

Boltzmann-
Gibbs
principle

Milton Jara

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 $d = 1$

The exclusion
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Spectral gap
inequality

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Density
fluctuations
and fractional
SBE

Second-order Boltzmann-Gibbs principle and applications

Milton Jara and Patrícia Gonçalves

IMPA, Rio de Janeiro and U. do Minho, Braga

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- **Fermi-Pasta-Ulam:** anomalous heat conduction for anharmonic oscillators in $d = 1$
- Basile-Bernardin-Olla: rigorous derivation of anomalous heat conduction for **stochastic** oscillators in $d = 1$
- Conservation of energy and momentum are important
- Basile-Olla-Spohn, J.-Komorowski-Olla: in the kinetic limit of BBO model, it appears the fractional heat equation

$$\partial_t u = -(-\Delta)^{3/4} u$$

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Density fluctuations and fractional SBE

- Energy fluctuations are governed by the **fractional stochastic Burgers equation**

$$\partial_t u = -c(-\Delta)^{3/4} u + a \nabla u^2 + \nabla \mathcal{W} \quad (\text{SFBE})$$

- White noise is formally invariant \rightarrow (SFBE) is **VERY ill-posed** (compare with KPZ equation)
- The stationary solution of (SFBE) (if exists and unique) satisfies

$$\lambda^{-1/2} u(\lambda^{3/2} t, \lambda x) \sim u(t, x)$$

- Self-similarity $\xrightarrow{?}$ scaling limit of a universality class

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- $p : \mathbb{Z} \rightarrow [0, +\infty)$ transition rate \rightarrow irreducible
- $\Omega = \{0, 1\}^{\mathbb{Z}}$ state space
- $\{\eta_t; t \geq 0\}$ exclusion process of jump rate $p(\cdot)$
- $\rho \in [0, 1]$; ν_ρ product measure in Ω of density ρ
- ν_ρ invariant, ergodic under η_t
- $\eta_0 \sim \nu_{1/2}$

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- Generator of η_t :

$$Lf(\eta) = \sum_{x,y \in \mathbb{Z}} p(y-x)\eta(x)\{1-\eta(y)\}\nabla_{xy}f(\eta)$$

- $\Lambda_\ell = \{-\ell, \dots, \ell\}$ box of radius ℓ
- The process η_t has a spectral gap of order $\ell^{-\alpha}$ if there is $K \in (0, +\infty)$ s.t.

$$\int f^2 d\nu_{1/2} \leq K\ell^\alpha \sum_{x,y \in \Lambda_\ell} p(y-x) \int (\nabla_{xy}f)^2 d\nu_{1/2}$$

for any $\ell \geq \ell_0$ and any f s.t. $\int f d\nu_\rho = 0$ for all $\rho \in [0, 1]$

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Let $\alpha \in (0, 2]$. We say that $p(\cdot)$ belongs to the domain of normal attraction of an α -stable law if:

i) for $\alpha = 2$,

$$\sum_{x \in \mathbb{Z}} p(x)x^2 < +\infty$$

ii) for $\alpha \in (0, 2)$,

$$\lim_{x \rightarrow +\infty} x^\alpha \sum_{\pm y \geq x} p(y) = c^\pm$$

and $c^+ + c^- \in (0, \infty)$.

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Theorem (Gonçalves, J.)

Let $\alpha \in (0, 2]$. Let $p(\cdot)$ be an irreducible transition rate on the domain of attraction of an α -stable law. Then the exclusion process η_t has a spectral gap of order $\ell^{-\alpha}$.

- Case $\alpha = 2$: Quastel and Diaconis, Saloff-Coste
- Building block for the second-order Boltzmann-Gibbs principle

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- $f : \Omega \rightarrow \mathbb{R}$ local, $\text{supp}(f) \subseteq \Lambda_\ell$

- τ_x : standard shift in Ω

$$\varphi_f(\rho) := \int f d\nu_\rho$$

$$\eta^\ell(x) := \frac{1}{2\ell+1} \sum_{|y| \leq \ell} \eta(x+y)$$

$$\mathcal{Q}_\ell(\eta) := (\eta^\ell(0) - \tfrac{1}{2})^2 - \frac{1}{4\ell}$$

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Theorem (Local Boltzmann-Gibbs principle)

Let $\alpha \in (1, 2]$. Let $p(\cdot)$ be an irreducible transition rate on the domain of attraction of an α -stable law. Let $f : \Omega \rightarrow \mathbb{R}$ be a local function s.t. $\varphi_f(\frac{1}{2}) = 0$. There exists $c = c(f, K)$ s.t.

$$\mathbb{E} \left[\left(\int_0^t \{f(\eta_s) - \varphi'_f(\frac{1}{2})(\eta_s^\ell(0) - \frac{1}{2})\} ds \right)^2 \right] \leq ct\ell^{\alpha-1}$$

for any $t \geq 0$ and any $\ell \in \mathbb{N}$.

