Behavioural Portfolio Choice

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This Talk ...

- Overview of recent progress on analytical/quantitative treatment of behavioural finance in asset allocation
- Highlight major challenges, and solutions
- Demonstrate that the solutions lead to new problems in both finance and mathematics

Expected Utility Theory

- Expected Utility Theory (EUT): To evaluate gambles (random variables, lotteries) and form preference
- Foundation laid by von Neumann and Morgenstern (1947)
- Axiomatic approach: completeness, transivity, continuity and independence
- Risk preference representable by expectation of utility function

Human Judgement Implied by Expected Utility Theory

- EUT: Dominant model for decision making under uncertainty, including financial asset allocation
- Basic tenets of human judgement implied by EUT in the context of asset allocation:
  - **Frame of problem**: Investors' preference is independent of how problem is stated (described, or framed)
  - **Source of satisfaction**: Investors evaluate assets according to final wealth
  - **Attitude towards risk**: Investors are always risk averse (concave utility)
  - **Beliefs about future**: Investors are able or willing to objectively evaluate probabilities of future returns

Frame Independence

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- People can see through all the different ways a problem might be described
- Frame independence: the foundation of neoclassical economics/finance
- Merton Miller: "If you transfer a dollar from your right pocket to your left pocket, you are no wealthier. Franco (Modigliani) and I proved that rigorously"

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- I paid immediately ... *filled with gratitude and joy*

A Trivial Game ...

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  - A: 75% chance to lose $7,600, 25% chance to gain $2,400

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  - B2: lose \$7,500 for sure

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  - B2: lose \$7,500 for sure
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- Choose between
  - A2: 75% chance to lose \$10,000, 25% chance to lose nothing
  - B2: lose \$7,500 for sure
  - A2 was more popular
- "$A \equiv A_1 + A_2 > B_1 + B_2 \equiv B$"!

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Loss Aversion: Losses Matter More

Paul Samuelson (1963): Choose between

- A: **Win** \$10,000 with 50% chance and **lose** \$5,000 with 50% chance
- B: Don't take this bet
- B was more popular
- **Loss aversion**: pain from a loss is more than joy from a gain of the same magnitude

Probability Distortion (Weighting): Lottery Ticket and Insurance

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Probability Distortion (Weighting): Lottery Ticket and Insurance

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  - A: Win \$5,000 with 0.1% chance
  - B: Win \$5 with 100% chance
  - A was more popular
- Choose between
  - A: Lose \$5,000 with 0.1% chance
  - B: Lose \$5 with 100% chance
  - This time: B was more popular

EUT Turned Upside Down

- **Frame of problem**: Investors' preference may be dependent of how problem is framed
- **Source of satisfaction**: Investors do not always evaluate assets according to final wealth
- **Attitude towards risk**: Investors are not always risk averse
- **Beliefs about future**: Investors are unable or sometimes unwilling to objectively evaluate probabilities of future returns

Cumulative Prospect Theory (CPT)

- **Reference point** (Kahneman and Tversky 1979) or **customary wealth** (Markowitz 1952) - choice of reference point is part of the framing

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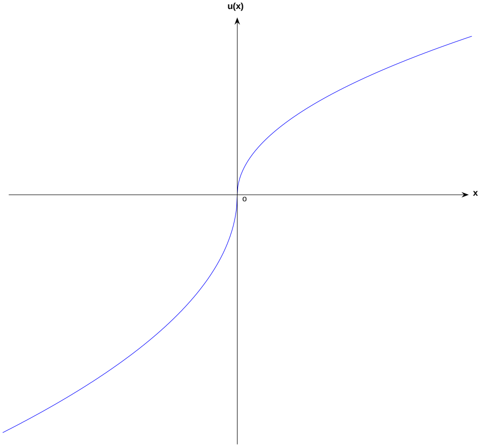
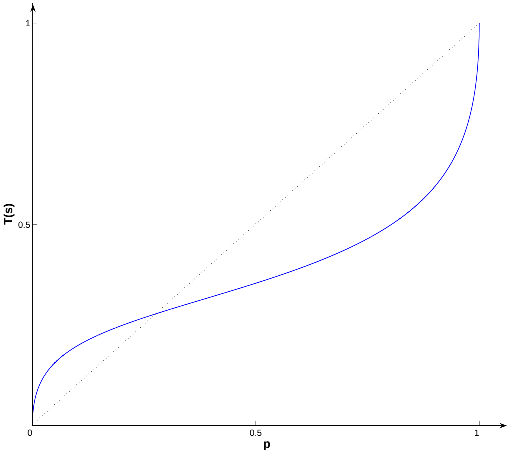
- **Probability distortions**
- Backbone of behavioral economics/finance theory

