Heat Conduction through rare collisions

François Huveneers

joint work with Stefano Olla

Université de Paris Dauphine, CEREMADE

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Deriving Fourier's law in two steps (I)

Try to obtain Fourier's law

$$\partial_t E \;=\; \partial_x\big(\kappa \,.\, \partial_x E\big)$$

from a Hamiltonian system by taking two successive limits.

Here

- $E \equiv E(x,t)$ is the local energy (in a macroscopic description),
- $\kappa \equiv \kappa(E)$ is the thermal conductivity of the material.

Only a purely 1-D dynamics will be considered in this talk

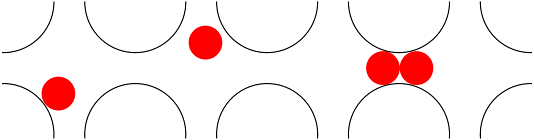
Deriving Fourier's law in two steps (II)

1. Microscopic hamiltonian system

Hamiltonian systems of N identical particles such that

- the interaction between particles is controlled by a small parameter $\epsilon > 0$,
- when the interaction is turned off ($\epsilon = 0$), the dynamics of single particles have good mixing properties.

Typical example:



Deriving Fourier's law in two steps (III)

2. Weak coupling limit: send $\epsilon \to 0$ and rescale time

Take the energy of atom k:

$$e_k(\mathbf{q},\mathbf{p}) \quad (= p_k^2/2 \text{ in our model}).$$

Find a non trivial $\alpha > 0$ such that

$$e_k\big(\,\mathbf{q}_\epsilon(\epsilon^{-\alpha}t)\,,\,\mathbf{p}_\epsilon(\epsilon^{-\alpha}t)\,\big) \xrightarrow{\mathrm{D}} \mathcal{E}_k(t) \quad \text{as} \quad \epsilon \to 0$$

where

- $(\mathcal{E}_k)_{1\leq k\leq N}$ is an autonomous Markov process:

$$\mathrm{d}\mathcal{E}_k \;=\; f(\mathcal{E}_1,\ldots,\mathcal{E}_N),$$

- The convergence $\xrightarrow{\mathrm{D}}$ is in distribution w.r.t. initial measure.

Remark: the number N of particles is constant: no space rescaling

Deriving Fourier's law in two steps (IV)

3. Diffusive limit: rescale space and time

Start from the stochastic process $(\mathcal{E}_k)_{1\leq k\leq N}$ and define

$$E_N(x,t) \;=\; \mathcal{E}_{x\,.\,N}\,(t\,.\,N^2)$$

Show, in some specific sense, that $E_N \to E$ as $N \to \infty$, with

$$\partial_t E \;=\; \partial_x\big(\kappa\,.\,\partial_x E\big).$$

4. Our contribution

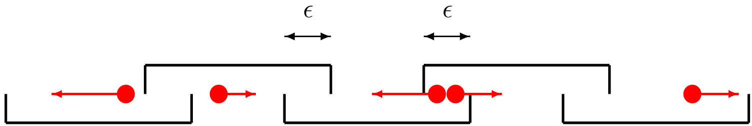
We accomplish rigorously this program starting from a 1-D almost hamiltonian microscopic system.

Main source of inspiration:

- Gaspard and Gilbert (2008)
- Liverani and Olla (2010)

The model (I)

1-D particles moving in their cells:



- One particle in each cell cell.
- Overlap region of size ϵ between near cells.
- Exchange of momenta through elastic collision in this region.

When $\epsilon \to 0$,

collisions should become rare,
but the interaction is still strong.

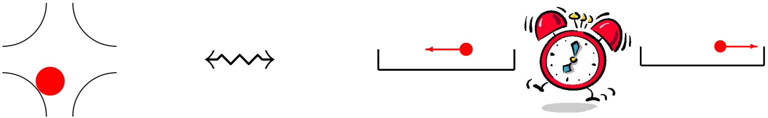
The model (II)

1. Dynamics of one isolated particle ($\epsilon = 0$)

Hamiltonian dynamics + Stochastic noise

The particle

- moves freely in its cell,
  and is reflected elastically at the boundaries of the cell.
- is equipped with an independent Poisson clock,
  which flips the sign of its velocity when it rings.



