Anomalous diffusion for a class of systems with two conservation laws

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- The nature and the exponents corresponding to this anomalous diffusion are unknown. Theoretical and numerical studies are controversial. Universality or dependance on the interaction potential?
- We will discuss here a simple system of coupled differential equations which have similar behaviors.

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- Let $V : \mathbb{R} \to [0, +\infty)$ be a smooth potential and consider the set of coupled differential equations

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- We interpret it as a fluctuating interface. The energy $\sum_x V(\eta_x)$ and the volume $\sum_x \eta_x$ are conserved by the dynamics:

$$\frac{d}{dt} V(\eta_x(t)) = -\nabla \left[j^e_{x-1,x}(t) \right], \quad \frac{d}{dt}\eta_x = -\nabla \left[j^v_{x-1,x}(t) \right]$$

with the associated instantaneous currents

$$j^e_{x,x+1} = -V'(\eta_x) V'(\eta_{x+1}), \quad j^v_{x,x+1} = -\left[V'(\eta_{x+1}) + V'(\eta_x) \right]$$

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- **Harmonic potential**: $V(\eta) = \eta^2$. The dynamics is linear and solvable by Fourier transform:

$$\frac{d\widehat{\eta}}{dt}(t,k) = i\omega(k)\,\widehat{\eta}(t,k), \quad \widehat{\eta}(t,k) = \sum_{x\in\mathbb{Z}} \eta_x(t)\, e^{2i\pi kx}.$$

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- The total energy current $J^e = \sum_{x\in\mathbb{Z}} j^e_{x,x+1}$ takes the simple form

$$J^e = \int_{\mathbb{T}} v_{\rm g}(k) E_k\, dk, \quad v_{\rm g}(k) = \omega'(k) = -4\pi\cos(2\pi k).$$

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- **Exponential potential**: $V(\eta) = e^{-\eta} + \eta - 1$. The system is completely integrable and known as the Kac-Van-Moerbecke (KVM) system. It is related to the Toda lattice, i.e. chain of oscillators with exponential interactions, which is also completely integrable.
- **Cubic,quartic potentials...**: $V(\eta) = \eta^2 + \alpha\eta^3 + \beta\eta^4$.

Equilibrium measures and thermodynamic relations

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  is invariant for the dynamics.
- Let $e(\beta,\lambda) = \mu_{\beta,\lambda}(V(\eta_0))$ and $v(\beta,\lambda) := \mu_{\beta,\lambda}(\eta_0)$ be the averaged energy and volume. Under suitable conditions, there is a one-to-one correspondence between the chemical potentials (β,λ) and the conserved quantities (e,v) through the thermodynamic entropy; we write $\beta := \beta(e,v)$, $\lambda := \lambda(e,v)$.

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- Start the system with the initial local Gibbs equilibrium state:

$$d\mu^N_{\beta_0,\lambda_0}(\eta) = \prod_{x \in \mathbb{T}_N} \frac{\exp\{-\beta_0(x/N)V(\eta_x) - \lambda_0(x/N)\eta_x\}}{Z(\beta_0(x/N), \lambda_0(x/N))} d\eta_x,$$

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- We expect that at time tN the system is close to a local Gibbs equilibrium state associated to macroscopic energy-volume profiles $(\mathfrak{e}_t, \mathfrak{v}_t)$, ie.

$$\eta(tN) \sim_{\mathrm{Law}} \prod_{x \in \mathbb{T}_N} \frac{\exp\{-\beta_t(x/N)V(\eta_x) - \lambda_t(x/N)\eta_x\}}{Z(\beta_t(x/N), \lambda_t(x/N))}\, d\eta_x,$$

with (β_t, λ_t) the chemical potentials associated to $(\mathfrak{e}_t, \mathfrak{v}_t)$.

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- In the sequel we restrict us to the (small but macroscopic) time interval where a smooth solution exists.

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Definition

We say that the infinite volume dynamics $\{\eta_x(t)\,;\, x \in \mathbb{Z}, t \geq 0\}$ is ergodic if the only time and space invariant probability measures ν on $\mathbb{R}^{\mathbb{Z}}$ with finite entropy density are mixtures of $\{\mu_{\beta,\lambda}, \beta > 0, \lambda \in \mathbb{R}\}$.

