L^2–Mixing properties of a heat conduction model

Alexander Grigo[1], Konstantin Khanin[1], Domokos Szasz[2]

[1]University of Toronto, [2]Budapest University of Technology

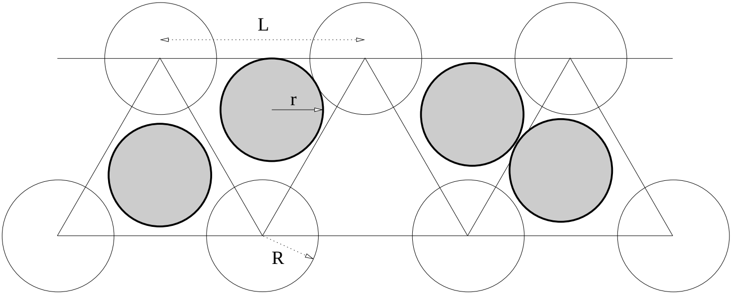
Workshop on the Fourier Law and Related Topics
Fields Institute, Toronto, Canada
April 8, 2011

Outline of the talk

- Description of the model; motivation
- Step one: Detailed analysis of a special case
- Step two: Spectral gap for the general case
- Example
- Conclusion

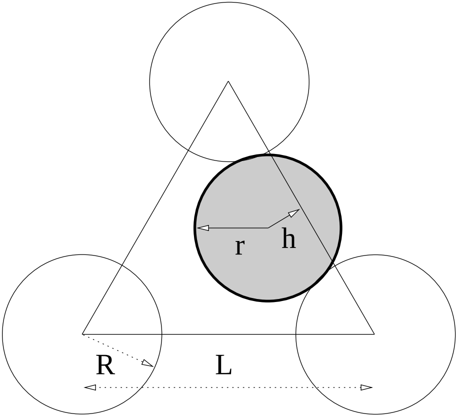
A concrete mechanical model

- Bunimovich, Liverani, Pellegrinotti, Suhov proved ergodicity for arbitrary number of (strongly) confined particles
- Gaspard, Gilbert gave a derivation of a hydrodynamic description in a certain limiting regime.



A concrete mechanical model: limiting system

- in strongly mixing systems return time statics are asymptotically exponentially distributed



A concrete mechanical model: limiting system

Two-step program to obtain a hydrodynamic description:

1. Rare interaction limit to obtain a master equation for a jump process
2. Hydrodynamic limit for the stochastic process corresponding to the master equation

A concrete mechanical model: limiting system

- State space: only the energies are left in the limit

$$X = (X_1, \dots, X_N) \in \mathbb{R}_+^N$$

- The limiting process has generator

$$\mathcal{L}A(X) = \sum_{i=1}^{N-1} \Lambda(X_i, X_{i+1}) \int_0^1 P\left(\frac{X_i}{X_i + X_{i+1}}, d\alpha\right) [A(T_{i,\alpha}X) - A(X)]$$

where

$$\Lambda(X_i, X_{i+1}) = \Lambda_+(X_i + X_{i+1})\, \Lambda_R\left(\frac{X_i}{X_i + X_{i+1}}\right)$$
$$\Lambda_+(s) = \sqrt{s}$$

- The total energy $X_1 + \ldots + X_N$ is preserved

A concrete mechanical model: limiting system

With $\alpha = \frac{X_i}{X_i + X_{i+1}}$

$$\frac{P(\beta, d\alpha)}{d\alpha} = \frac{3}{2} \frac{1 \wedge \sqrt{\frac{\alpha \wedge (1-\alpha)}{\beta \wedge (1-\beta)}}}{\frac{1}{2} + \beta \vee (1-\beta)}$$

$$\Lambda_R(\beta) = \frac{\sqrt{2\pi}}{6} \frac{\frac{1}{2} + \beta \vee (1-\beta)}{\sqrt{\beta \vee (1-\beta)}}$$

$$T_{i,\alpha} = \left(\begin{array}{c|cc|c} 1 & & & \\ \hline & \alpha & \alpha & \\ & 1-\alpha & 1-\alpha & \\ \hline & & & 1 \end{array}\right) \in \mathbb{R}^{N \times N}$$

