Constructing Markov Processes with Dependent Jumps by Multivariate Subordination: Applications to Multi-Name Credit-Equity Modeling

Fields Institute

(Fields Quantitative Finance Seminar)

Rafael Mendoza-Arriaga
McCombs School of Business

Joint work with: Vadim Linetsky

Introduction

- We develop a new class of multi-name unified credit-equity models that jointly model the stock prices of multiple firms, as well as their default events,

Introduction

- We develop a new class of multi-name unified credit-equity models that jointly model the stock prices of multiple firms, as well as their default events,
  - We construct a multi-dimensional Markov semimartingale by applying a multivariate subordination of jump-to-default extended constant elasticity of variance (JDCEV) diffusions.

Introduction

- We develop a new class of multi-name unified credit-equity models that jointly model the stock prices of multiple firms, as well as their default events,
  - We construct a multi-dimensional Markov semimartingale by applying a multivariate subordination of jump-to-default extended constant elasticity of variance (JDCEV) diffusions.
- Each of the stock prices experiences state-dependent jumps with the leverage effect (arrival rates of large jumps increase as the stock price falls), including the possibility of a jump to zero (jump to default).

Introduction

- We develop a new class of multi-name unified credit-equity models that jointly model the stock prices of multiple firms, as well as their default events,
  - We construct a multi-dimensional Markov semimartingale by applying a multivariate subordination of jump-to-default extended constant elasticity of variance (JDCEV) diffusions.
- Each of the stock prices experiences state-dependent jumps with the leverage effect (arrival rates of large jumps increase as the stock price falls), including the possibility of a jump to zero (jump to default).
- Some of the jumps are idiosyncratic to each firm, while some are either common to all firms (systematic), or common to a subgroup of firms.

Introduction

- We develop a new class of multi-name unified credit-equity models that jointly model the stock prices of multiple firms, as well as their default events,
  - We construct a multi-dimensional Markov semimartingale by applying a multivariate subordination of jump-to-default extended constant elasticity of variance (JDCEV) diffusions.
- Each of the stock prices experiences state-dependent jumps with the leverage effect (arrival rates of large jumps increase as the stock price falls), including the possibility of a jump to zero (jump to default).
- Some of the jumps are idiosyncratic to each firm, while some are either common to all firms (systematic), or common to a subgroup of firms.
- For the two-firm case, we obtain analytical solutions for credit derivatives and equity derivatives, such as basket options, in terms of eigenfunction expansions associated with the relevant subordinated semigroups.

Unifying Credit-Equity Models

The Jump to Default Extended Diffusions (JDED)

Before moving on to use time changes to construct models with jumps, we review the Jump-to-Default Extended Diffusion framework (JDED)

Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases}$$

(ζ default time)

We assume *absolute priority:* the stock holders do not receive any recovery in the event of default

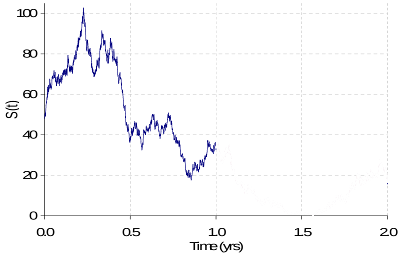
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases}$$

(ζ default time)

Stock Price



Model the pre-default stock dynamics under an EMM $\mathbb{Q}$ as:

$$d\tilde{S}_t = [\mu + \ k(\tilde{S}_t)\ ]\tilde{S}_t dt + \sigma(\tilde{S}_t)\,\tilde{S}_t dB_t$$

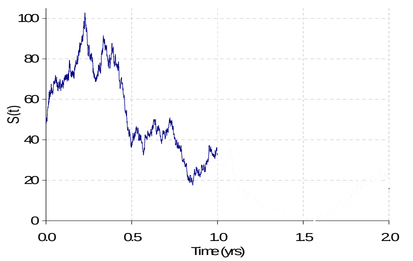
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \le t \end{cases}$$

(ζ default time)

Stock Price



Model the pre-default stock dynamics under an EMM $\mathbb{Q}$ as:

$$d\tilde{S}_t = [\underbrace{\mu} + k(\tilde{S}_t)]\tilde{S}_t dt + \sigma(\tilde{S}_t)\tilde{S}_t dB_t$$

$\Rightarrow \mu = r - q$. *Drift*: short rate r minus the dividend yield q

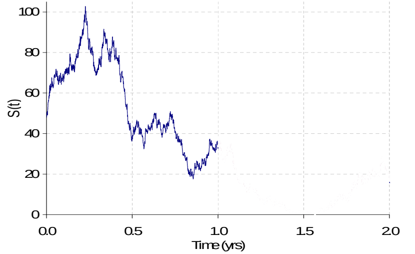
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \le t \end{cases}$$

