

# Long range correlations in non-equilibrium systems

T. Bodineau,

*Joint works with*

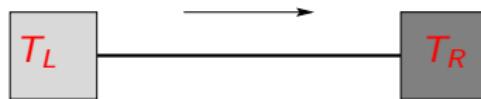
*B. Derrida, J. Lebowitz, V. Lecomte, F. van Wijland*

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# Outline

- Stochastic dynamics & Invariant measure
- Correlations and non-equilibrium phase transitions
- Equivalence of ensembles for non-equilibrium systems

## Open systems with reservoirs



Heat reservoirs  $T_L \neq T_R$

Current flowing through the system :

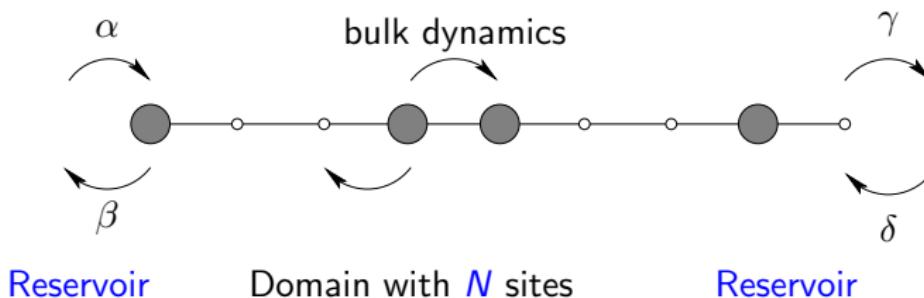
$\left\{ \begin{array}{l} \text{Heat} \\ \text{Electrons} \\ \text{Particles} \end{array} \right.$

### Questions.

- Structure of the steady state
- Long range correlations

# Stochastic particle systems

Particles:  $\eta(t) = \{\eta_i(t)\}_{i \leq N} \in \{0, 1\}^N$



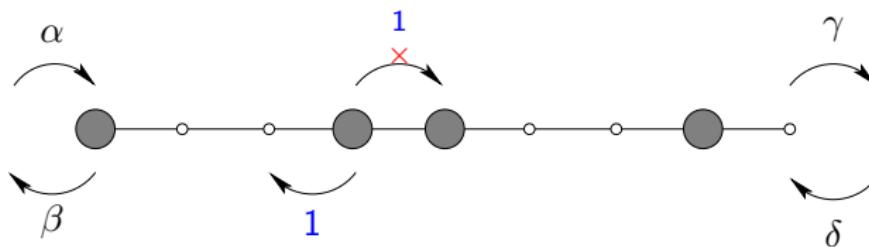
Left reservoir acting at site 0:

- ⇒ particle creation at rate  $\alpha$
- ⇒ particle annihilation at rate  $\beta$

Right reservoir acting at site  $N$

# Stochastic particle systems

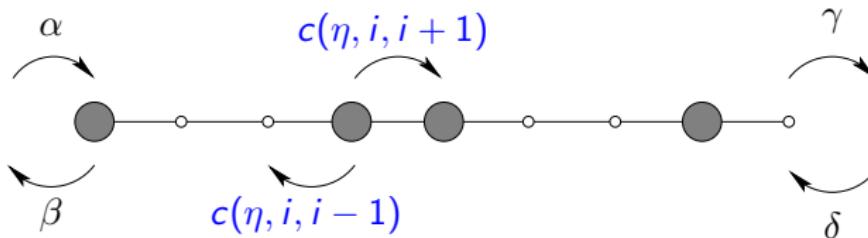
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- Symmetric Simple Exclusion Process (SSEP)  
 $(\alpha, \beta) = (\gamma, \delta)$  Reversible markov chain.  
Steady state = product Bernoulli measure
- $(\alpha, \beta) \neq (\gamma, \delta)$  Non reversible markov chain.