So the noise

- mimics a chaotic dynamics inside each cell,
- acts on every particle individually,
- preserves the energy of each particle.

The model (III)

2. Adding the interaction ($\epsilon > 0$)

- The energy of particles only contains a kinetic term:

$$e_k = p_k^2/2.$$

- 1-D elastic collisions between equal masses:

exchange of momenta and thus exchange of energy

- So, if the initial energies are

$$0 < \mathrm{e}_1 < \cdots < \mathrm{e}_{N'} < +\infty \qquad (N' \leq N),$$

then they are forever the only reachable energies.

Weak coupling limit (I)

1. Definitions and Assumptions

Define

- The energy at rescaled time

$$\mathcal{E}_k^\epsilon(t) := e_k^\epsilon(\epsilon^{-1}t) = \frac{1}{2}\left(p_k^\epsilon\right)^2(\epsilon^{-1}t).$$

- For $a, b \in \mathbb{R}_+$, let

$$\gamma(a,b) := \frac{1}{2}\max\left\{\sqrt{2a}, \sqrt{2b}\right\}.$$

Assume

- The initial measure has some regularity.
- Energies are such that $0 < e_1 < \cdots < e_{N'} < +\infty$.
- Energies lie in some subset of $\mathbb{R}_+^{N'}$ of full Lebesgue measure.

Weak coupling limit (II)

2. Theorem (S. Olla, F.H.)

Under these assumptions,

- There exists a cad-lag process

$$\mathcal{E} = (\mathcal{E}_k)_{1\leq k\leq N} \ \text{ with values in } \ \{\mathsf{e}_1, \dots, \mathsf{e}_{N'}\}^N$$

  such that

$$\mathcal{E}^\epsilon = (\mathcal{E}^\epsilon_k)_{1\leq k\leq N} \xrightarrow{\mathrm{D}} \mathcal{E} = (\mathcal{E}_k)_{1\leq k\leq N} \quad \text{as} \quad \epsilon \to 0.$$

- The probability distribution $\overline{\mathsf{P}}_t(\,\cdot\,)$ of $\mathcal{E}(t)$ solves the equation

$$\partial_t \overline{\mathsf{P}}_t(e_1, \dots, e_N) = \sum_{k=1}^{N-1} \gamma(e_k, e_{k+1}) \Big(\overline{\mathsf{P}}_t(\dots, e_{k+1}, e_k, \dots) - \overline{\mathsf{P}}_t(\dots, e_k, e_{k+1}, \dots) \Big)$$

Hydrodynamic limit from the weak coupling limit

The following cases at least can be obtained

- Only two different energies: $\mathrm{e}_1 < \mathrm{e}_2$. Then

$$\partial_t \overline{\mathrm{P}}_t(\,\cdot\,) \;=\; \overline{\gamma}\,.\,\sum_{k=1}^{N-1} \Big(\overline{\mathrm{P}}_t(\dots, e_{k+1}, e_k, \dots) - \overline{\mathrm{P}}_t(\dots, e_k, e_{k+1}, \dots) \Big)$$

  with $\overline{\gamma} := \sqrt{2\mathrm{e}_2}$. Hydrodynamic limit for the SSEP:

$$\partial_t E(x,t) \;=\; \overline{\gamma}\,.\,\partial_x^2 E(x,t).$$

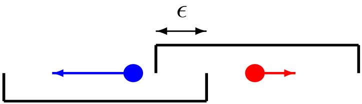
- N' independent of N different energies: $\mathrm{e}_1 < \cdots < \mathrm{e}_{N'}$.
  Heat equation with non-constant coefficient:

$$\partial_t E(x,t) \;=\; \partial_x \big(\kappa\,.\,\partial_x E(x,t)\big).$$

Heuristic of the theorem (I)

The intuition of the theorem follows Gaspard and Gilbert.