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- We introduce the empirical energy and empirical volume ($q \in [0,1)$)

$$\begin{pmatrix} \mathcal{E}_N(t,q) \\ \mathcal{V}_N(t,q) \end{pmatrix} = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \mathbf{1}_{[x/N,(x+1)/N)}(q) \begin{pmatrix} V(\eta_x(tN)) \\ \eta_x(tN) \end{pmatrix}.$$

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- Following ideas developed in [Olla-Varadhan-Yau'93], we have

Proposition

If the infinite volume dynamics is ergodic then the empirical energy and the empirical volume converges, in the smooth regime, to the system of conservation laws

$$\begin{cases} \partial_t \mathfrak{e} - \partial_q \tau^2 = 0, \\ \partial_t \mathfrak{v} - 2\partial_q \tau = 0. \end{cases}$$

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The proof relies on arguments developed in [Fritz-Funaki-Lebowitz'94] in the context of chains of anharmonic oscillators.

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The stochastic energy-volume conserving dynamics $\{\hat{\eta}(t)\}$ is ergodic. In the hyperbolic time scaling, the empirical energy and empirical volume converges (in the smooth regime), to

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Observe that the noise does not affect the form of the hydrodynamic equations.
The derivation beyond the shocks is considerably more difficult (cf. [Fritz-Tóth' 04], not applicable here).

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- If we have a super diffusive behavior, it evolves in a time scale of order $tN^{1+\alpha}$, $\alpha < 1$.
- A more tractable quantity than the fluctuations field is the diffusivity $\mathcal{D}_{\beta,\lambda}(t)$.

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The normalized current is

$$\widehat{J}_{x,x+1} = J_{x,x+1} - \mathfrak{J}(\bar{\xi}) - (D\mathfrak{J})(\bar{\xi})\left(\xi_x - \bar{\xi}\right).$$

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Proposition

If $V(r) = r^2$ (harmonic) or $V(r) = e^{-r} + r - 1$ (KVM system) then the transport is ballistic, in the sense that

$$\mathcal{D}_{\beta,\lambda}(t) \ge c_{\lambda,\beta}\, t$$

Diffusivity

Theorem (B., Stoltz'11)

Consider the stochastic energy-volume conserving model $\hat{\eta}(t)$ with harmonic potential. Then, the diffusivity is of order $\sqrt{t}$ (it can be explicitly computed), i.e. the system is super diffusive.

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Consider the stochastic energy-volume conserving model $\hat{\eta}(t)$ with harmonic potential. Then, the diffusivity is of order $\sqrt{t}$ (it can be explicitly computed), i.e. the system is super diffusive.

The proof relies on arguments developed in [Basile-B.-Olla '06] for chain of oscillators with energy-momentum conserving noise.

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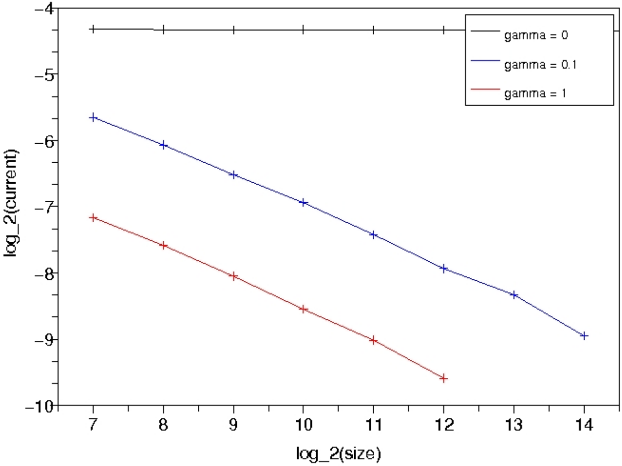
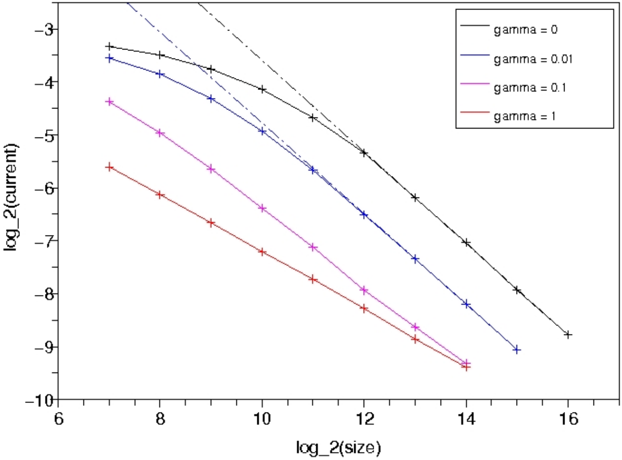
- We are interested in the dependance with respect to N of the averaged energy flux $\langle j^e_{x,x+1} \rangle_{ss,\gamma}$. A diffusive behavior corresponds to $\langle j^e_{x,x+1} \rangle_{ss,\gamma} \sim N^{-1}$; a super diffusive behavior to $\langle j^e_{x,x+1} \rangle_{ss,\gamma} \sim N^{-1+\alpha}$, $0 < \alpha \leq 1$.