A concrete mechanical model: limiting system

- In the limit as $N \to \infty$ and $\xi = i/N$, $t = N^2 \tau$ the empirical process

$$\sum_{i=1}^{N} \frac{1}{N} \delta_{X_i(t)}$$

should converge to a process with density $u(\xi, \tau)$ solving

$$\partial_\tau u(\xi, \tau) = \partial_\xi (\text{const} \sqrt{u(\xi, \tau)} \, \partial_\xi u(\xi, \tau))$$

- This was studied by Gaspard, Gilbert

A concrete mechanical model: limiting system

Mathematical problems:

- Non-gradient structure
- The rates are not bounded away from zero

Questions that will be addressed in this talk:

- Existence of stationary distributions
- Rates of convergence as function of system size

Special case

Consider $\Lambda = 1$ and some $P = P(d\alpha)$

$$\mathcal{L}A(x) = \sum_{i=1}^{N-1} \int_0^1 P(d\alpha)\,[A(T_{i,\alpha}x) - A(x)]$$

Assumption and notation

- We assume that $\int P(d\alpha)\,\alpha = \frac{1}{2}$.
- The simplexes $\mathcal{S}_{\epsilon,N} = \{x : \sum_{i=1}^{N} \frac{1}{N} x_i = \epsilon\}$ are invariant.

Remark

Proving a spectral gap even for this special case is non-trivial as is well known from the Kac model (1956). (McKean 1966, Diaconis and Saloff-Coste 2000, Janvresse 2001, Carlen, Carvalho and Loss 2000-2008)

Lower bound

For any initial $\mathrm{X}(0) \in \mathcal{S}_{\epsilon,N}$

$$\| \mathbb{E}\, \mathrm{X}(t) - x_\epsilon^\infty \| \leq e^{-t\,\Delta_N} \, \| \mathrm{X}(0) - x_\epsilon^\infty \| \quad \text{where} \quad x_\epsilon^\infty = \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon \end{pmatrix}$$

$$\text{and} \qquad \Delta_N = 2 \sin^2 \frac{\pi}{2\,N}$$

for all $t \geq 0$.

Remark

This inequality is sharp, and thus shows that convergence to equilibrium cannot occur at a rate faster than $\mathcal{O}(N^{-2})$.

Spectral gap

Theorem ($L^2_{\pi_{\epsilon,N}}$–spectral gap for reversible $\pi_{\epsilon,N}$)

Suppose that P satisfies $\int P(d\alpha)\,\alpha = \frac{1}{2}$ and $\sigma_P^2 < \frac{1}{4}$. If the stationary distribution $\pi_{\epsilon,N}$ of $\mathrm{X}(t)$ on $\mathcal{S}_{\epsilon,N}$ is reversible, then

$$\sigma(\mathcal{L}) \subset \Big(-\infty, -\frac{1}{2}\,[1 - 4\,\sigma_P^2]\ \sin^2\Big[\frac{\pi}{N+2}\Big]\Big] \cup \{0\}$$

for the spectrum of $\mathcal{L}$ acting as a selfadjoint, bounded negative semi-definite operator on $L^2_{\pi_{\epsilon,N}}$, and 0 is a simple eigenvalue corresponding to the constant eigenfunction.

Spectral gap

In what sense should we expect convergence?

- If $P(d\alpha)$ has a density, then we should expect convergence in total variation.
- If $P(d\alpha)$ has a density, then we should expect convergence in $L^2(\pi)$.
- If $P(d\alpha) = \delta_{1/2}(d\alpha)$ then the convergence is not in total variation, and $L^2(\pi)$ is trivial.

To handle general P we need a topological structure.
Idea of the proof:

1. Construct a special metric on $\mathcal{S}_{\epsilon,N}$.
2. Establish weak convergence in Vaserstein distance.
3. Use reversibility to obtain a spectral gap in L^2.