(ζ default time)

Stock Price



Model the pre-default stock dynamics under an EMM $\mathbb{Q}$ as:

$$d\tilde{S}_t = [\mu + \ k(\tilde{S}_t)\ ]\tilde{S}_t dt + \underbrace{\sigma(\tilde{S}_t)} \tilde{S}_t dB_t$$

$\Rightarrow$ $\sigma(S)$. State dependent *volatility*

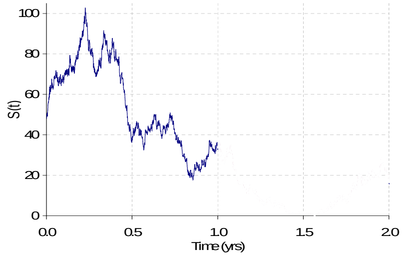
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases}$$

(ζ default time)

Stock Price



Model the pre-default stock dynamics under an EMM $\mathbb{Q}$ as:

$$d\tilde{S}_t = [\mu + \underbrace{k(\tilde{S}_t)}\,]\tilde{S}_t dt + \sigma(\tilde{S}_t)\,\tilde{S}_t dB_t$$

$\Rightarrow$ $k(S)$. State dependent *default intensity*

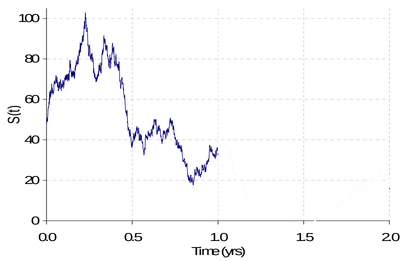
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \le t \end{cases}$$

(ζ default time)

Stock Price



Model the pre-default stock dynamics under an EMM $\mathbb{Q}$ as:

$$d\tilde{S}_t = [\mu + \underbrace{k(\tilde{S}_t)}\,]\tilde{S}_t dt + \sigma(\tilde{S}_t)\,\tilde{S}_t dB_t$$

$\Rightarrow$ $k(S)$. State dependent *default intensity*

- Compensates for the *jump-to-default* and ensures the discounted martingale property

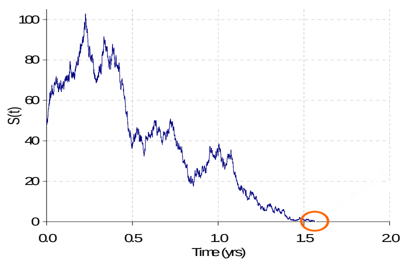
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases}$$

(ζ default time)

Stock Price



If the diffusion $\tilde{S}_t$ can hit zero:

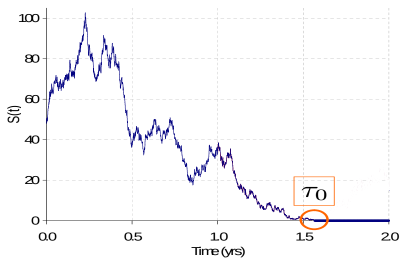
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases}$$

(ζ default time)

Stock Price



If the diffusion $\tilde{S}_t$ can hit zero:

$\Rrightarrow$ Bankruptcy at the first hitting time of zero,

$$\tau_0 = \inf \left\{ t : \tilde{S}_t = 0 \right\}$$

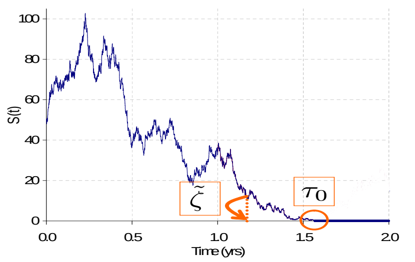
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \le t \end{cases}$$

(ζ default time)

Stock Price



Prior to τ_0 default could also arrive by a jump-to-default $\tilde{\zeta}$ with default intensity $k(\tilde{S})$,

$$\tilde{\zeta} = \inf\left\{ t \in [0, \tau_0] : \int_0^t k(\tilde{S}_u) \ge e \right\}, \quad e \approx Exp(1)$$

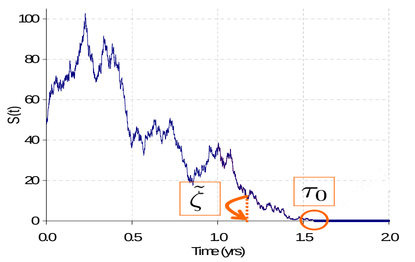
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases}$$

(ζ default time)