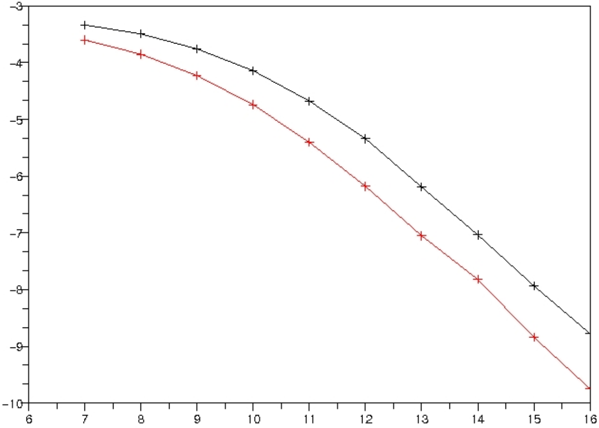
Numerical simulations: $V(r) = r^2$



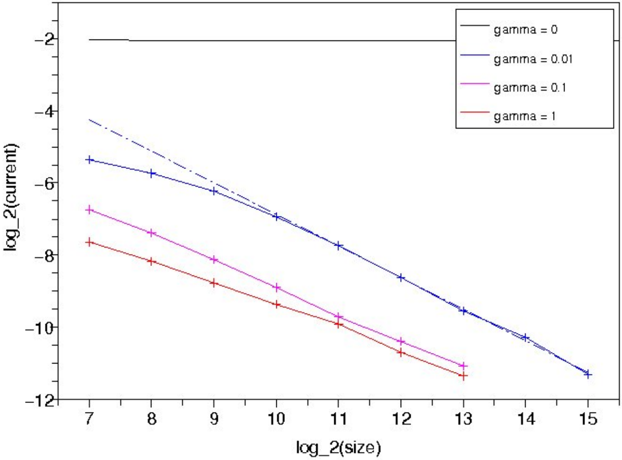
Numerical simulations: $V(r) = r^2 + r^4$



Numerical simulations: $V(r) = r^2 + r^3 + r^4$ / $V(r) = r^2 + r^4$



Numerical simulations: $V(r) = e^{-r} + r - 1$



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- More surprising is the fact that α increase with γ.
- A similar behavior has been observed for chains of oscillators perturbed by an energy-momentum conserving noise [Basile et al. '07, Iacobucci et al. '10].

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For any $\gamma > 0$, the diffusivity $\mathcal{D}_{\lambda,\beta}(t)$ is at least of order $t^{1/4}$ and at most of order $t^{1/2}$, i.e. it is super diffusive. Moreover, until time scaling $tN^{4/3}$, energy fluctuations are trivial.

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Observe that in this case, the system of conservation laws is

$$\begin{cases} \partial_t \mathfrak{e} - \partial_q (\mathfrak{v} - \mathfrak{e})^2 = 0, \\ \partial_t \mathfrak{v} - 2\partial_q (\mathfrak{v} - \mathfrak{e}) = 0. \end{cases}$$

Hence, $(\mathfrak{e} - \mathfrak{v} + 1)$ evolves according to the autonomous Burgers equation.

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- For the conserving stochastic model, prove the superdiffusive behavior for a general class of potentials.
- For the conserving stochastic model, prove that the divergence exponent depends (or not depends) on γ.
- Randomness can also be introduced in the model:

$$m_x d\eta_x = V'(\eta_{x+1}) - V'(\eta_{x-1})$$

  What is the effect of the randomness on the transport properties? Is it a normal conductor as soon as V is anharmonic?

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- The equations of motion can be rewritten as

$$
\begin{aligned}
dr_i &= \Big(V'(p_i) - V'(p_{i-1})\Big)dt,\\
dp_i &= \Big(V'(q_{i+1} - q_i) - V'(q_i - q_{i-1})\Big)dt.
\end{aligned}
$$

Correspond to the Hamiltonian:

$$
H(q,p) = \sum_i V(q_i - q_{i-1}) + \sum_i V(p_i),
$$