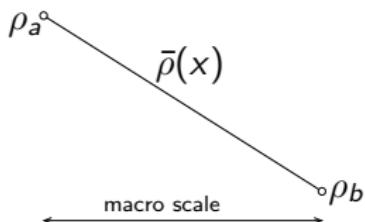
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 Steady state = product Bernoulli measure
- $(\alpha, \beta) \neq (\gamma, \delta)$  Non reversible markov chain.
- More general dynamics (Kawasaki dynamics)

# SSEP invariant measure



Linear density profile  $\bar{\rho}(x)$

$$\rho_a = \frac{\alpha}{\alpha+\beta}, \quad \rho_b = \frac{\delta}{\delta+\gamma}$$

Truncated two-point correlation function

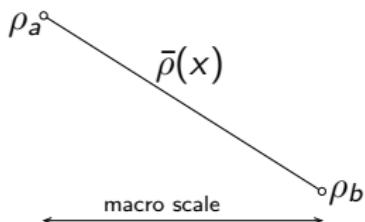
$$i < j, \quad \langle \eta_i; \eta_j \rangle = \langle \eta_i \eta_j \rangle - \langle \eta_i \rangle \langle \eta_j \rangle = \frac{1}{N} C^{\text{open}}\left(\frac{i}{N}, \frac{j}{N}\right)$$

with

$$0 \leq x < y \leq 1, \quad C^{\text{open}}(x, y) = -(\rho_a - \rho_b)^2 x(1 - y).$$

- ⇒ Current =  $\rho_a - \rho_b$
- ⇒ Local equilibrium : Correlations are of order  $1/N$
- ⇒ Global contribution :  $\sum_{i \neq j} \langle \eta_i; \eta_j \rangle = -\frac{N}{12}$

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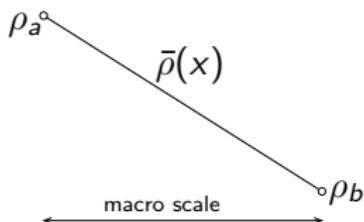
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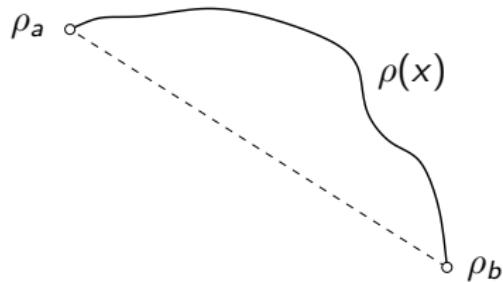
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# Derivation of the correlations

- Exact microscopic computations [Spohn]
- Density large deviations [Derrida, Lebowitz, Speer]

$$\mathcal{F}(\rho) = \lim_{N \rightarrow \infty} -\frac{1}{N} \log \langle \text{observing } \rho \rangle$$



Macroscopic density profile:

$$0 \leq x \leq 1, \quad \rho(x)$$

Large deviations :  $\langle \text{observing } \rho \rangle \simeq \exp(-N\mathcal{F}(\rho))$

Let  $\lambda$  be a smooth function in  $[0, 1]$

$$\frac{1}{N} \log \left\langle \exp \left( \sum_{i=1}^N \lambda \left( \frac{i}{N} \right) \eta_i \right) \right\rangle \rightarrow \mathcal{G}(\lambda)$$

with  $\mathcal{G}(\lambda) = \sup \left\{ \int_0^1 dx \rho(x) \lambda(x) - \mathcal{F}(\rho) \right\}$ .