Proof of convergence in Vaserstein-2 distance cont'd

Recall that the definition of the Vaserstein-p distance is

$$\rho_p(\mu,\nu) = \inf_{\substack{X\sim\mu \\ Y\sim\nu}} [\mathbb{E}\,\mathrm{d}(X,Y)^p]^{\frac{1}{p}} \qquad \text{and set} \qquad \rho(\mu,\nu) \equiv \rho_1(\mu,\nu)$$

where μ and ν are two probability measures on a compact metric space $(\mathcal{S},\mathrm{d})$. Furthermore, for $p = 1$ the duality

$$\rho(\mu,\nu) = \inf_{\substack{X\sim\mu \\ Y\sim\nu}} \mathbb{E}\,\mathrm{d}(X,Y) = \sup_{f\,:\,\mathrm{Lip}(f)\leq 1} \mu(f) - \nu(f)$$

follows by the Kantorovich-Rubinstein theorem.

Proof of convergence in Vaserstein-2 distance cont'd

- Let d and d' be two equivalent distances, then

$$\rho(\mu_t, \nu_t) \le c\, e^{-\gamma\, t} \iff \rho'(\mu_t, \nu_t) \le c'\, e^{-\gamma\, t}$$

- Therefore, to obtain a strict contraction

$$\rho(\mathsf{X}(t,x), \mathsf{X}(t,x')) \le \mathrm{d}(x,x')\, e^{-\gamma\, t} \quad \text{for all} \quad x, x' \in \mathcal{S}_{\epsilon,N}$$

  we need a carefully chosen distance!
- The induced Euclidean distance on $\mathcal{S}_{\epsilon,N}$ does not work.

Proof of convergence in Vaserstein-2 distance cont'd

Recall the definition of the matrices $T_{i,\alpha}$

$$T_{i,\alpha} = \left(\begin{array}{c|cc|c} 1 & & & \\ \hline & \alpha & \alpha & \\ & 1-\alpha & 1-\alpha & \\ \hline & & & 1 \end{array}\right) , \qquad i = 1, \ldots, N-1 \, .$$

Key observation:

$$\begin{pmatrix} \alpha & \alpha \\ 1-\alpha & 1-\alpha \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{super-stable}$$

$$\begin{pmatrix} \alpha & \alpha \\ 1-\alpha & 1-\alpha \end{pmatrix} \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} = 1 \cdot \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} \qquad \text{neutral}$$

Proof of convergence in Vaserstein-2 distance cont'd

- Any $x \in \mathcal{S}_{\epsilon,N}$ can be written as

$$x = \epsilon\,\mathbf{1} + \sum_{i=1}^{N-1} u_i\,[\mathbf{e}_i - \mathbf{e}_{i+1}]$$

for some $u \in \mathbb{R}^{N-1}$.

- $\mathcal{S}_{\epsilon,N} \subset \mathbb{R}_+^N$ is in one-to-one correspondence with the set $\{u \in \mathbb{R}^{N-1} : -\epsilon \le u_1,\ u_{i-1} \le \epsilon + u_i,\ u_{N-1} \le \epsilon\}$.
- Conversely, $\epsilon = \sum_{i=1}^N \frac{1}{N}\, x_i$, and u is the solution to the discrete Poisson equation with Dirichlet boundary conditions

$$u_{i-1} - 2\,u_i + u_{i+1} = x_{i+1} - x_i \qquad \text{for} \qquad i = 1, \ldots, N-1$$

where we formally set $u_0 \equiv u_N \equiv 0$.

