Fourier law and the weak coupling limit

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As already explained, the main technical difficulty in the derivation of the *Heat equation* or the *Fourier law* lies in establishing some form of *local equilibrium*

To simplify the problem one can study a weak coupling:

many independent systems with small interactions.

Consider a finite region $\Lambda \subset \mathbb{Z}^\nu$, a Hamiltonian $\mathcal{H}_x(q_x, p_x)$ on a manifold $\mathcal{M}_x$ at each site $x \in \Lambda$ and the total Hamiltonian

$$\mathcal{H}^\varepsilon(q,p) = \sum_x \mathcal{H}_x(q_x, p_x) + \varepsilon \sum_{|x-y|=1} V(q_x, q_y).$$

If the single site dynamics has good mixing properties and ε is small enough, then one expects to have the local systems always close to equilibrium.

Caveat: It cannot be always so simple.

If $\mathcal{H}_x \sim \varepsilon$, then the perturbation may be no longer small !

To simplify even further the problem one can try to make sense of the limit $\varepsilon \to 0$, and hope that too small energies will never occur.

More precisely. Let $E_x = \mathcal{H}_x$ be the energy at site x,

$$\frac{d}{dt} E_x = \varepsilon \sum_{|x-y|=1} \nabla V(q_x, q_y)(p_x + p_y) =: \varepsilon \sum_{|x-y|=1} j_{x,y}.$$

The microcanical measure is symmetric in p, hence in equilibrium $\mathbb{E}(j_{x,y}) = 0$.

The effective exchange of energy is due to fluctuations

It is then natural to consider the random variables $E_x^\varepsilon(t) = E_x(\varepsilon^{-2}t)$.

The randomness being in the initial condition on the q, p variables at fixed energies.

$$E_x^\varepsilon(t) = E_x^\varepsilon(0) + \varepsilon \int_0^{t\varepsilon^{-2}} \sum_{|x-y|=1} j_{xy}(s)ds.$$

Thus the limit $\varepsilon \to 0$ looks like some kind of CLT.

Work in Collaboration with S.Olla

Let $\mathcal{M}_x = \mathbb{R}^{2d}$ and $U, V \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ strictly convex

$$\mathcal{H}_x = \frac{p_x}{2} + U(q_x).$$

In this case the single site system has poor ergodic properties. To deal with this we add a noise on the velocities preserving the Kinetic energy and prove that the resulting system has good mixing properties.

Theorem 1 (Olla, L.) *The process* $\{E_x^\varepsilon\}$ *converges weakly to a limit* $\{\mathcal{E}_x\}$ *satisfying the mesoscopic SDE*

$$d\mathcal{E}_x = \sum_{|x-y|=1} \boldsymbol{a}(\mathcal{E}_x, \mathcal{E}_y)dt + \sum_{|x-y|=1} \boldsymbol{b}(\mathcal{E}_x, \mathcal{E}_y)dB_{x,y}$$

where $\boldsymbol{a}(\mathcal{E}_x, \mathcal{E}_y) = -\boldsymbol{a}(\mathcal{E}_y, \mathcal{E}_x)$, $\boldsymbol{b}(\mathcal{E}_x, \mathcal{E}_y) = \boldsymbol{b}(\mathcal{E}_y, \mathcal{E}_x)$ *and* $B_{x,y} = -B_{y,x}$ *are independent standard random walks.*
In addition, $\boldsymbol{a}, \boldsymbol{b}^2 \in \mathcal{C}^\infty((0,\infty)^2)$ *and*

$$\boldsymbol{b}^2(\mathcal{E}_x, \mathcal{E}_y) = A\mathcal{E}_x\mathcal{E}_y(1 + \mathcal{O}(\mathcal{E}_x + \mathcal{E}_y))$$
$$\boldsymbol{a}(\mathcal{E}_x, \mathcal{E}_y) = A(\mathcal{E}_x - \mathcal{E}_y)(1 + \mathcal{O}(\mathcal{E}_x + \mathcal{E}_y))$$

Work in Collaboration with D.Dolgopyat

Let $\mathcal{M}_x = T^*M$, M being a d-dimensional manifold with strictly negative curvature

$$\mathcal{H}_x = \frac{p_x^2}{2}.$$

The system is mixing (Dolgopyat (1998), Liverani (2004)).

But mixing is not so good: mixing rate goes to zero when energy goes to zero! (Harder to study)

Theorem 2 (Dolgopyat, L.) *For $d \geq 3$, $\{E_x^\varepsilon\}$ converges weakly to a limit $\{\mathcal{E}_x\}$ satisfying the mesoscopic SDE*

$$d\mathcal{E}_x = \sum_{|x-y|=1} \boldsymbol{a}(\mathcal{E}_x, \mathcal{E}_y)dt + \sum_{|x-y|=1} \boldsymbol{b}(\mathcal{E}_x, \mathcal{E}_y)dB_{x,y}$$

where $\boldsymbol{a}(\mathcal{E}_x, \mathcal{E}_y) = -\boldsymbol{a}(\mathcal{E}_y, \mathcal{E}_x)$, $\boldsymbol{b}(\mathcal{E}_x, \mathcal{E}_y) = \boldsymbol{b}(\mathcal{E}_y, \mathcal{E}_x)$ and $B_{x,y} = -B_{y,x}$ are independent standard random walks. In addition, $\boldsymbol{a}, \boldsymbol{b}^2 \in \mathcal{C}^\infty((0,\infty)^2)$ and, for $\mathcal{E}_x \leq \mathcal{E}_y$,

$$\boldsymbol{b}^2(\mathcal{E}_x, \mathcal{E}_y) = \frac{A\mathcal{E}_x}{\sqrt{2\mathcal{E}_y}} + \mathcal{O}\left(\mathcal{E}_x^{\frac{3}{2}}\mathcal{E}_y^{-1}\right)$$

$$\boldsymbol{a}(\mathcal{E}_x, \mathcal{E}_y) = \frac{A(d-2)}{2\sqrt{2\mathcal{E}_y}} + \mathcal{O}\left(\mathcal{E}_x^{\frac{1}{2}}\mathcal{E}_y^{-1}\right),$$

In both cases we can prove that zero is unreachable, i.e. if at time zero all the energies are strictly positive, then they will remain strictly positive for all times. This implies that the SDE have a unique solution (uniqueness of the Martingale problem).

Next step

Consider the above SDE in a region $\Lambda_L = [-L, L]^\nu \subset \mathbb{Z}^\nu$, and, for each $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^\nu, \mathbb{R})$, define the random variable

$$\tilde{\mathcal{E}}_L(t, \varphi) = L^{-\nu} \sum_{x \in \Lambda_L} \mathcal{E}_x(L^2 t) \varphi(L^{-1} x)$$

Prove that (Hydrodynamics limit) $\tilde{\mathcal{E}}_L(t, \varphi)$ converges weakly to $\int \rho(t, x) \varphi(x)$ where (Heat equation)

$$\partial \rho_t = \operatorname{div}(D \nabla \rho),$$

for some diffusion coefficient $D \in \mathcal{C}^1(\mathbb{R}^\nu, GL(\nu, \mathbb{R}))$.