For  $\lambda$  small

$$\begin{aligned} \mathcal{G}(\lambda) &= \int_0^1 dx \bar{\rho}(x) \lambda(x) + \frac{1}{2} \int_0^1 dx \bar{\rho}(x)(1 - \bar{\rho}(x))\lambda(x)^2 \\ &\quad + \frac{1}{2} \int_0^1 \int_0^1 dx dy C^{\text{open}}(x, y)\lambda(x)\lambda(y) \end{aligned}$$

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# Hydrodynamic limit

Diffusive hydrodynamic scaling  
 $(x = i/N, t = \tau/N^2) \Rightarrow \begin{cases} \{0, N\} \hookrightarrow [0, 1] \\ \eta_i(\tau) \hookrightarrow \rho_{(x,t)} \end{cases}$

Typical density:  $\partial_t \bar{\rho}_{(x,t)} = \partial_x (D(\bar{\rho}_{(x,t)}) \partial_x \bar{\rho}_{(x,t)})$

An arbitrary macroscopic evolution  $(\rho_{(x,t)})$  with  $t \in [0, T]$

$$\langle \text{observing } (\rho_{(x,t)}) \rangle \approx \exp(-N \mathcal{I}_{[0,T]}(\rho))$$

then

$$\mathcal{I}_{[0,T]}(\rho) = \int_0^T dt \int_0^1 dx \frac{\left( \nabla^{-1}(\partial_t \rho(x, t) - \partial_x(D(\rho_{(x,t)}) \partial_x \rho_{(x,t)})) \right)^2}{2\sigma(\rho_{(x,t)})}$$

[Kipnis, Olla, Varadhan] .....

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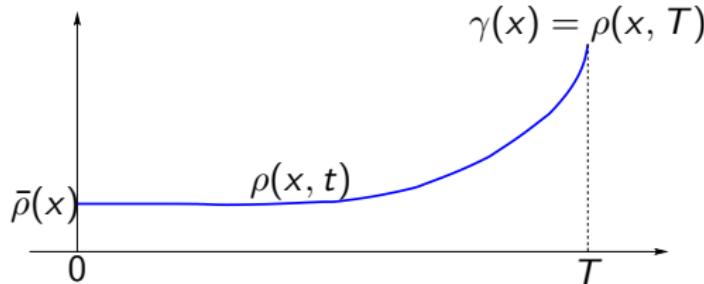
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# A dynamical approach to compute the steady state

$\langle \text{observing the density } \gamma(x) \rangle = \langle \text{observing } \gamma(x) \text{ at time } T \rangle$

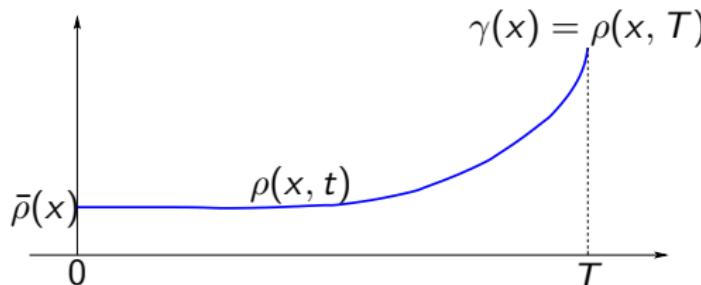
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$$\langle \text{observing the density } \gamma(x) \rangle = \langle \text{observing } \gamma(x) \text{ at time } T \rangle \\ \simeq \exp(-N \inf_{\rho, q} \mathcal{I}_{[0, T]}(\rho, q))$$



$$\mathcal{F}(\gamma) = \lim_T \inf \{\mathcal{I}_{[0, T]}(\rho, q); \quad \rho(x, 0) = \bar{\rho}(x), \quad \rho(x, T) = \gamma(x)\}$$

[Bertini, De Sole, Gabrielli, Jona-lasinio, Landim]

## Two-point correlation function

$$\mathcal{F}(\gamma) = \lim_{T \rightarrow \infty} \inf \left\{ \mathcal{I}_{[0, T]}(\rho, q); \quad \rho(x, 0) = \bar{\rho}(x), \quad \rho(x, T) = \gamma(x) \right\}$$

Perturbation around the steady state:  $\gamma(x) = \bar{\rho}(x) + \varepsilon \varphi(x)$   
Optimal trajectory at the second order in  $\varepsilon$