Proof of convergence in Vaserstein-2 distance cont'd

- ϵ is conserved, so that $\mathsf{U}(t)$ is Markov

$$\mathsf{X}(t) = \epsilon\,\mathbf{1} + \sum_{i=1}^{N-1} \mathsf{U}_i(t)\,[\mathbf{e}_i - \mathbf{e}_{i+1}]$$

- the generator reads

$$\hat{\mathcal{L}}_{\epsilon,N} A(u) = \Lambda \sum_{i=1}^{N-1} \int P(d\alpha)\,[A(\hat{T}^{\epsilon}_{i,\alpha} u) - A(u)]$$

where

$$\hat{T}^{\epsilon}_{i,\alpha} u - u = [(1-\alpha)\,u_{i-1} + \alpha\,u_{i+1} + (2\,\alpha - 1)\,\epsilon - u_i]\,\mathbf{e}_i$$

with the convention $u_0 \equiv u_N \equiv 0$.

Proof of convergence in Vaserstein-2 distance cont'd

Definition (Adapted metric)

Let x and x' be any two initial points on $\mathcal{S}_{\epsilon,N}$ then

$$x = \epsilon\,\mathbf{1} + \sum_{i=1}^{N-1} u_i\,[\mathbf{e}_i - \mathbf{e}_{i+1}]\,, \qquad x' = \epsilon\,\mathbf{1} + \sum_{i=1}^{N-1} u'_i\,[\mathbf{e}_i - \mathbf{e}_{i+1}]$$

and we define

$$\hat{\mathrm{d}}(u,u') := \Big[\sum_{i=1}^{N-1} (u_i - u'_i)^2\Big]^{\frac{1}{2}}\,, \qquad \mathrm{d}(x,x') = \hat{\mathrm{d}}(u,u')$$

- $\max_{u,u'\in\mathcal{S}_{\epsilon,N}} \hat{\mathrm{d}}(u,u') \leq \epsilon\,N\,\sqrt{N-1}$
- $\hat{T}^{\epsilon}_{i,\alpha}$ is super stable in direction i.

Proof of convergence in Vaserstein-2 distance cont'd

Consider the bivariate Markov process $(\mathrm{U}(t), \mathrm{U}'(t))$ on $\mathcal{S}_{\epsilon,N} \times \mathcal{S}_{\epsilon,N}$

$$\bar{\mathcal{L}}A(u,u') = \sum_{i=1}^{N-1} \int P(d\alpha)\,[A(\hat{T}^{\epsilon}_{i,\alpha}u,\, \hat{T}^{\epsilon}_{i,\alpha}u') - A(u,u')]$$

Proposition (Average contraction rate)

For any two u and u',

$$\bar{\mathcal{L}}\hat{\mathrm{d}}(u,u')^2 \le -[1 - 4\,\sigma_P^2]\,\sin^2\left[\frac{\pi}{N+2}\right]\hat{\mathrm{d}}(u,u')^2$$

where σ_P^2 denotes the variance of P.

Proof of convergence in Vaserstein-2 distance cont'd

Sketch of the proof:

- It is straightforward to verify

$$
\begin{aligned}
\bar{\mathcal{L}}\hat{\mathrm{d}}(u,u')^2 = &-\left[\frac{1}{4} - \sigma_P^2\right] [u - u']^T\, \mathcal{C}^{(N-1)}\, [u - u'] \\
&- \left[\frac{1}{4} + \sigma_P^2\right] \left([u_1 - u_1']^2 + [u_{N-1} - u_{N-1}']^2\right)
\end{aligned}
$$

$$
\mathcal{C}^{(N-1)} = \left(\begin{array}{c|cccc|c}
2 & 0 & -1 & 0 & 0 & \\
\hline
0 & 2 & 0 & -1 & 0 & \\
-1 & 0 & 2 & 0 & -1 & \\
 & & \ddots & & & \\
 & -1 & 0 & 2 & 0 & -1 \\
 & 0 & -1 & 0 & 2 & 0 \\
\hline
 & 0 & 0 & -1 & 0 & 2
\end{array}\right) \in \mathbb{R}^{(N-1)\times(N-1)}
$$

- The spectrum of $\mathcal{C}^{(N-1)}$ is explicitly computable. $\square$

Proof of convergence in Vaserstein-2 distance cont'd

Proposition (Rate of convergence in Vaserstein-2 distance)