- For $\alpha < 1$, any local function satisfies a CLT

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Theorem (Second-order Boltzmann-Gibbs principle)

Let $\alpha \in (1, 2]$. Let $p(\cdot)$ be an irreducible transition rate on the domain of attraction of an α -stable law. Let $f : \Omega \rightarrow \mathbb{R}$ be a local function s.t. $\varphi_f(\frac{1}{2}) = \varphi'_f(\frac{1}{2}) = 0$. There exists a constant $c := c(f, \rho)$ such that

$$\mathbb{E} \left[\left(\int_0^t \sum_{x \in \mathbb{Z}} \tau_x \{ f(\eta_s) - \varphi''_f(\frac{1}{2}) \mathcal{Q}_\ell(\eta_s) \} h_x ds \right)^2 \right] \leq c t \ell^{\alpha-1} \sum_{x \in \mathbb{Z}} h_x^2$$

for any $t \geq 0$, any $\ell \in \mathbb{N}$ and any $h \in \ell^2(\mathbb{Z})$.

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- $p(z) = p^s(z) + p^a(z)$ for every $z \in \mathbb{Z}$
- $p^s(\cdot)$: irreducible, symmetric, on the domain of attraction of an $3/2$ -stable law
- $p^a(\cdot)$: finite range, $\sum_z z p^a(z) = a \neq 0$

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- $\mathcal{S}(\mathbb{R})$: Schwartz space of test functions
- $\mathcal{S}'(\mathbb{R})$: space of tempered distributions
- $n \in \mathbb{N}$: scaling parameter
- $\mathcal{Y}_t^n(\cdot)$: density fluctuation field

$$\mathcal{Y}_t^n(F) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} (\eta_{tn^{3/2}}(x) - \frac{1}{2}) F\left(\frac{x}{n}\right)$$

for $F \in \mathcal{S}(\mathbb{R})$

Density fluctuations and fractional SBE

Boltzmann-Gibbs principle

Milton Jara

Outline

Anomalous heat conduction in $d = 1$

The exclusion process

Spectral gap inequality

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Theorem

The process $\{\mathcal{Y}_t^n; t \in [0, T]\}$ is tight with respect to the J_1 -Skorohod topology in $\mathcal{D}([0, T]; \mathcal{S}'(\mathbb{R}))$. Any limit point of $\{\mathcal{Y}_t^n; t \in [0, T]\}$ is a stationary energy solution of the stochastic fractional Burgers equation

$$\partial \mathcal{Y} = -c(-\Delta)^{3/4} \mathcal{Y} + a \nabla \mathcal{Y}^2 + \sqrt{\frac{c}{2}} \nabla \mathcal{W}.$$

Energy solutions

Boltzmann-
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- Approximating the quadratic field: $\{\iota_\varepsilon; \varepsilon \in (0, 1)\}$ approximation of the identity

$$\mathcal{A}_{s,t}^\varepsilon(F) = \int_s^t \int_{\mathbb{R}} \langle (\mathcal{Y}_s * \iota_\varepsilon)^2, \nabla F_{t'} \rangle dt'$$

- Energy condition: There exists constant κ s.t.

$$E[(\mathcal{A}_{s,t}^\varepsilon(F) - \mathcal{A}_{s,t}^{\varepsilon'}(F))^2] \leq \kappa \max\{\varepsilon, \varepsilon'\} \int_s^t \|F_{t'}\|^2 dt'$$

for any $0 \leq s < t \leq T$, any $0 < \varepsilon, \varepsilon' < 1$ and any $F : [0, T] \rightarrow \mathcal{S}(\mathbb{R})$.

- Energy condition + “loose a priori bounds” \Rightarrow quadratic field $\mathcal{A}_{s,t}(\cdot)$ well defined

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$\{\mathcal{Y}_t; t \in [0, T]\}$ is a stationary energy solution if:

- i) $\{\mathcal{Y}_t; t \in [0, T]\}$ satisfies the energy condition
- ii) The process

$$\begin{aligned}\mathcal{M}_t(F) = & \mathcal{Y}_t(F_t) - \mathcal{Y}_0(F_0) - \int_0^t \mathcal{Y}_s((\partial_s - c(-\Delta)^{3/4})F_s)ds \\ & - \mathcal{A}_{0,t}(F)\end{aligned}$$

is a martingale of variance $\int_0^t \frac{1}{4} \|\nabla F_s\|^2 ds$.

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