S-shaped Function



Probability Distortion Function



Behavioural Portfolio Choice *à la* Prospect Theory

$$
\begin{array}{ll}
\underset{X}{\text{Max}} & \int_0^{\infty} w_+ \left(P\left(u_+\left((X-B)_+\right) > x\right)\right) dx \\
& \quad - \int_0^{\infty} w_- \left(P\left(u_-\left((X-B)_-\right) > x\right)\right) dx \\
\text{Subject to} & E[\rho X] = x_0
\end{array}
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Berkelaar, Kouwenberg and Post (2004), Jin and Zhou (2008)

Mathematical Challenges

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Mathematical Challenges

- Expected utility: stochastic control/HJB, martingale/convex duality
- Prospect model: ???
  - Nonconcave in X: convex duality fails
  - Nonlinear expectation with Choquet integration: time-consistency or HJB fails

Our Model (Again)

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Deriving Optimal Solution: Divide and Conquer

We do "**divide and conquer**"

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Gain Part Problem (GPP): A problem with parameters (A, x_+):

$$\begin{array}{ll} \text{Maximize} & V_+(X) = \int_0^{+\infty} w_+(P\{u_+(X) > y\})dy \\ \text{subject to} & \begin{cases} E[\rho X] = x_+, \\ X \geq 0, \text{ a.s.}, \\ X = 0, \text{ a.s. on } A^C, \end{cases} \end{array} \tag{1}$$

where $x_+ \geq (x_0 - E[\rho B])^+$ (≥ 0) and $A \in \mathcal{F}_T$ with $P(A) \leq 1$

- Define its optimal value to be $v_+(A, x_+)$

Divide and Conquer (Cont'd)

Loss Part Problem (LPP): A problem with parameters (A, x_+):

$$\begin{array}{ll} \text{Minimise} & V_-(X) = \int_0^{+\infty} w_-(P\{u_-(X) > y\})dy \\ \text{subject to} & \begin{cases} E[\rho X] = x_+ - x_0 + E[\rho B], \\ X \geq 0, \text{ a.s.}, \\ X = 0, \text{ a.s. on } A, \\ X \text{ is bounded a.s.}, \end{cases} \end{array} \tag{2}$$

where $x_+ \geq (x_0 - E[\rho B])^+$ and $A \in \mathcal{F}_T$ with $P(A) \leq 1$

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where $x_+ \geq (x_0 - E[\rho B])^+$ and $A \in \mathcal{F}_T$ with $P(A) \leq 1$

- Define its optimal value to be $v_-(A, x_+)$

Then, in Step 2 we solve

$$\begin{array}{ll} \text{Maximize} & v_+(A, x_+) - v_-(A, x_+) \\ \text{subject to} & \begin{cases} A \in \mathcal{F}_T, \ x_+ \geq (x_0 - E[\rho B])^+, \\ x_+ = 0 \text{ when } P(A) = 0, \\ x_+ = x_0 - E[\rho B] \text{ when } P(A) = 1. \end{cases} \end{array} \tag{3}$$

Yes It Works

Theorem

Given X^*, define $A^* := \{\omega : X^* \geq 0\}$ and $x_+^* := E[\rho(X^*)^+]$. Then X^* is optimal for the behavioural problem iff (A^*, x_+^*) are optimal for Problem (3) and $(X^*)^+$ and $(X^*)^-$ are respectively optimal for Problems (1) and (2) with parameters (A^*, x_+^*).