Let $\mathrm{U}(t)$ *and* $\mathrm{U}'(t)$ *be any two Markov chains generated by* $\hat{\mathcal{L}}$ *on* $\mathcal{S}_{\epsilon,N}$*. Then*

$$\begin{aligned} \rho_2(\mathrm{U}(t),\mathrm{U}'(t)) &\le \rho_2(\mathrm{U}(0),\mathrm{U}'(0)) \exp\left(-\frac{1}{2}[1-4\,\sigma_P^2]\sin^2\left[\frac{\pi}{N+2}\right]t\right) \\ &\le \epsilon\, N\sqrt{N-1}\exp\left(-\frac{1}{2}[1-4\,\sigma_P^2]\sin^2\left[\frac{\pi}{N+2}\right]t\right) \end{aligned}$$

holds for all t*.*

- If $\sigma_P^2 < \frac{1}{4}$, then there exists a unique stationary distribution $\pi_{\epsilon,N}$ on each $\mathcal{S}_{\epsilon,N}$.
- This rate of convergence is again $\mathcal{O}(N^{-2})$, and thus optimal.

Proof of L^2 spectral gap

- A direct consequence (Kantorovich-Rubinstein duality) of the one-step contraction of the metric is the following:

Lemma (Lipschitz contraction)

Let $A\colon \mathcal{S}_{\epsilon,N} \to \mathbb{R}$ be a Lipschitz continuous function with respect to the distance $\mathrm{d}(.,.)$, and set $A_t(x) = \mathbb{E}[A(X(t))\,|\,X(t) = x]$ for all $t \geq 0$ and $x \in \mathcal{S}_{\epsilon,N}$. Then A_t is Lipschitz continuous with Lipschitz constant

$$\mathrm{Lip}(A_t) \leq \mathrm{Lip}(A)\, \exp\Big(-\frac{1}{2}\,[1 - 4\,\sigma_P^2]\,\sin^2\Big[\frac{\pi}{N+2}\Big]\,t\Big)$$

for all $t \geq 0$.

Proof of L^2 spectral gap

- If $\pi_{\epsilon,N}$ is reversible, then $\mathcal{L}$ is a self-adjoint bounded operator in $L^2_{\pi_{\epsilon,N}}$.
- the constant functions are eigenfunctions to the eigenvalue 0.
- All Lipschitz constants get contracted by $e^{\mathcal{L}}$ by a uniform rate $\frac{1}{2}\,[1-4\,\sigma_P^2]\,\sin^2[\frac{\pi}{N+2}]$.
- Spectral calculus then shows that the spectral gap of $\mathcal{L}$ is estimated by $\frac{1}{2}\,[1-4\,\sigma_P^2]\,\sin^2[\frac{\pi}{N+2}]$. $\square$

Summary:

- Weak convergence at rate $\mathcal{O}(N^{-2})$ for any P.
- Spectral gap $\mathcal{O}(N^{-2})$ for reversible π.

Spectral Gap

- Recall the general setup:

$$\mathcal{L}A(\mathrm{X}) = \sum_{i=1}^{N-1} \Lambda(\mathrm{X}_i, \mathrm{X}_{i+1}) \int_0^1 P\left(\frac{\mathrm{X}_i}{\mathrm{X}_i + \mathrm{X}_{i+1}}, d\alpha\right) [A(T_{i,\alpha}\mathrm{X}) - A(\mathrm{X})]$$

  where

$$\Lambda(\mathrm{X}_i, \mathrm{X}_{i+1}) = \Lambda_+(\mathrm{X}_i + \mathrm{X}_{i+1})\, \Lambda_R\left(\frac{\mathrm{X}_i}{\mathrm{X}_i + \mathrm{X}_{i+1}}\right)$$

$$\Lambda_+(s) = \sqrt{s}$$

- Idea: Use a perturbation result to establish the spectral gap.