➡ Expansion of  $\mathcal{F}$  wrt  $\varepsilon$

## ABC

$q$	1
$AB \rightarrow BA$	$BA \rightarrow AB$
$BC \rightarrow CB$	$CB \rightarrow BC$
$CA \rightarrow AC$	$AC \rightarrow AC$

Densities:  $r_A, r_B, r_C$ System on the ring  $\{1, \dots, N\}$ Segregation occurs for  $q < 1$  :  $AAAAA BBBBB CCCCC$ 

[Evans, Kafri, Kuduvily, Mukamel]

[Kafri, Biron, Evans, Mukamel]

[Clincy, Derrida, Evans]

[Ayyer, Carlen, Lebowitz, Mohanty, Mukamel, Speer]

[Bertini, De Sole, Gabrielli, Jona-lasinio, Landim]

# ABC: Hydrodynamic equations

Weak drift :  $q = \exp(-\frac{\beta}{N})$

$$\begin{cases} \partial_\tau \rho_A(x, \tau) = \partial_x^2 \rho_A(x, \tau) + \beta \partial_x (\rho_A(x, \tau)(2\rho_B(x, \tau) + \rho_A(x, \tau) - 1)) \\ \partial_\tau \rho_B(x, \tau) = \partial_x^2 \rho_B(x, \tau) + \beta \partial_x (\rho_B(x, \tau)(1 - \rho_B(x, \tau) - 2\rho_A(x, \tau))) \\ \rho_A(x, \tau) + \rho_B(x, \tau) + \rho_C(x, \tau) = 1 \end{cases}$$

with fixed mean densities  $r_A, r_B, r_C$ .

The constant profiles  $\bar{\rho}_A(x) = r_A, \bar{\rho}_B(x) = r_B, \bar{\rho}_C(x) = r_C$  are stationary solutions.

Phase transition.

$\beta < \beta_c$  : Steady state density profiles are constant  $= r_A, r_B, r_C$

$\beta > \beta_c$  : Space dependent steady state densities : segregation

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# ABC: Two-point correlation function

$r_A = r_B = r_C = 1/3$ . [Evans, Kafri, Koduvely, Mukamel]

- ⇒ Explicit expression for the steady state (reversibility)
- ⇒ Large deviation function

General case : Non equilibrium system

When the phase transition is 2nd order:  $\beta_c = \frac{2\pi}{\sqrt{1-2(r_A^2+r_B^2+r_B^2)}}$

$$\beta < \beta_c \quad \langle A_i; A_j \rangle = \langle A_i A_j \rangle - \langle A_i \rangle \langle A_j \rangle = \frac{1}{N} C_{AA} \left( \frac{i}{N} - \frac{j}{N} \right)$$

$$C_{AA}(x) = -r_A(1-r_A) - \frac{\alpha}{\sin(\frac{\pi\beta}{\beta_c})} \left( \beta \cos \left( \frac{\pi\beta}{\beta_c} (2x-1) \right) - \frac{\beta_c}{\pi} \sin \left( \frac{\pi\beta}{\beta_c} \right) \right)$$

$$\lim_{\beta \rightarrow \beta_c} C_{AA}(x) = \infty$$

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# Phase transition WASEP

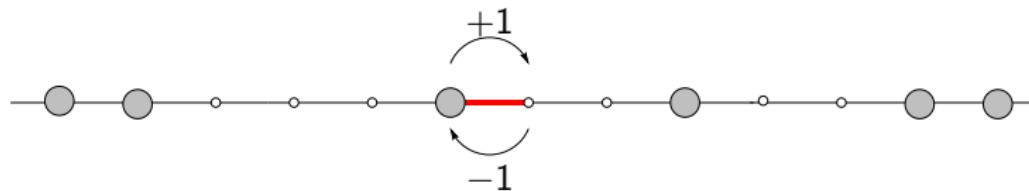
WASEP on a ring  $\{1, N\}$

$$\text{jump rates} = \begin{cases} 1 + \frac{\nu}{N} & \bullet \curvearrowright \circ \\ 1 - \frac{\nu}{N} & \circ \curvearrowleft \bullet \end{cases}$$