Solution Flow

- Solve GPP for any parameter (A, x_+), getting optimal solution $X_+(A, x_+)$ and optimal value $v_+(A, x_+)$

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- Solve Step 2 problem and get optimal (A^*, x_+^*)

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- Solve LPP for any parameter (A, x_+), getting optimal solution $X_-(A, x_+)$ and optimal value $v_-(A, x_+)$
- Solve Step 2 problem and get optimal (A^*, x_+^*)
- Then $X_+(A^*, x_+^*) - X_-(A^*, x_+^*)$ solves the behavioral model

Simplification

Recall Step 2 problem

$$v_+(A, x_+) - v_-(A, x_+)$$

optimisation over a set of *random events* A: hard to handle

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optimisation over a set of *random events* A: hard to handle

Theorem

For any feasible pair (A, x_+) *of Problem (3), there exists* $c \in [\underline{\rho}, \bar{\rho}]$ *such that* $\bar{A} := \{\omega : \rho \leq c\}$ *satisfies*

$$v_+(\bar{A}, x_+) - v_-(\bar{A}, x_+) \geq v_+(A, x_+) - v_-(A, x_+). \qquad (4)$$

Simplification

Recall Step 2 problem

$$v_+(A, x_+) - v_-(A, x_+)$$

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- Use $v_+(c, x_+)$ and $v_-(c, x+)$ to denote $v_+(\{\omega : \rho \leq c\}, x_+)$ and $v_-(\{\omega : \rho \leq c\}, x_+)$ respectively

Simplification (Cont'd)

- Problem (3) is equivalent to

$$
\begin{array}{ll}
\text{Maximize} & v_+(c, x_+) - v_-(c, x_+) \\
\text{subject to} & \begin{cases} \underline{\rho} \le c \le \bar{\rho}, \quad x_+ \ge x_0^+, \\ x_+ = 0 \text{ when } c = \underline{\rho}, \\ x_+ = x_0 \text{ when } c = \bar{\rho}. \end{cases}
\end{array}
\tag{5}
$$

Choquet Maximisation and Beyond

GPP specialises a general maximisation problem involving Choquet integral:

$$\begin{array}{ll} \underset{X}{\text{Maximise}} & C(X) := \int_0^{+\infty} w(P(u(X) > y))dy \\ \text{subject to} & E[\rho X] = a, \quad X \geq 0, \end{array} \tag{6}$$

where $a \geq 0$, $w(\cdot) : [0,1] \mapsto [0,1]$ non-decreasing, differentiable with $w(0) = 0$, $w(1) = 1$, and $u(\cdot)$ strictly concave, strictly increasing, twice differentiable with $u(0) = 0$, $u'(0) = +\infty$, $u'(+\infty) = 0$

Quantile Formulation

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  - *monotonicity* of supremium value with respect to initial wealth a

Rewriting $C(X)$

$$\begin{aligned} C(X) &= \int_0^{+\infty} w(P(u(X) > y))dy \\ &= \int_0^{+\infty} u(x)d[-w(1-F_X(x))] \\ &= \int_0^{+\infty} u(x)w'(1-F_X(x))dF_X(x) \\ &= \int_0^{1} u(G(z))w'(1-z)dz \\ &= E[u(G(Z))w'(1-Z)], \end{aligned} \tag{7}$$

where $Z \sim U(0,1)$ and $G = F_X^{-1}$ (quantile function)

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- We change decision variable from X (r.v.) to G (quantile)
- ... by which we recover linear expectation and concavity (if $u(\cdot)$ is concave)!