Spectral Gap

If $\pi_{\epsilon,N}$ is reversible, then

$$\mathcal{D}_{\epsilon,N}(A) = \int \pi_{\epsilon,N}(dx)\, A(x)\, [-\mathcal{L}A](x)$$

has the representation

$$\mathcal{D}_{\epsilon,N}(A) = \frac{1}{2} \int \pi_{\epsilon,N}(dx) \sum_{i=1}^{N-1} \Lambda_+(x_i + x_{i+1})\, \Lambda_R\Big(\frac{x_i}{x_i + x_{i+1}}\Big) \cdot$$
$$\cdot \int P\Big(\frac{x_i}{x_i + x_{i+1}}, d\alpha\Big)\, [A(T_{i,\alpha}x) - A(x)]^2$$

The spectral gap has the variational characterization

$$\mathrm{gap}(\mathcal{L}) = \inf\left\{ \frac{\mathcal{D}_{\epsilon,N}(A)}{\mathrm{Var}(A)} : A \in L^2_{\pi_{\epsilon,N}}\,, \quad \mathrm{Var}(A) \neq 0 \right\}$$

Spectral Gap

Proposition (Comparison)

Fix $\epsilon > 0$ and N, and let $\pi_{\epsilon,N}$ be a reversible stationary distribution of $\mathcal{L}$ on $\mathcal{S}_{\epsilon,N}$.

(i) $\Lambda_+(x_i + x_{i+1})\,\Lambda_R(\frac{x_i}{x_i+x_{i+1}}) \geq \Lambda^-_{\epsilon,N}$.

(ii) $P(\frac{x_i}{x_i+x_{i+1}}, .) \geq \beta\, P^\star(.)$ *for some* $\beta > 0$ *and some probability measure* $P^\star$ *on* $[0,1]$ *with mean* $\int P^\star(d\alpha)\,\alpha = \frac{1}{2}$ *and variance* $\sigma^2_{P^\star} < \frac{1}{4}$,

(iii) *For the above choice of* $P^\star$ *the unique stationary distribution* $\pi^\star_{\epsilon,N}$ *of* $\mathcal{L}^\star$ *on* $\mathcal{S}_{\epsilon,N}$ *is reversible.*

(iv) $C^-_{\epsilon,N} \leq \frac{\pi_{\epsilon,N}(dx)}{\pi^\star_{\epsilon,N}(dx)} \leq C^+_{\epsilon,N}$ *for some* $0 < C^-_{\epsilon,N} \leq C^+_{\epsilon,N} < \infty$

$$\sigma(\mathcal{L}) \subset \Big(-\infty, -\beta\,\frac{C^-_{\epsilon,N}}{C^+_{\epsilon,N}}\,\Lambda^-_{\epsilon,N}\,\frac{1}{2}\,[1 - 4\,\sigma^2_{P^\star}]\,\sin^2\Big[\frac{\pi}{N+2}\Big]\Big] \cup \{0\}$$

Spectral Gap

Proof:

- $L^2_{\pi_{\epsilon,N}} = L^2_{\pi^\star_{\epsilon,N}}$
- Dirichlet form comparison

$$\mathcal{D}_{\epsilon,N}(A) = \frac{1}{2}\int \pi_{\epsilon,N}(dx) \sum_{i=1}^{N-1} \Lambda_+(x_i + x_{i+1})\,\Lambda_R\Big(\frac{x_i}{x_i + x_{i+1}}\Big)\cdot$$
$$\cdot \int P\Big(\frac{x_i}{x_i + x_{i+1}}, d\alpha\Big)\,[A(T_{i,\alpha}x) - A(x)]^2$$
$$\geq \beta\, C^-_{\epsilon,N}\, \Lambda^-_{\epsilon,N}\, \mathcal{D}^\star_{\epsilon,N}(A)$$

- Variance comparison

$$\mathrm{Var}(A) = \inf_{c\in\mathbb{R}} \int \pi_{\epsilon,N}(dx)\,[A(x) - c]^2 \leq C^+_{\epsilon,N}\ \mathrm{Var}^\star(A)$$