Stock Price



Prior to τ_0 default could also arrive by a jump-to-default $\tilde{\zeta}$ with default intensity $k(\tilde{S})$,

$$\tilde{\zeta} = \inf\left\{t \in [0, \tau_0] : \int_0^t k(\tilde{S}_u) \geq e\right\}, \quad e \approx Exp(1)$$

$\Rightarrow$ At time $\tilde{\zeta}$ the stock price S_t jumps to zero and the firm defaults on its debt

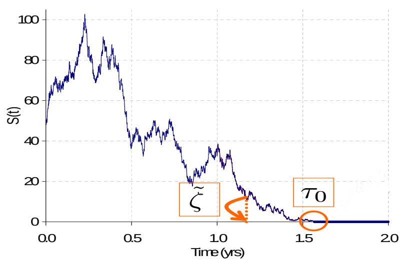
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases}$$

(ζ default time)

Stock Price



The default time ζ is the earliest of:

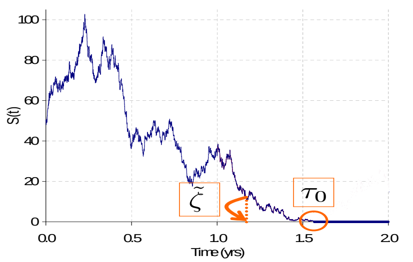
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases}$$

(ζ default time)

Stock Price



The default time ζ is the earliest of:

1. The stock hits level zero by diffusion: τ_0

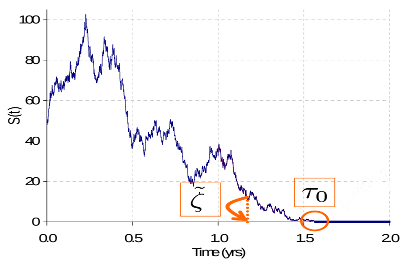
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases}$$

(ζ default time)

Stock Price



The default time ζ is the earliest of:

1. The stock hits level zero by diffusion: τ_0
2. The stock jumps to zero from a positive value: $\tilde{\zeta}$

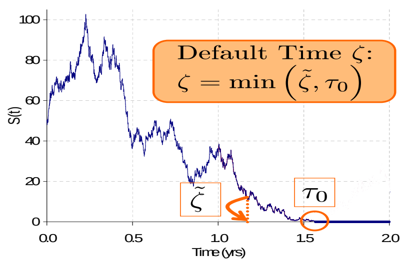
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases}$$

(ζ default time)

Stock Price



The default time ζ is the earliest of:

1. The stock hits level zero by diffusion: τ_0
2. The stock jumps to zero from a positive value: $\tilde{\zeta}$

$$\zeta = \min\left(\tilde{\zeta}, \tau_0\right)$$

Time Changed Process Construction

$$S_t = X_{\mathcal{T}_t}$$

- X_t is a background process (e.g. JDED)
- $\mathcal{T}_t$ is a random clock process independent of X_t

Time-Changed Process $S_t = X_{\mathcal{T}_t}$

Time Changed Process Construction

$$S_t = X_{\mathcal{T}_t}$$

- X_t is a background process (e.g. JDED)
- $\mathcal{T}_t$ is a random clock process independent of X_t

Random Clock $\{\mathcal{T}_t, t \geq 0\}$

Non-decreasing RCLL process starting at $\mathcal{T}_0 = 0$ and $\mathbb{E}[\mathcal{T}_t] < \infty$.

- We are interested in T.C. with analytically tractable Laplace Transform (LT):

$$\mathcal{L}(t, \lambda) = \mathbb{E}\left[e^{-\lambda \mathcal{T}_t}\right] < \infty$$

Lévy Subordinators with L.T. $\mathcal{L}(t, \lambda) = e^{-\phi(\lambda)t} \Rrightarrow$ induce jumps

Three Parameter Lévy measure:

$$\nu(ds) = Cs^{-Y-1}e^{-\eta s}ds$$

where $\quad C > 0, \;\; \eta > 0, \;\; Y < 1$

- C changes the time scale of the process (simultaneously modifies the intensity of jumps of all sizes)
- Y controls the small size jumps
- η defines the decay rate of big jumps

Examples of Lévy Subordinators

Three Parameter Lévy measure:

$$\nu(ds) = Cs^{-Y-1}e^{-\eta s}ds$$

$$\text{where} \qquad C > 0, \;\; \eta > 0, \;\; Y < 1$$

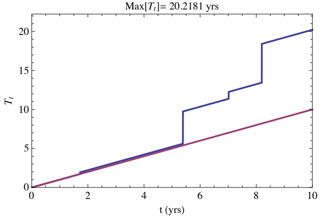
- C changes the time scale of the process (simultaneously modifies the intensity of jumps of all sizes)
- Y controls the small size jumps
- η defines the decay rate of big jumps