$\mu_N$  stationary measure:  
Bernoulli with density  $\bar{\rho}$

Integrated current through the edge  $(i, i+1)$ :

$Q_{[0,\tau]} = \text{Number of jumps from } i \text{ to } i+1 \text{ during } [0, \tau]$   
– Number of jumps from  $i+1$  to  $i$  during  $[0, \tau]$

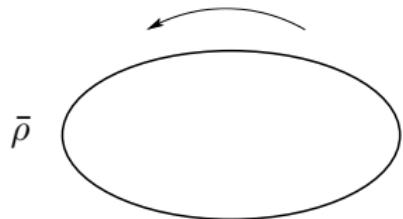


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Mean current:

$$\mathbb{E}_{[0,\tau]}^\nu \left( \frac{Q_{[0,\tau]}}{\tau} \right) = \frac{\nu}{N} 2\bar{\rho}(1-\bar{\rho}) = \frac{\nu}{N} \sigma(\bar{\rho})$$

$-T$  $time = 0$  $T$ 

## Definition

For a current  $\mathcal{J}$ , the **constrained measure** is given by

$$\mu_{\mathcal{J},N}(F(\eta)) = \lim_T \mathbb{E}\left(F(\eta(t=0)) \mid \frac{1}{2T} Q_{[-T,T]} = \mathcal{J}\right)$$

The two-point correlation function is

$$\mu_{\mathcal{J},N}(\eta_i; \eta_j) = \mu_{\mathcal{J},N}(\eta_i; \eta_j) - \mu_{\mathcal{J},N}(\eta_i)\mu_{\mathcal{J},N}(\eta_j)$$

For large currents  $\Leftrightarrow$  [Popkov, Schütz, Simon]

For  $\mathcal{J} > q_c^* = \nu\sigma(\bar{\rho})\sqrt{1 - \frac{\pi^2}{2\nu^2\sigma(\bar{\rho})}}$

The correlations scale for large  $N$  as

$$\mu_{\mathcal{J},N}(\eta_i; \eta_j) = \frac{1}{N} \frac{\sigma(\bar{\rho})}{2} \left( -1 + \sum_{k \geq 1} 2C_k \cos\left(2\pi k \frac{i-j}{N}\right) \right)$$

with  $\sigma(\rho) = \rho(1-\rho)$

$$C_k = -1 + \frac{1}{\sqrt{1 - \frac{\sigma''}{8\sigma\pi^2 k^2}(\mathcal{J}^2 - \nu^2\sigma^2)}}$$

**Consequence.** When  $\mathcal{J} \rightarrow q_c^*$  the correlations blow ( $C_1 \rightarrow \infty$ ).  
Precursor of the macroscopic clustering which occurs after the transition.

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# Phase transition on a ring

Large deviations on the ring  $\{1, N\}$  for the WASEP

$$\lim_T \lim_N \frac{1}{TN^2} \log \mathbb{P}_{[0, TN^2]}^\nu \left( \frac{Q_{TN^2}}{TN^2} \sim \frac{\mathcal{J}}{N} \right) = G^\nu(\mathcal{J})$$

with  $G^\nu(\mathcal{J}) = \lim_{T \rightarrow \infty} \inf_\rho \left\{ \frac{1}{T} \mathcal{I}_{[0, T]}^\nu(\rho) \right\}$

$$\mathcal{I}_{[0, T]}^\nu(\rho) = \int_0^T dt \int_0^1 dx \frac{(q_{(x,t)} + \frac{1}{2}\rho'_{(x,t)} - \nu\sigma(\rho_{(x,t)}))^2}{2\sigma(\rho_{(x,t)})}$$

with  $\partial_t \rho = -\partial_x q$  and  $\mathcal{J} = \frac{1}{T} \int_0^T dt \int_0^1 dx q_{(x,t)}$

