, v_\perp) \in X$, where τ_A is the first return time to A.

- "Illumination Property." [Eckmann and Jaquet 2006]
- $r_1, r_2 \in$ same thermostat $\Rightarrow \exists$ path $r \rightsquigarrow r'$.
- There exist M and a path from any r to any r' in $\leq M$ steps.
- Can boost $v_\perp$ to the desired value $v'_\perp$.
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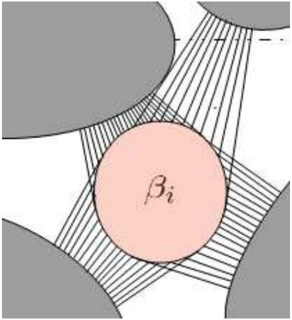

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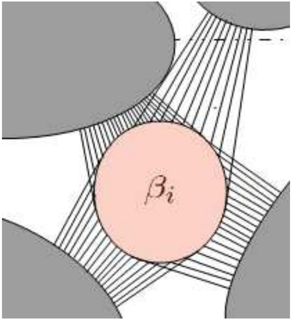

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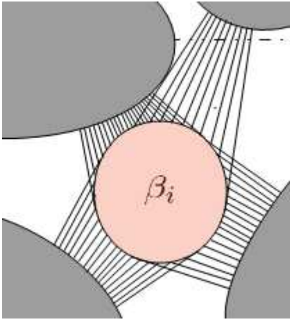

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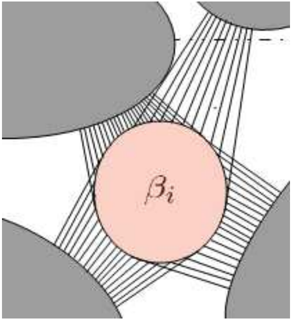

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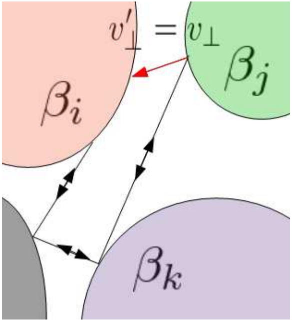

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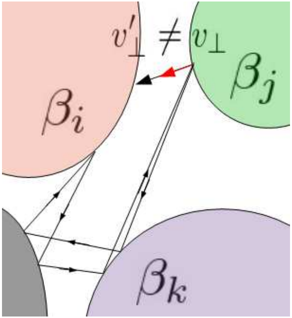

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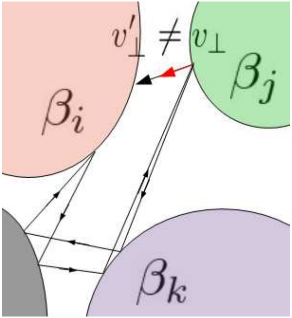

Aperiodicity

- Fix $C = \{(r, v_\perp) : v_\perp^{\min} < v_\perp < v_\perp^{\max}\}$ let m denote the normalized Lebesgue measure on C.

Lemma 2

Φ is aperiodic. That is
$\gcd\{n \geq 1 | \exists \eta : \mathcal{P}^n((r, v_\perp), \cdot) \geq \eta m \quad \forall (r, v_\perp) \in C\} = 1.$

- There exist N and a path from any $(r, v_\perp)$ to any $(r', v'_\perp)$ taking precisely N steps
  - such that all angles of incidence and reflection are bounded away from $\pm\frac{\pi}{2}$
- There exists N such that for any $(r, v_\perp) \in C$, $\mathcal{P}^N((r, v_\perp), C) \geq \eta m(C)$.
- From $(r, v_\perp) \in C$ can always jump to some $(r', v'_\perp) \in C$ and then apply the above: $\mathcal{P}^{N+1}((r, v_\perp), C) \geq \eta' m(C)$
- $\gcd(N, N+1) = 1$, so Φ is aperiodic.

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- From $(r, v_\perp) \in C$ can always jump to some $(r', v'_\perp) \in C$ and then apply the above:
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- $\gcd(N, N+1) = 1$, so Φ is aperiodic.

Aperiodicity

- Fix $C = \{(r, v_\perp) : v_\perp^{\text{min}} < v_\perp < v_\perp^{\text{max}}\}$
  let m denote the normalized Lebesgue measure on C.