Lévy-Khintchine formula

$$\mathcal{L}(t, \lambda) = e^{-\phi(\lambda)t}$$

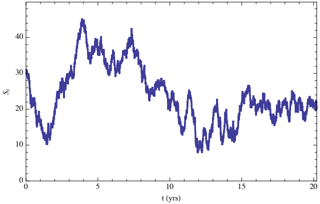
$$\text{where} \qquad \phi(\lambda) = \begin{cases} \gamma\lambda - C\Gamma(-Y)[(\lambda+\eta)^Y - \eta^Y], & Y \neq 0 \\ \gamma\lambda + C\ln(1+\lambda/\eta), & Y = 0 \end{cases}$$

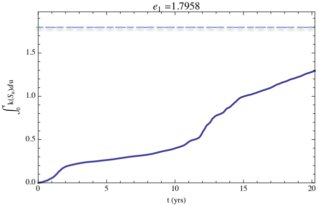
Time-Changed Process



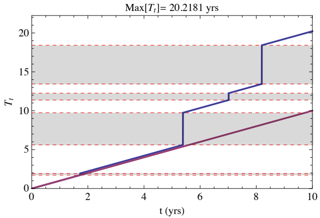
- T_t CPP with exp. arrival rate = 1/3 (per year) and exp. Jump size = 2 (yrs)
- The Time Changed Process is constructed by subordinating a JDCEV process with T_t as:

$$S_{Tt}=X(T_t)$$



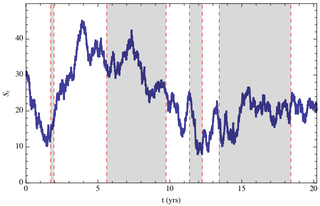


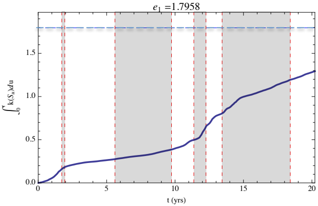
Time-Changed Process



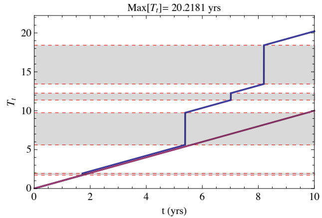
- T_t CPP with exp. arrival rate = 1/3 (per year) and exp. Jump size = 2 (yrs)
- The Time Changed Process is constructed by subordinating a JDCEV process with T_t as:

$$S_{T_t}=X(T_t)$$



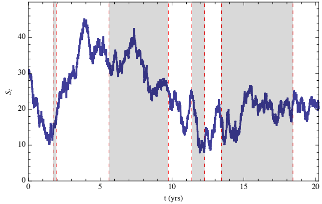


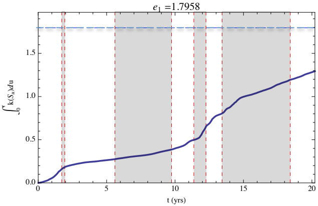
Time-Changed Process

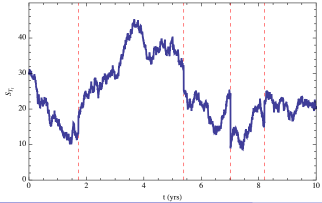


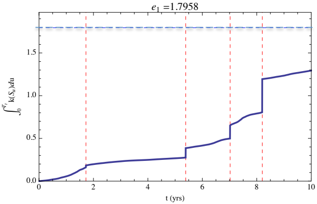
- T_t CPP with exp. arrival rate = 1/3 (per year) and exp. Jump size = 2 (yrs)
- The Time Changed Process is constructed by subordinating a JDCEV process with T_t as:

$$S_{T_t}=X(T_t)$$

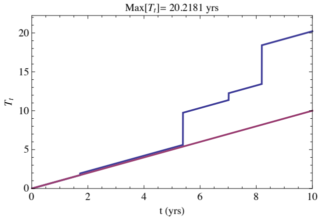








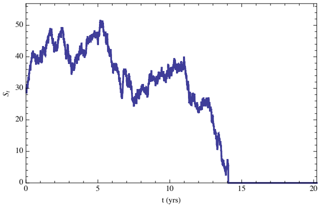
Time-Changed Process

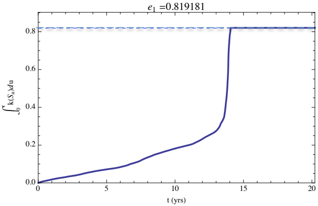


We kill the background process at:

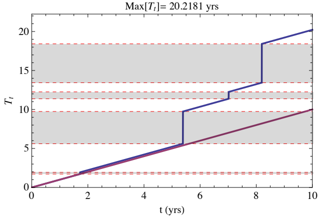
$$\zeta = \min(\zeta^*, \tau_0)$$

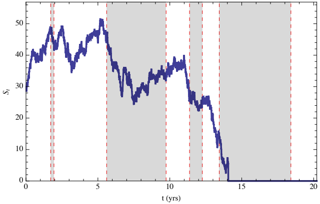
- *How about default for the Time-Changed process?*





Time-Changed Process



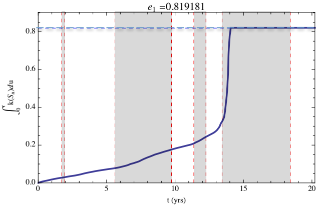


We kill the background process at:

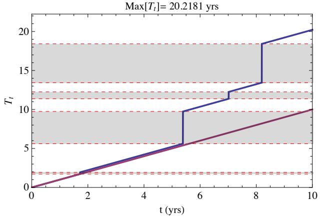
$$\zeta = \min(\zeta^*, \tau_0)$$

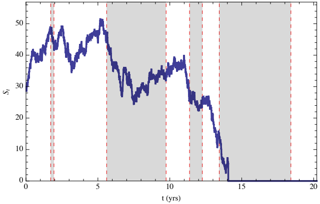
- *How about default for the Time-Changed process?*

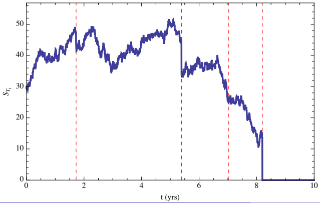
$$\tau_D = \inf\{ t : T_t \geq \zeta \}$$



Time-Changed Process







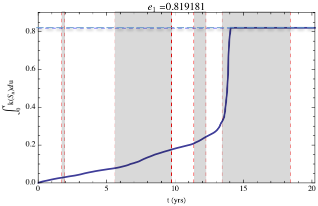
We kill the background process at:

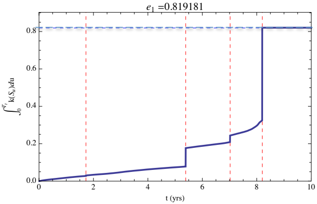
$$\zeta = \min(\zeta^*, \tau_0)$$

- *How about default for the Time-Changed process?*

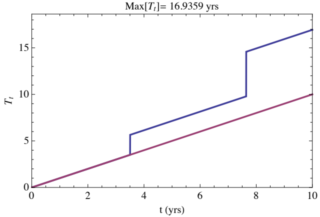
$$\tau_D = \inf\{ t : T_t \geq \zeta \}$$

In this case right after the jump time!





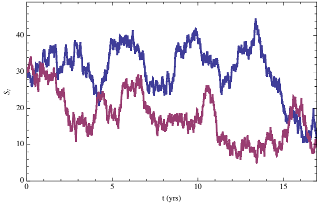
Time-Changed Process

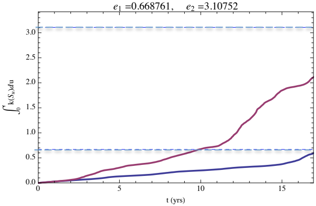


Multiple Firms* --- *The trivial case!

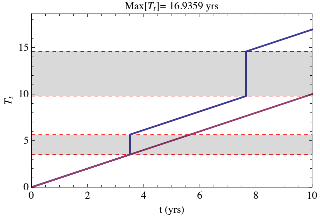
Take two firms subordinated with the same subordinator T_t:

- $S^1_t = X^1(T_t)$ *firm 1.*
- $S^2_t = X^2(T_t)$ *firm 2.*





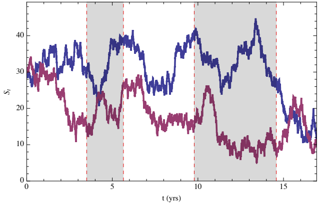
Time-Changed Process

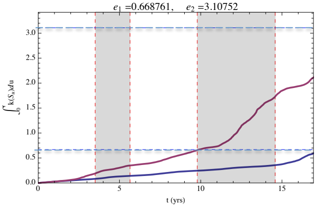


Multiple Firms* --- *The trivial case!