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- Hence

$$E[\rho X] = a \Leftrightarrow E\left[F_\rho^{-1}(1-Z)G(Z)\right] = a$$
$$\Leftrightarrow \int_0^1 F_\rho^{-1}(1-z)G(z)dz = a$$

Choquet Maximisation Rewritten

Rewrite Problem (6) as follows

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Maximise}} & \tilde{C}(G(\cdot)) = \int_0^1 u(G(z))w'(1-z)dz \\ \text{subject to} & \int_0^1 F_\rho^{-1}(1-z)G(z)dz = a, \; G(\cdot) \in \mathbb{G}, \; G(0+) \geq 0. \end{array} \tag{8}$$

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- Accommodating $G(\cdot) \in \mathbb{G}$ can be technically tricky
- If $G^*(\cdot)$ is optimal then $X^* = G^*(1 - F_\rho(\rho))$: optimal terminal cash flow is anti-comonotonic w.r.t. pricing kernel ρ

Quantile Formulation: History

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- Convex/concave distortion: Schied (2004, 2005), Dana (2005), Carlier and Dana (2005)
- S-shaped distortion: Jin and Zhou (2008)
- A general framework developed in He and Zhou (2009) for possibly non-concave utility function and non-convex/concave distortions

Also Covers...

- Goal achieving (Browne 1999, 2000; He and Zhou 2009)
- Yaari's model (He and Zhou 2009)
- SP/A model (He and Zhou 2010)
- Mean-risk model with coherent risk measure (He, Jin and Zhou 2010)
- Markowitz problem with probability distortions (Bi, Zhong and Zhou 2010)
- "Distorted" optimal stopping (Xu and Zhou 2010)
- Insurance contract with rank dependent utility (Bernard, He, Yan and Zhou 2010)

Choquet Minimization: Combinatorial Optimisation in Function Spaces

LPP specialises a general Choquet minimisation problem:

$$\begin{array}{ll} \underset{X}{\text{Minimise}} & C(X) := \int_0^{+\infty} w(P(u(X) > y))dy \\ \text{subject to} & E[\rho X] = a, \;\; X \geq 0, \end{array} \tag{9}$$

where $a \geq 0$, $w(\cdot) : [0,1] \mapsto [0,1]$ non-decreasing, differentiable with $w(0) = 0$, $w(1) = 1$, and $u(\cdot)$ strictly increasing, concave, strictly concave at 0, with $u(0) = 0$

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- To *minimise* a *concave* functional: wrong direction!

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- Solution must have a very different structure compared with the gain counterpart
- Lagrange fails
- Solution should be a "corner point solution": essentially a combinatorial optimisation in an infinite dimensional space

Characterising Corner Point Solutions

Proposition

The optimal solution to (9), if it exists, must be in the form $G^*(z) = q(b)\mathbf{1}_{(b,1)}(z)$, $z \in [0,1)$, *with some* $b \in [0,1)$ *and* $q(b) := \frac{a}{E[\rho \mathbf{1}_{\{F_\rho(\rho) > b\}}]}$. *Moreover, in this case, the optimal solution is* $X^* = G^*(F_\rho(\rho))$.

- One only needs to find an optimal *number* $b \in [0,1)$

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- One only needs to find an optimal *number* $b \in [0,1)$
- ... which motivates introduction of the following problem

$$\begin{array}{ll} \underset{b}{\text{Minimise}} & f(b) := \int_0^1 u(G(z))w'(1-z)dz \\ \text{subject to} & G(\cdot) = \frac{a}{E[\rho \mathbf{1}_{\{F_\rho(\rho)>b\}}]}\mathbf{1}_{(b,1]}(\cdot), \;\; 0 \le b < 1. \end{array}$$

Solving Loss Part Problem

Theorem

Problem (9) admits an optimal solution if and only if the following problem

$$\min_{0 \le c < \bar{\rho}} u\left(\frac{a}{E[\rho \mathbf{1}_{\{\rho > c\}}]}\right) w(P(\rho > c))$$

admits an optimal solution c^*, in which case the optimal solution to (9) is $X^* = \frac{a}{E[\rho \mathbf{1}_{\{\rho > c^*\}}]} \mathbf{1}_{\rho > c^*}$.

