Quenched results for random Lorentz Tubes

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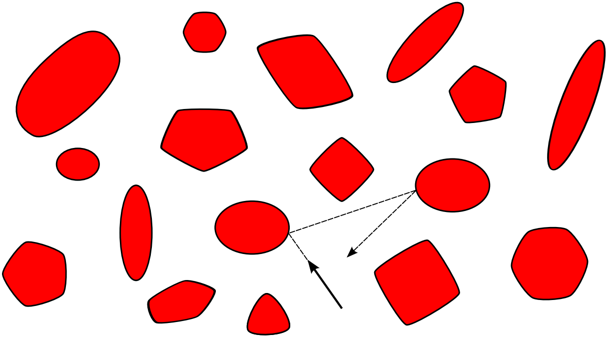
(joint papers with **G. Cristadoro, M. Seri, M. Degli Esposti, S. Troubetzkoy**)

Workshop on the Fourier Law and Related Topics
Fields Institute, Toronto, April 4-8, 2011

Thematic Program on Dynamics and Transport in Disordered Systems

Preliminaries

Lorentz Gas: d-dimensional billiard system in the complement of a (possibly aperiodic) array of semi-dispersing scatterers.



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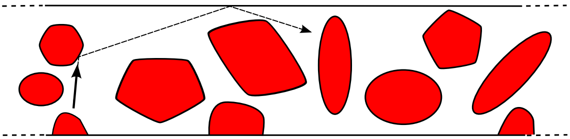

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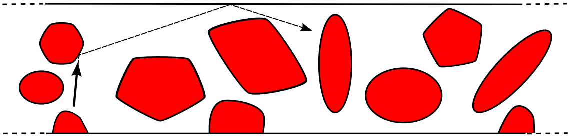

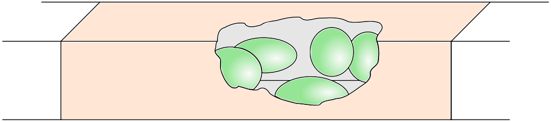

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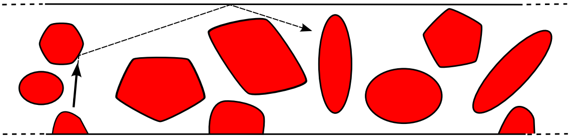

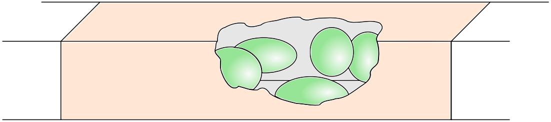

Specialize (for the moment) to $d = 2$.

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- want to prove typicality of above properties in a large class of LTs.

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Question: Define "most".

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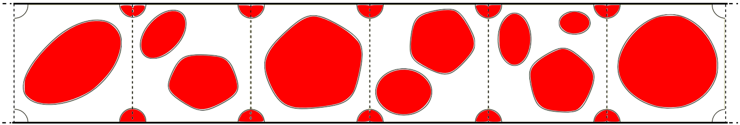

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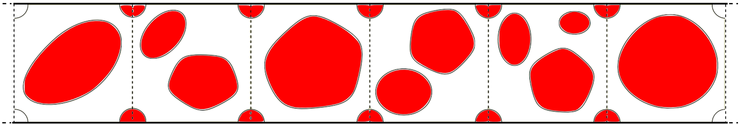

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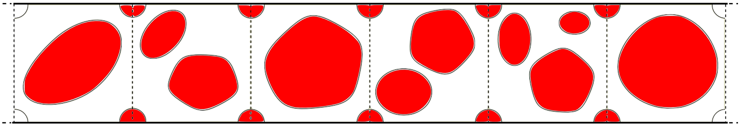

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E.g.: i.i.d. local configurations, $\Pi = \pi^{\otimes\mathbb{Z}}$ (π probability on Ω)

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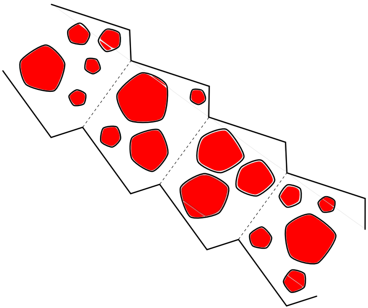

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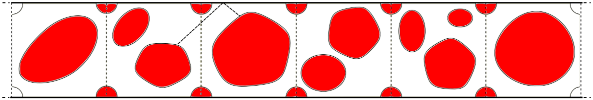

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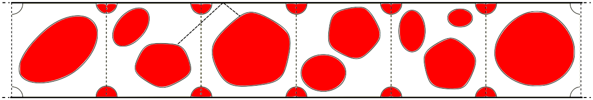

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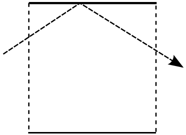
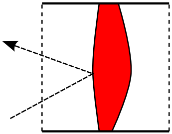
Want to avoid:





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(Infinite — though locally finite — horizon can be worked out *(L, Troubetzkoy 2011)* but requires stronger assumptions.)