Lemma 2

Φ is aperiodic. That is
$\gcd\{n \geq 1 | \exists \eta : \mathcal{P}^n((r, v_\perp), \cdot) \geq \eta m \quad \forall (r, v_\perp) \in C\} = 1.$

- There exist N and a path from any $(r, v_\perp)$ to any $(r', v'_\perp)$ taking precisely N steps
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Potential V: heuristics

Lemma

$$\mathcal{P}V(r,v_\perp) \leq \gamma V(r,v_\perp) + b\mathbf{1}_C(r,v_\perp), \quad \forall (r,v_\perp) \in X$$

- $\mathcal{P}V(r,v_\perp) = \int_{-\pi/2}^{\pi/2} V(r',v'_\perp)\sqrt{\frac{\beta_i}{\pi}}\frac{v_\perp}{\cos^2(\varphi)}e^{-\beta_i v_\perp^2 \tan^2(\varphi)}d\varphi$
- $v'_\perp = \frac{\cos(\varphi')}{\cos(\varphi)}v_\perp$
- $v_\perp$ very large: v_t is negligible. $\varphi \approx 0 \Rightarrow v'_\perp \approx \cos(\varphi')v_\perp \leq v_\perp$.
- $v_\perp$ very small: v_t is the main contribution. $\Rightarrow v'_\perp \approx \cos(\varphi')v_t$.
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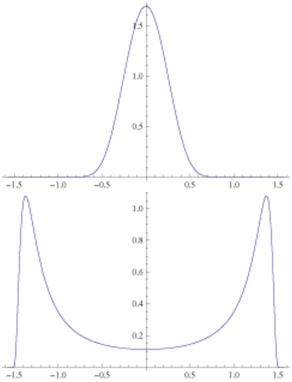

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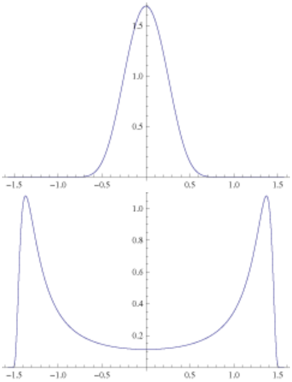

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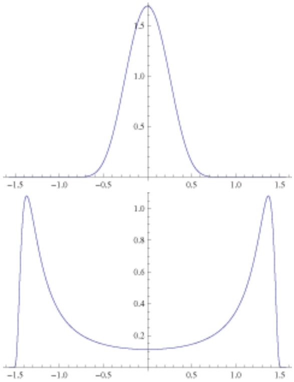

Potential V: try dependence on $v_\perp$ only

Lemma

$$\mathcal{P}V(v_\perp) \le \gamma V(v_\perp) + b\mathbf{1}_C(v_\perp), \quad \forall v_\perp \in [0,\infty)$$

- $V(v_\perp) \sim \begin{cases} v_\perp^a, & v_\perp > v_\perp^{\max}; \\ v_\perp^{-b}, & v_\perp < v_\perp^{\min}. \end{cases}$
- With such a potential do not get γ uniformly less than 1 (as $v_\perp \to \infty$)

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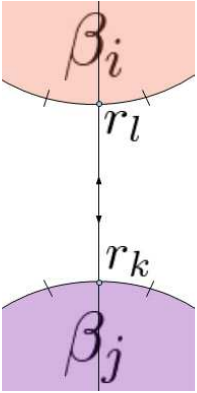

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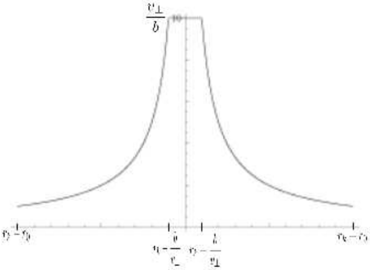
Lemma

$$\mathcal{P}V(r,v_\perp) \le \gamma V(r,v_\perp) + b\mathbf{1}_C(r,v_\perp), \quad \forall (r,v_\perp) \in X$$

- For small enough r_0, large enough b and $v_\perp^c \gg v_\perp^{\max}$, let $V(r,v_\perp) = \begin{cases} g_{v_\perp}(r)v_\perp^2, & v_\perp > v_\perp^{\max} \\ g_{v_\perp}(r)v_\perp^{-3/4}, & v_\perp < v^{\min}) \perp . \end{cases}$



$$g_{v_\perp}(r) = \begin{cases} \frac{v_\perp}{b}, & r \in (r_k - \frac{b}{v_\perp}, r_k + \frac{b}{v_\perp}); \\ \frac{1}{|x|}, & ; r \in (r_k - r_0, r_k + r_0) \setminus (r_k - \frac{b}{v_\perp}, r_k + \frac{b}{v_\perp}) \\ 1, & \textit{otherwise}. \end{cases}$$

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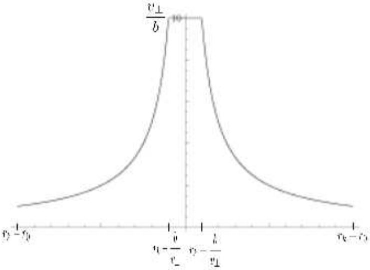

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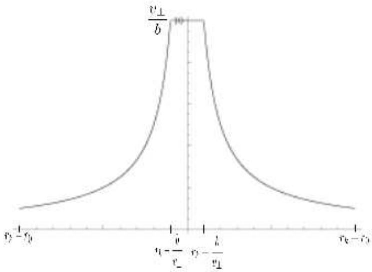

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