- $\mathrm{gap} \geq \beta\,\frac{C^-_{\epsilon,N}}{C^+_{\epsilon,N}}\,\Lambda^-_{\epsilon,N}\mathrm{gap}^\star$ $\square$

Reversible product measures

- The results we obtained are for reversible $\pi_{\epsilon,N}$.
- $\mathcal{S}_{\epsilon,N}$ is a simplex, hence not with respect to product measures $\mu(dx) = \nu(dx_1) \cdots \nu(dx_N)$
- When is such a μ reversible on $\mathbb{R}_+^N$?
- For mechanical systems this is (the projection of) the well known Boltzmann-Gibbs distribution.

Reversible product measures

Lemma (Reversible product measures and system size)

The product measure $\mu(dx) = \nu(dx_1)\cdots\nu(dx_N)$ is reversible for $\mathrm{X}(t)$ for some N if and only if it is reversible for $N = 2$.

Proof.

Since the generator is a sum of pair interactions, reversibility holds iff

$$\int_{\mathbb{R}_+^2} \nu(dx_1)\,\nu(dx_2)\,\Lambda_+(x_1+x_2)\,\Lambda_R\Big(\frac{x_1}{x_1+x_2}\Big)\int P\Big(\frac{x_1}{x_1+x_2},d\alpha\Big)\cdot$$
$$\cdot\,\psi(\alpha\,[x_1+x_2],(1-\alpha)\,[x_1+x_2],x_1,x_2)$$
$$=\int_{\mathbb{R}_+^2} \nu(dx_1)\,\nu(dx_2)\,\Lambda_+(x_1+x_2)\,\Lambda_R\Big(\frac{x_1}{x_1+x_2}\Big)\int P\Big(\frac{x_1}{x_1+x_2},d\alpha\Big)\cdot$$
$$\cdot\,\psi(x_1,x_2,\alpha\,[x_1+x_2],(1-\alpha)\,[x_1+x_2])$$

for any (non-negative) test function $\psi\colon \mathbb{R}_+^2\times\mathbb{R}_+^2\to\mathbb{R}$. $\square$

Reversible product measures

Corollary (Reversible product measures and rate functions)

The product measure μ is reversible for some rate functions Λ_R and Λ_+ with $\Lambda_+(\eta) > 0$ for all $\eta > 0$ if and only if it reversible for any such choice for Λ_+ (while Λ_R is kept fixed).

Reversibility holds if and only if

$$\int_{\mathbb{R}_+^2} \nu(dx_1)\,\nu(dx_2)\,\Lambda_R\Big(\frac{x_1}{x_1+x_2}\Big)\int P\Big(\frac{x_1}{x_1+x_2}, d\alpha\Big)\,\eta\Big(x_1+x_2, \alpha, \frac{x_1}{x_1+x_2}\Big)$$
$$= \int_{\mathbb{R}_+^2} \nu(dx_1)\,\nu(dx_2)\,\Lambda_R\Big(\frac{x_1}{x_1+x_2}\Big)\int P\Big(\frac{x_1}{x_1+x_2}, d\alpha\Big)\,\eta\Big(x_1+x_2, \frac{x_1}{x_1+x_2}, \alpha\Big)$$

Reversible product measures

Theorem (Reversible product measures)

Suppose that the Markov chain on $[0,1]$ *with transition kernel* $P(\beta, d\alpha)$ *has a unique invariant distribution, say* $p(.)$. *Then* $\mu(dx) = \nu(dx_1)\cdots\nu(dx_N)$ *is reversible if and only if either one holds:*