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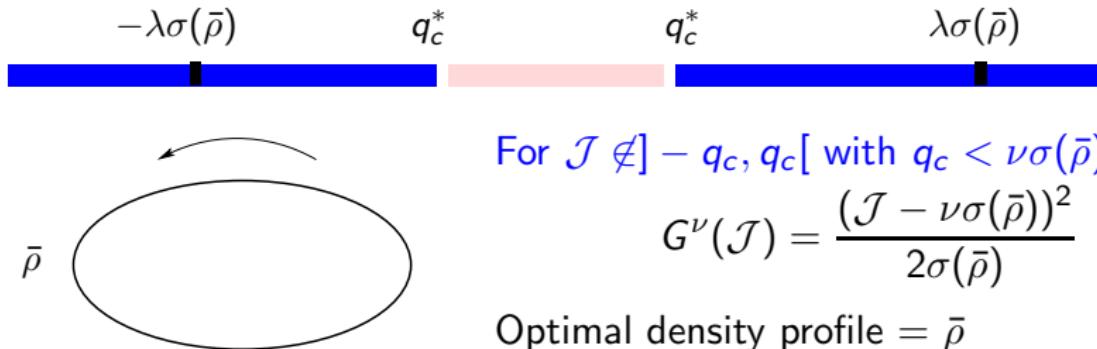
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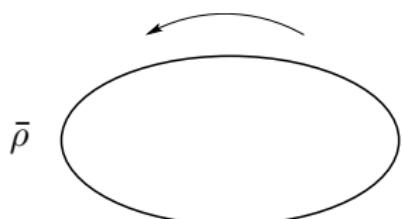
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# Phase transition on a ring



Stability of the flat profile  $\bar{\rho}$

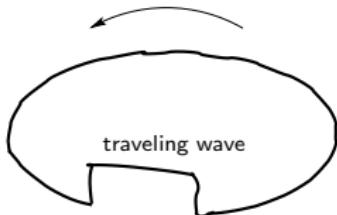
# Phase transition on a ring



For  $\mathcal{J} \notin ] - q_c, q_c [$  with  $q_c < \nu\sigma(\bar{\rho})$

$$G^\nu(\mathcal{J}) = \frac{(\mathcal{J} - \nu\sigma(\bar{\rho}))^2}{2\sigma(\bar{\rho})}$$

Optimal density profile =  $\bar{\rho}$



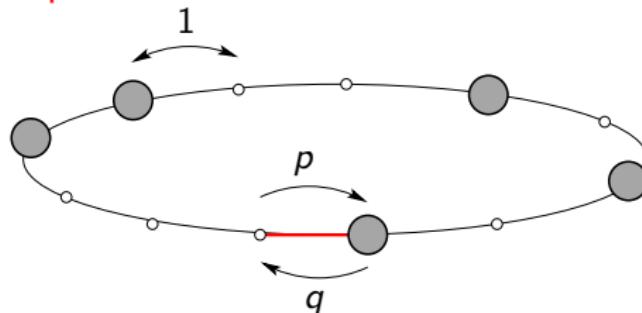
$$q \in ] - q_c^*, q_c^* [, \quad G^\nu(\mathcal{J}) < \frac{(\mathcal{J} - \nu\sigma(\bar{\rho}))^2}{2\sigma(\bar{\rho})}$$

$$\text{with } q_c^* = \nu\sigma(\bar{\rho})\sqrt{1 - \frac{\pi^2}{2\nu^2\sigma(\bar{\rho})}}$$

# SSEP driven by a Battery

Periodic system:  $\eta(t) = \{\eta_i(t)\}_{i \leq N} \in \{0, 1\}^N$

Total number of particles conserved



At the Battery:  $p \langle \eta_N(1 - \eta_1) \rangle = q \langle \eta_1(1 - \eta_N) \rangle$

Local Equilibrium :