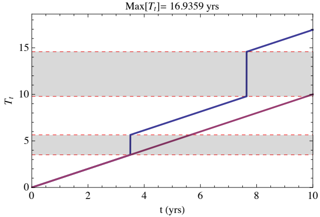
Take two firms subordinated with the same subordinator T_t:

- $S^1_t = X^1(T_t)$ *firm 1.*
- $S^2_t = X^2(T_t)$ *firm 2.*





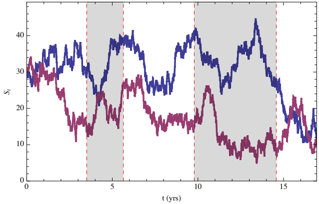
Time-Changed Process

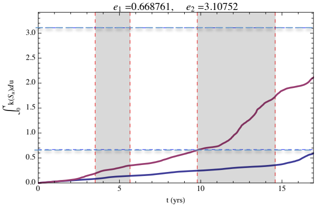


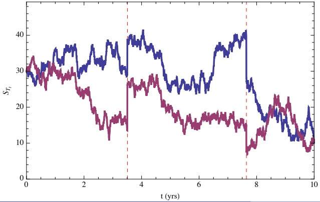
Multiple Firms --- The trivial case!

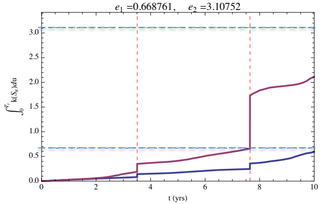
Take two firms subordinated with the same subordinator T_t:

- $S^1_t = X^1(T_t)$ *firm 1.*
- $S^2_t = X^2(T_t)$ *firm 2.*

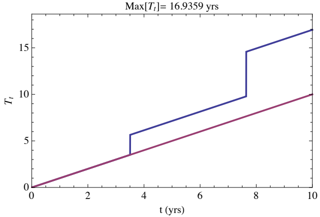


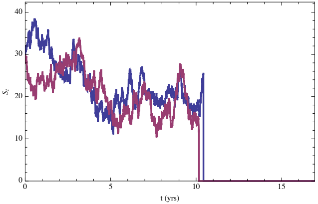






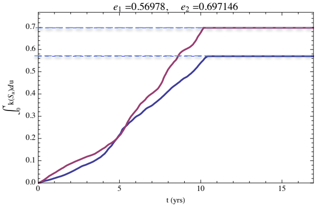
Time-Changed Process



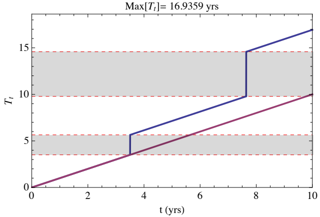


Multiple Firms* --- *The trivial case!

How about default?



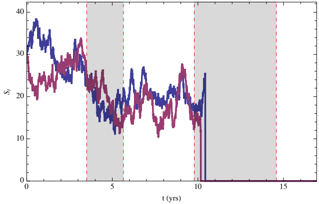
Time-Changed Process

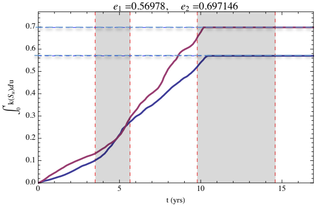


Multiple Firms* --- *The trivial case!

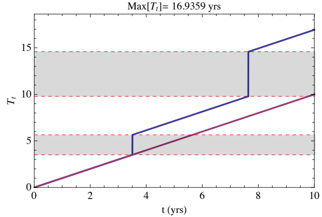
How about default?

- $\tau_D^1 = \inf \{ t : T_t \geq \zeta^1 \}$
- $\tau_D^2 = \inf \{ t : T_t \geq \zeta^2 \}$





Time-Changed Process

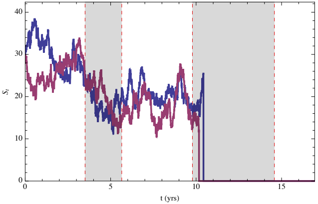


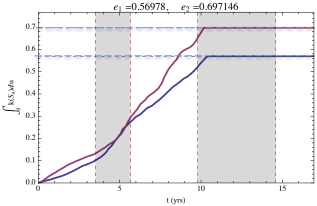
Multiple Firms --- The trivial case!

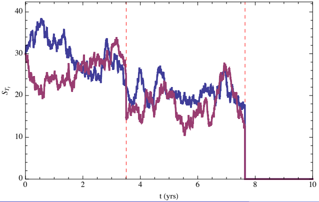
How about default?

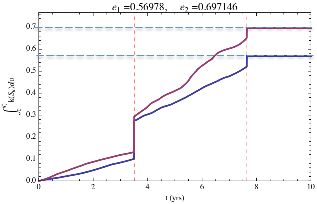
- $\tau_D^1 = \inf\{ t : T_t \geq \zeta^1 \}$
- $\tau_D^2 = \inf\{ t : T_t \geq \zeta^2 \}$

In this case they default simultaneously!