1. *There exists an* $\epsilon > 0$ *such that* $\nu(dx_1) = \delta(\epsilon, dx_1)$, $p(d\alpha) = \delta(\frac{1}{2}, d\alpha)$, *and* $P(\frac{1}{2}, d\alpha) = \delta(\frac{1}{2}, d\alpha)$.
2. *There exists an* $\epsilon > 0$ *and a* $d > 0$ *such that*

$$\nu(dx_1) = \frac{dx_1}{\epsilon}\left[\frac{x_1}{\epsilon}\right]^{\frac{d}{2}-1}\frac{e^{-\frac{x_1}{\epsilon}}}{\Gamma(\frac{d}{2})}$$

$$p(d\beta) = d\beta\,[\beta\,(1-\beta)]^{\frac{d}{2}-1}\,\frac{\Gamma(d)}{\Gamma(\frac{d}{2})^2}\,\Lambda_R(\beta)\,\frac{1}{Z}$$

$$\int p(d\beta)\int P(\beta, d\alpha)\,\psi(\alpha,\beta) = \int p(d\beta)\int P(\beta, d\alpha)\,\psi(\beta,\alpha)$$

Reversible product measures

Proof:

- We only need to consider $N = 2$ and $\Lambda_+ = 1$
- Conditioning μ on the sum implies that reversibility holds iff

$$\int \nu_R(s, d\beta)\, \Lambda_R(\beta) \int P(\beta, d\alpha)\, \eta(\alpha, \beta) = \int \nu_R(s, d\beta)\, \Lambda_R(\beta) \int P(\beta, d\alpha)\, \eta(\beta, \alpha)$$

  for ν_+–almost every s
- In particular $p(d\beta) = \frac{1}{Z}\, \nu_R(s, d\beta)\, \Lambda_R(\beta)$ for ν_+–almost every s,
- Uniqueness of p shows that $\nu_R(s, d\beta)$ is independent of s.
- The constants and Gamma distributions are the only possible solutions. $\square$

Example – continued

The 3-dimension billiard chain considered by Gaspard and Gilbert:

- Explicit computation:

$$\frac{P(\beta, d\alpha)}{d\alpha} = \frac{3}{2} \, \frac{1 \wedge \sqrt{\frac{\alpha \wedge (1-\alpha)}{\beta \wedge (1-\beta)}}}{\frac{1}{2} + \beta \vee (1-\beta)}$$

$$\Lambda_R(\beta) = \frac{\sqrt{2\pi}}{6} \, \frac{\frac{1}{2} + \beta \vee (1-\beta)}{\sqrt{\beta \vee (1-\beta)}} \,, \qquad \Lambda_+(s) = \sqrt{s}$$

Example – continued

Lemma

If $\Lambda_+(s)$ is replaced by any non-negative continuous function, which is bounded away from zero, then the following hold for any N and ϵ.

1. *The product measure $\mu(dx) = \nu(dx_1)\cdots\nu(dx_N)$ with $\nu(dx_1) = \frac{dx_1}{\epsilon}\sqrt{\frac{x_1}{\epsilon}}\frac{2\,e^{-\frac{x_1}{\epsilon}}}{\sqrt{\pi}}$ is the unique reversible product measure for $X(t)$.*
2. *On every $\mathcal{S}_{\epsilon,N}$ there exists a unique stationary distribution $\pi_{\epsilon,N}$. This measure is obtained by conditioning $\mu(dx)$.*
3. *The spectrum $\sigma(\mathcal{L})$ of the generator $\mathcal{L}$ acting on $L^2_{\pi_{\epsilon,N}}$ satisfies*

$$\sigma(\mathcal{L}) \subset \Big(-\infty, -C\,\sin^2\Big[\frac{\pi}{N+2}\Big]\Big] \cup \{0\}$$

for some constant C, which depends on the choice of Λ_+.

Conclusion and Future work

- We showed weak convergence at rate $\mathcal{O}(N^{-2})$ for the state independent setting.
- We introduced a special metric which allowed to obtain L^2 spectral gaps for reversible measures.
- We obtained L^2 spectral bounds for the reversible state-dependent process, assuming a lower bound on the rate function.
- We classified all reversible product measures.
- Modulo the cut-off we obtain spectral bounds for the billiard chain model.
- Hydrodynamics can be done for the state-independent process (linear heat equation, gradient system)

Ongoing work is on the hydrodynamic limit for the state dependent process, as well as removing the lower bound on the rate function.