Jin and Zhou's Solution

Consider a mathematical programme in (c, x_+):

$$\begin{aligned}\text{Maximise} \quad & E\left[u_+\left((u'_+)^{-1}\left(\frac{\lambda(c,x_+)\rho}{w'_+(F(\rho))}\right)\right)w'_+(F(\rho))\mathbf{1}_{\rho\le c}\right] \\ & -u_-\left(\frac{x_+-(x_0-E[\rho B])}{E[\rho\mathbf{1}_{\rho>c}]}\right)w_-(1-F(c))\end{aligned}$$

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where $\lambda(c, x_+)$ satisfies $E\left[(u'_+)^{-1}\left(\frac{\lambda(c,x_+)\rho}{w'_+(F(\rho))}\right)\rho\mathbf{1}_{\rho\le c}\right] = x_+$

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Optimal solution (Jin and Zhou 2008) (under mild technical conditions)

$$X^* = \left[(u'_+)^{-1}\left(\frac{\lambda\rho}{w'_+(F(\rho))}\right) + B\right]\mathbf{1}_{\rho\le c^*} - \left[\frac{x_+^* - (x_0 - E[\rho B])}{E[\rho\mathbf{1}_{\rho>c^*}]} - B\right]\mathbf{1}_{\rho>c^*}$$

Interpretations, Implications, and Applications

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- "Principal guaranteed fund"
- "Minimum compensation and bonus scheme" in HR management (Chang and Zhou 2010, Chang, Cvitanic, Zhou 2010)

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- Jin and Zhou (2009)

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- ... which suggests, from loss-control or regulatory perspective, a model with *a priori* bound on potential losses
- It would (indirectly) limit leverage and hence magnitude of greed
- The new model formulated and solved (Jin, Zhang, Zhou 2009)

$$
\begin{array}{ll}
\text{Maximize} & V(X-B) \\
\text{subject to} & \begin{cases} E[\rho X] = x_0 \\ X \geq B - L \\ X \text{ is an } \mathcal{F}_T - \text{random variable} \end{cases}
\end{array}
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- Hope and fear are measured via *curvatures* of probability distortions at lower and higher ends respectively
- Via the quantile formulation, we have solved this model completely
- Extreme fear prevents agent from taking too much risks and he has to secure a positive payoff level

Quantifying Hope and Fear

- A portfolio choice problem in continuous time featuring hope, fear and aspiration
- Hope and fear are present simultaneously, captured by a reverse S-shaped distortion function
- Hope and fear are measured via *curvatures* of probability distortions at lower and higher ends respectively
- Via the quantile formulation, we have solved this model completely
- Extreme fear prevents agent from taking too much risks and he has to secure a positive payoff level
- The hope is reflected by the gambling nature of the strategy

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- These functions are dramatically different from those in a neoclassical model to systematically capture certain aspects of irrationalities such as risk-seeking, and hope and fear

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- We use CPT and specific value functions as the carrier for exploring the "predictable irrationalities"
- *Quantifying behavioural finance*: research is in its infancy, yet potential is unlimited – or so we believe

Collaborators and Papers

Hanqing Jin (Oxford), Xuedong He (Columbia), Zuoquan Xu (HK Poly), Jaksa Cvitanic (CalTech), Hualei Chang (Goldman Sachs), Yifei Zhong (Oxford), Song Zhang (Peking U), Junna Bi (Nankai), Carol Bernard (Waterloo), Jia-an Yan (CAS), Thaleia Zariphoupoulou (Oxford/UT Austin)

- H. Jin and X. Zhou, "Behavioral portfolio selection in continuous time", *Mathematical Finance*, Vol. 18 (2008), pp. 385-426.
- X. He and X. Zhou, "An analytical treatment of portfolio choice under prospect theory", to appear in *Management Science*.
- X. He and X. Zhou, "Portfolio choice via quantiles", to appear in *Mathematical Finance*.
- H. Jin and X. Zhou, "Greed, leverage, and potential losses: A prospect theory perspective", to appear in *Mathematical Finance*.

Thank You!

http://people.maths.ox.ac.uk/ zhouxy/article.htm