Multiple Firms – The **Not-So-Trivial** case

- Consider two firms, now running in two different random clocks

$$S_t^1 = X^1(\mathcal{T}_t^1) \quad \textit{firm 1}$$

$$S_t^2 = X^2(\mathcal{T}_t^2) \quad \textit{firm 2}$$

where $\mathcal{T}_t^i$ $i = 1, 2$; are dependent (correlated) subordinators.

Multiple Firms – The **Not-So-Trivial** case

- Consider two firms, now running in two different random clocks

$$S_t^1 = X^1(\mathcal{T}_t^1) \quad \textit{firm 1}$$

$$S_t^2 = X^2(\mathcal{T}_t^2) \quad \textit{firm 2}$$

  where $\mathcal{T}_t^i$ $i = 1, 2$; are dependent (correlated) subordinators.
- Note: although it is feasible to visualize it, it requires much more creativity! (and many slides!)

Multiple Firms – The **Not-So-Trivial** case

- Consider two firms, now running in two different random clocks

$$S_t^1 = X^1(\mathcal{T}_t^1) \quad \textit{firm 1}$$

$$S_t^2 = X^2(\mathcal{T}_t^2) \quad \textit{firm 2}$$

  where $\mathcal{T}_t^i$ $i = 1, 2$; are dependent (correlated) subordinators.

- Note: although it is feasible to visualize it, it requires much more creativity! (and many slides!)
- Why is this not so trivial?

Multiple Firms – The **Not-So-Trivial** case

- Consider two firms, now running in two different random clocks

$$S_t^1 = X^1(\mathcal{T}_t^1) \quad \textit{firm 1}$$

$$S_t^2 = X^2(\mathcal{T}_t^2) \quad \textit{firm 2}$$

  where $\mathcal{T}_t^i$ $i = 1, 2$; are dependent (correlated) subordinators.
- Note: although it is feasible to visualize it, it requires much more creativity! (and many slides!)
- Why is this not so trivial?
  - When we use a single subordinator $\mathcal{T}_t$ all we require to model n firms is an n dimensional Markov process,

$$(S_t^1, S_t^2, ..., S_t^n) = (X^1(\mathcal{T}_t), X^2(\mathcal{T}_t), ..., X^n(\mathcal{T}_t)) = \mathbf{X}_{\mathcal{T}_t}$$

    In this case, the all coordinates of the "vector" jump together at the same time and for the same time length!

Multiple Firms – The **Not-So-Trivial** case

- Consider two firms, now running in two different random clocks

$$S_t^1 = X^1(\mathcal{T}_t^1) \quad \textit{firm 1}$$

$$S_t^2 = X^2(\mathcal{T}_t^2) \quad \textit{firm 2}$$

  where $\mathcal{T}_t^i$ $i = 1, 2$; are dependent (correlated) subordinators.
- Note: although it is feasible to visualize it, it requires much more creativity! (and many slides!)
- Why is this not so trivial?
  - When we use a single subordinator $\mathcal{T}_t$ all we require to model n firms is an n dimensional Markov process,

$$(S_t^1, S_t^2, ..., S_t^n) = (X^1(\mathcal{T}_t), X^2(\mathcal{T}_t), ..., X^n(\mathcal{T}_t)) = \mathbf{X}_{\mathcal{T}_t}$$

    In this case, the all coordinates of the "vector" jump together at the same time and for the same time length!
  - When we use an n-dimensional subordinator $\mathcal{T}_t = (\mathcal{T}_t^1, \mathcal{T}_t^2, ..., \mathcal{T}_t^n)$ we require an n-parameter Markov process,

$$(S_t^1, S_t^2, ..., S_t^n) = (X^1(\mathcal{T}_t^1), X^2(\mathcal{T}_t^2), ..., X^n(\mathcal{T}_t^3)) = \mathbf{X}_{(\mathcal{T}_t^1, \mathcal{T}_t^2, ..., \mathcal{T}_t^n)}$$

    In this case, only some coordinates of the vector may jump together and, if they do, they may jump for different time lengths!

Multiple Firms – The **Not-So-Trivial** case

- Consider two firms, now running in two different random clocks

$$S_t^1 = X^1(\mathcal{T}_t^1) \quad \textit{firm 1}$$

$$S_t^2 = X^2(\mathcal{T}_t^2) \quad \textit{firm 2}$$

where $\mathcal{T}_t^i$ $i = 1, 2$; are dependent (correlated) subordinators.