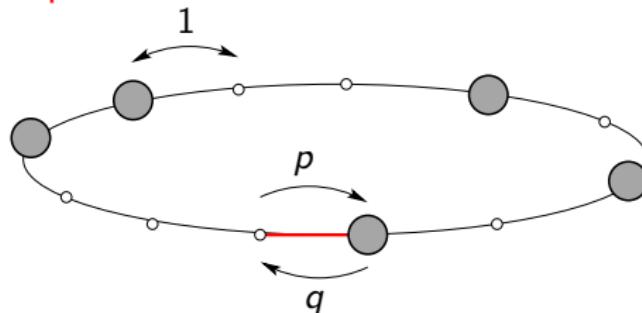
⇒ Product measure with different densities across the battery

$$p\langle \eta_N \rangle (1 - \langle \eta_1 \rangle) = q\langle \eta_1 \rangle (1 - \langle \eta_N \rangle)$$

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Periodic system:  $\eta(t) = \{\eta_i(t)\}_{i \leq N} \in \{0, 1\}^N$

Total number of particles conserved



At the Battery:  $p \langle \eta_N(1 - \eta_1) \rangle = q \langle \eta_1(1 - \eta_N) \rangle$

Local Equilibrium :

⇒ Product measure with different densities across the battery

$$p\langle \eta_N \rangle (1 - \langle \eta_1 \rangle) = q\langle \eta_1 \rangle (1 - \langle \eta_N \rangle)$$

# SSEP driven by a Battery

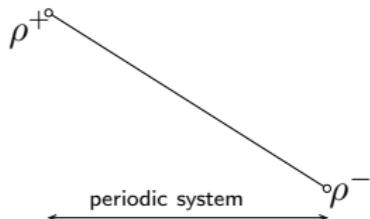
**Steady State:** Mean density  $\bar{\rho}$

A diagram illustrating the steady state condition. A horizontal double-headed arrow labeled "periodic system" connects two points on a downward-sloping line. The top point is labeled  $\rho^+$  and the bottom point is labeled  $\circ\rho^-$ . Above the line, the equation  $\rho^+ + \rho^- = 2\bar{\rho}$  is written in blue. Below the line, another equation in blue shows the product of the densities:  $\rho^+ \rho^- (1 - \rho^+) = q \rho^- (1 - \rho^-)$ .

- Same density profile as in the open system
- At the leading order the stationary measure is locally a product measure.

# SSEP driven by a Battery

**Steady State:** Mean density  $\bar{\rho}$



$$\rho^+ + \rho^- = 2\bar{\rho}$$

$$p \rho^- (1 - \rho^+) = q \rho^+ (1 - \rho^-)$$

## Questions.

- ➊ Equivalence of ensembles
- ➋ Long range correlations ?
  - ⇒ Non-equilibrium + Micro-canonical constraint

# Hydrodynamic limit

Heat equation:  $\partial_t \rho(x, t) = \Delta \rho(x, t)$

Non linear boundary conditions:

$$p \rho(1, t)(1 - \rho(0, t)) = q \rho(0, t)(1 - \rho(1, t)), \quad \partial_x \rho(0, t) = \partial_x \rho(1, t)$$

Fluctuation around the steady state :  $\rho(x, t) = \bar{\rho}(x) + \varepsilon f(x, t)$

$$\partial_t \rho(x, t) = \Delta \rho(x, t), \quad f(1, t) = af(0, t), \quad \partial_x \rho(0, t) = \partial_x \rho(1, t)$$

with  $a = \frac{\rho^+(1-\rho^+)}{\rho^-(1-\rho^-)}$ .