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6. the first-return map to any boundary component is K-mixing (hence mixing).

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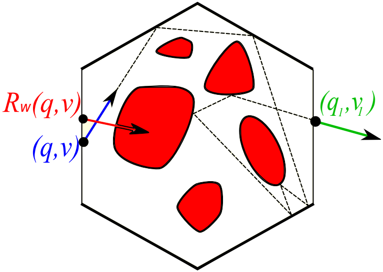
  (right exit: $\epsilon = +1$; left exit: $\epsilon = -1$)



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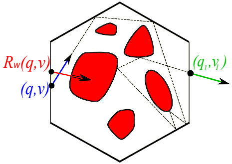

- $F(x, \ell) := (R_{\ell_0}(x)\, ,\, \sigma^{\epsilon(x,\ell_0)}(\ell)) \in \mathcal{N} \times \Omega^{\mathbb{Z}}$
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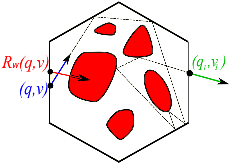

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$F : \mathcal{N} \times \Omega^{\mathbb{Z}} \longrightarrow \mathcal{N} \times \Omega^{\mathbb{Z}}$ preserves probability measure $\mu_0 \times \Pi$
($\mu_0 =$ Liouville measure on $\mathcal{N}$; $\Pi =$ random law for global conf'n)

$(\mathcal{N} \times \Omega^{\mathbb{Z}}, F, \mu_0 \times \Pi)$ called point of view of the particle (PVP)

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Discrete itinerary: $S_n(x, \ell) = \sum_{k=0}^{n-1} \epsilon \circ F^k(x, \ell) \quad (S_0(x, \ell) \equiv 0)$

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Classical Theorem (e.g., Atkinson 1976)

If (Σ, F, λ) is ergodic and $\epsilon : \Sigma \longrightarrow \mathbb{R}$ is integrable, the (1D) cocycle (S_n) of ϵ is recurrent, i.e.,

$$\liminf_{n\to\infty} |S_n| = 0, \quad \lambda\text{-almost everywhere}$$

if and only if $\int_\Sigma \epsilon \, d\lambda = 0$.

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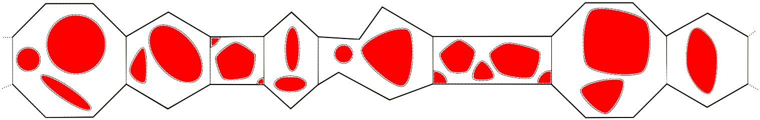
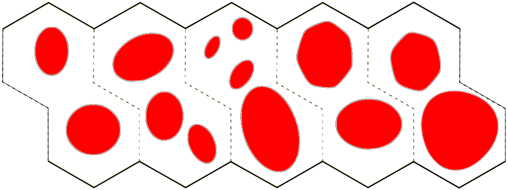
Shape of cell can be random too:



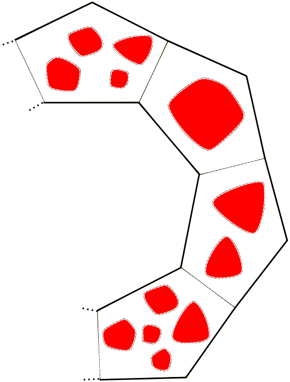
Extensions II

Gates can comprise more than one side:



Extensions III

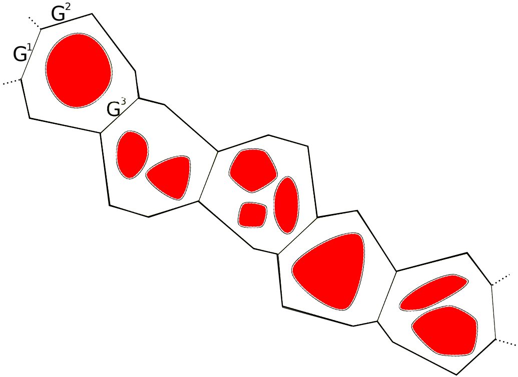
More general isometries can be used in lieu of translations:



In 2D, use Riemann sheets to avoid self-intersections

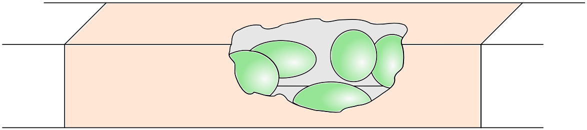
Extensions IV

Choice of gate can be random too:



Higher dimension

Most important extension: Dimension three and up



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- Every orbit has a non-grazing collision every so often (uniform hyperbolicity more delicate in $d \geq 3$, cf. astigmatism, etc.).