Let us take only two particles. Initially



Particle 1 has energy e_1, Particle 2 has energy e_2.

When $\epsilon \to 0$,

$$\begin{gathered}\text{collisions rarefy}\\ \Downarrow \\ \text{particles evolve almost independently}\\ \Downarrow \\ \text{they reach quickly their own equilibrium.}\end{gathered}$$

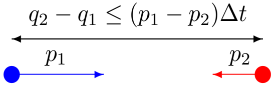
For $e = e_1$ or $e = e_2$, this measure is given by

$$\mu_e(u) = \frac{1}{2} \sum_{p=\pm\sqrt{2e}} \int_0^1 u(q,p)\,\mathrm{d}q$$

Heuristic of the theorem (II)

For $q_1 < q_2$, a collision occurs in a small time interval $\Delta t > 0$ if

$p_1 > p_2$ and



The energy exchange rate should be given by

$$\begin{aligned}
\tilde{\gamma} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \, \mathrm{P}\left(\text{a collision occurs during the time interval } \Delta t\right) \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{1}{4} \sum_{\substack{p_1 = \pm\sqrt{2e_1} \\ p_2 = \pm\sqrt{2e_2}}} \int \chi_{\mathbb{R}_+}(p_1 - p_2) \cdot \chi_{[0,(p_1-p_2)\Delta t]}(q_2 - q_1) \, \mathrm{d}q_1 \mathrm{d}q_2 \\
&= \epsilon \, . \, \frac{1}{2} \max\left\{\sqrt{2e_1}, \sqrt{2e_2}\right\}
\end{aligned}$$

Thus

- The non trivial time rescaling should be by a factor ϵ^{-1}.
- The asymptotic exchange rate should be the γ in the theorem.

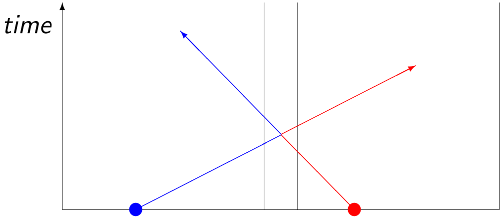
Comments on the hypotheses (I)

Deterministic trajectory matters on microscopic time scales since:

- it persists during time $\Delta t \sim 1$ with probability $\mathrm{e}^{-\lambda \Delta t} \sim 1$ (λ the parameter of the Poisson clocks),
- the interaction is strong.

1. Regularity of the initial measure

A Dirac distribution w.r.t. positions could produce a discontinuity at $t = 0$ (in the limit $\epsilon \to 0$ in the rescaled time)

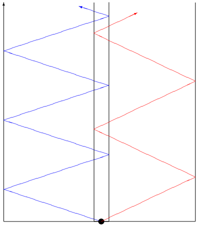


These particles collide with probability ~ 1, even when $\epsilon \to 0$.

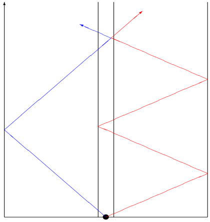
Comments on the hypotheses (II)

2. Condition on the momenta

After a collision, for the deterministic trajectory:



$|p_2| = \alpha|p_1|$: "typical"



$|p_2| = 2|p_1|$: "exceptional"

Correlations for (microscopic) times $\Delta t \sim 1$ are different:

- Typical case: $\mathrm{P}\big(\text{recollision in } \Delta t \sim 1\big) \sim \epsilon$ (noise)
- Atypical case: $\mathrm{P}\big(\text{recollision in } \Delta t \sim 1\big) \sim 1$ (deterministic)

Our theorem is shown for the typical case.

A similar theorem could hold in the atypical case, with another γ.

About the proof (I)

1. Basic ideas

As in Olla and Liverani:
coupled dynamics is seen as perturbation of the uncoupled one

In practice here:

- look at the dynamics during a time $\epsilon^{-1}\tau$,
- expand it in the number of collision per particle,
- the term of order 0 corresponds to the uncoupled dynamics,
- evaluate the 1st order term in the limit $\epsilon \to 0$,
- show that the 2d order is $\mathcal{O}(\tau^2)$.