Green's function

$$G_t(x, y) = \exp(-t\Delta)$$

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## Green's function

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# Two-point correlation function

Current :  $\mathcal{J} = \rho^+ - \bar{\rho}^-$

$$0 \leq x \leq y \leq 1$$

$$\begin{aligned} C^{\text{bat}}(x, y) = & -\frac{2}{(\mathbf{a}+1)^2} \left( \frac{\sigma(\bar{\rho}^+) + \sigma(\bar{\rho}^-)}{2} + \frac{\mathcal{J}^2}{3} \right) (\mathbf{a}x + 1 - x)(\mathbf{a}y + 1 - y) \\ & - 2\mathcal{J}^2 \int_0^\infty dt \int_0^1 dz G_t(x, z) G_t(y, z) - G_t(x, 0) G_t(y, 0) \end{aligned}$$

$$C^{\text{open}}(x, y) = -\mathcal{J}^2 \Delta_{\text{Dirichlet}}^{-1}(x, y)$$

⇒ Similar structure, Singularity at the battery.

⇒ Small current  $\mathcal{J}$  :

$$C^{\text{bat}}(x, y) = -\frac{\sigma(\bar{\rho})}{2} \left( 1 + \mathcal{J} \frac{\sigma'(\bar{\rho})}{\sigma(\bar{\rho})} (-1 + x + y) \right)$$

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Half-filling:  $\bar{\rho} = 1/2$

$$0 \leq x \leq y \leq 1$$

$$C^{\text{bat}}(x, y) = -\frac{1}{2}\sigma(\bar{\rho}^+) + \frac{1}{12}\mathcal{J}^2 - \mathcal{J}^2 \left( \frac{1}{2}(x+y)(1-(x+y)) + x \right)$$

$$+ 8\mathcal{J}^2 \sum_{k,n \geq 1}^{\infty} \frac{\cos(2\pi nx)\cos(2\pi ky)}{(2\pi)^2(k^2+n^2)}$$

$$C^{\text{open}}(x, y) = -\mathcal{J}^2 x(1-y)$$

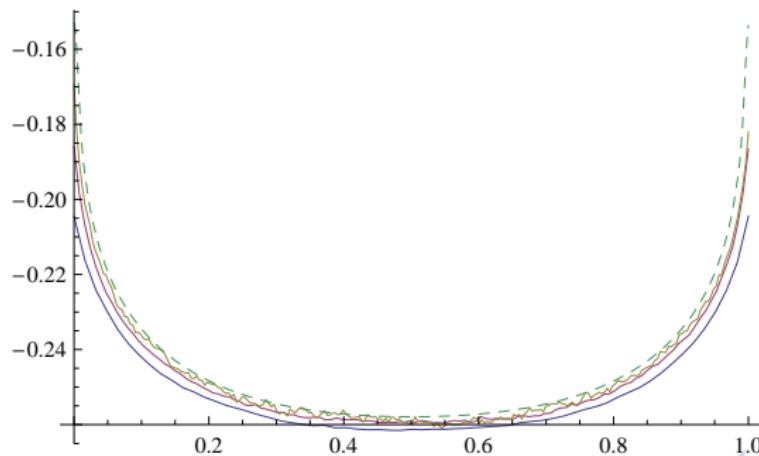
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$i \rightarrow N\langle \eta_i; \eta_{i+1} \rangle$

System sizes:  
 $N = 64, 128, 256$

SSEP on a ring with a varying field.  $E(x)$  for  $x \in [0, 1]$

Jump rates between  $\begin{cases} i \rightarrow i+1 : & \eta_i(1 - \eta_{i+1}) \exp\left(\frac{1}{N}E\left(\frac{i}{N}\right)\right) \\ i+1 \rightarrow i : & \eta_{i+1}(1 - \eta_i) \exp\left(-\frac{1}{N}E\left(\frac{i}{N}\right)\right) \end{cases}$

Expression for the two-point correlation function.

The battery model can be recovered when  $E$  converges to a Dirac.

Zero range with battery

Driven by reservoirs. [De Masi, Ferrari]

Product measure with local density  $\bar{\rho}(i)$

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Product measure with local density  $\bar{\rho}(i)$  conditioned to a fixed density

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# Conclusion

- Long range correlations in non-equilibrium models
- Long range correlations and non-equilibrium phase transitions
- Equivalence of ensembles