Let us look at this in some more details...

About the proof (II)

2. Formula to be established

Let us take only two particles. Take μ an initial measure s.t.

- μ gives energy e_1 to particle 1,
- μ gives energy e_2 to particle 2,
- μ is uniform in position and sign of the velocity.

Let also

- $\mathcal{P}_\epsilon^t \mu$: evolution of μ by the coupled dynamics,
- $\mathcal{P}_0^t \mu$: evolution of μ by the uncoupled dynamics.

For $\tau \to 0$ and $\epsilon << \tau$, we wish to get

$$\mathcal{P}_\epsilon^{\epsilon^{-1}\tau}\mu - \mathcal{P}_0^{\epsilon^{-1}\tau}\mu \;=\; \tau\,.\,\gamma(e_1,e_2)\,.\,\big(\mu(e_2,e_1) - \mu(e_1,e_2)\big) \;+\; \mathcal{O}(\tau^2)$$

Roughly speaking, this implies our theorem.

About the proof (III)

3. Duhamel expansion

Write

$$\mathcal{P}_\epsilon^t \mu = \mathrm{e}^{L_\epsilon t}\mu \quad \text{and} \quad \mathcal{P}_0^t \mu = \mathrm{e}^{L_0 t}\mu.$$

Actually, μ is such that $\mathcal{P}_0^t\mu = \mu$, and so

$$\begin{aligned}
\mathrm{e}^{L_\epsilon \epsilon^{-1}\tau}\mu - \mathrm{e}^{L_0\epsilon^{-1}\tau}\mu &= \int_0^{\epsilon^{-1}\tau} \mathrm{e}^{sL_\epsilon}(L_\epsilon - L_0)\mu \,\mathrm{d}s \\
&= \int_0^{\epsilon^{-1}\tau} \mathrm{e}^{sL_0}(L_\epsilon - L_0)\mu \,\mathrm{d}s \\
&\quad + \int_0^{\epsilon^{-1}\tau} \mathrm{d}s \int_0^s \mathrm{e}^{(s-r)L_\epsilon}(L_\epsilon - L_0)\mathrm{e}^{rL_0}(L_\epsilon - L_0)\mu \,\mathrm{d}r
\end{aligned}$$

- the first term in the RHS can be evaluated "explicitly",
- the rest term is shown to be $\mathcal{O}(\tau^2)$,
- $L_\epsilon - L_0$ formal but can be made rigorous (time discretization).

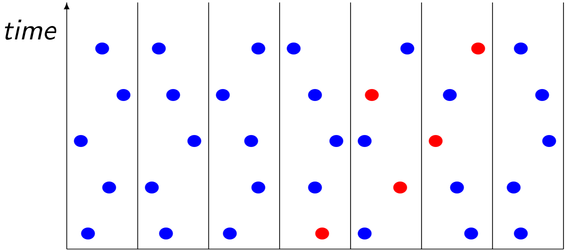
Beyond the two step strategy ?

Now, fix $\epsilon > 0$ and let $N \to \infty$.

Can we say something when taking a diffusive limit directly from our microscopic model ?

Very few yet, except in one (anecdotic) case:

Only one fast particle in a sea of slow particles



Due to chocks, the high energy describes some random walk.

Beyond the two step strategy ?

This walk can be understood:

- Take the "point of view of the particle":
  shift the dynamics s.t. the fast particle stays in the origin cell.
- Let X_s be the shifted dynamics.
- For X_s, the energy in each cell is now fixed.
- So, X_s can be seen to have space-time mixing properties independent of the system size.

So, the question reduces to the analysis of

$$\frac{1}{\sqrt{t}} \int_0^t f \circ X_s \, \mathrm{d}s$$

with

$$f \;=\; |p_1 - p_0| \,.\, \delta_0(q_1 - q_0) \;-\; |p_0 - p_{-1}| \,.\, \delta_0(q_0 - q_{-1})$$

with X_s having a spectral gap of order 1.