- Note: although it is feasible to visualize it, it requires much more creativity! (and many slides!)
- Why is this not so trivial?
  - When we use a single subordinator $\mathcal{T}_t$ all we require to model n firms is an n dimensional Markov process,

    $$(S_t^1, S_t^2, ..., S_t^n) = (X^1(\mathcal{T}_t), X^2(\mathcal{T}_t), ..., X^n(\mathcal{T}_t)) = \mathbf{X}_{\mathcal{T}_t}$$

    In this case, the all coordinates of the "vector" jump together at the same time and for the same time length!
  - When we use an n-dimensional subordinator $\mathcal{T}_t = (\mathcal{T}_t^1, \mathcal{T}_t^2, ..., \mathcal{T}_t^n)$ we require an n-parameter Markov process,

    $$(S_t^1, S_t^2, ..., S_t^n) = (X^1(\mathcal{T}_t^1), X^2(\mathcal{T}_t^2), ..., X^n(\mathcal{T}_t^3)) = \mathbf{X}_{(\mathcal{T}_t^1, \mathcal{T}_t^2, ..., \mathcal{T}_t^n)}$$

    In this case, only some coordinates of the vector may jump together and, if they do, they may jump for different time lengths!
- We proceed to describe our modeling framework in more detail.

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$

- Independent Diffusions X^i.
  - We take n independent, time-homogeneous, non-negative diffusion processes starting from positive values $X_0^i = S_0^i > 0$ (initial stock prices at time zero) and solving stochastic differential equations:

$$dX_t^i = (\mu_i + k_i(X_t^i))X_t^i \, dt + \sigma_i(X_t^i)X_t^i \, dB_t^i$$

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, \ldots, n.$$

- Independent Diffusions X^i.
  - We take n independent, time-homogeneous, non-negative diffusion processes starting from positive values $X_0^i = S_0^i > 0$ (initial stock prices at time zero) and solving stochastic differential equations:

$$dX_t^i = (\mu_i + k_i(X_t^i))X_t^i\, dt + \sigma_i(X_t^i)X_t^i\, dB_t^i$$

  - $\sigma_i(x)$ is the state-dependent instantaneous volatility

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, \ldots, n.$$

- Independent Diffusions X^i.
  - We take n independent, time-homogeneous, non-negative diffusion processes starting from positive values $X_0^i = S_0^i > 0$ (initial stock prices at time zero) and solving stochastic differential equations:

$$dX_t^i = (\mu_i + k_i(X_t^i))X_t^i \, dt + \sigma_i(X_t^i)X_t^i \, dB_t^i$$

  - $\sigma_i(x)$ is the state-dependent instantaneous volatility
  - $\mu_i + k_i(x)$ is the state-dependent instantaneous drift, $\mu_i \in \mathbb{R}$ are constant parameters

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$

- Independent Diffusions X^i.
  - We take n independent, time-homogeneous, non-negative diffusion processes starting from positive values $X_0^i = S_0^i > 0$ (initial stock prices at time zero) and solving stochastic differential equations:

$$dX_t^i = (\mu_i + k_i(X_t^i))X_t^i \, dt + \sigma_i(X_t^i)X_t^i \, dB_t^i$$

  - $\sigma_i(x)$ is the state-dependent instantaneous volatility
  - $\mu_i + k_i(x)$ is the state-dependent instantaneous drift, $\mu_i \in \mathbb{R}$ are constant parameters
  - B_t^i are n independent standard Brownian motions.

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$

- Multivariate Time Change $\mathcal{T}$.
  - $\mathcal{T}$ is an n-dimensional subordinator: A *n-dimensional subordinator* is a Lévy process in $\mathbb{R}_+^n = [0, \infty)^n$ that is increasing in each of its coordinates.

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases} , \quad i = 1, ..., n.$$

- Multivariate Time Change $\mathcal{T}$.
  - $\mathcal{T}$ is an n-dimensional subordinator: A *n-dimensional subordinator* is a Lévy process in $\mathbb{R}_+^n = [0, \infty)^n$ that is increasing in each of its coordinates.
  - The (n-dimensional) Laplace transform of a n-dimensional subordinator is given by (here $u_i \geq 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^n u_i v_i$):

$$\mathbb{E}[e^{-\langle \mathbf{u}, \mathcal{T}_t \rangle}] = e^{-t\phi(\mathbf{u})}$$

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, \ldots, n.$$

- Multivariate Time Change $\mathcal{T}$.
  - $\mathcal{T}$ is an n-dimensional subordinator: A *n-dimensional subordinator* is a Lévy process in $\mathbb{R}_+^n = [0, \infty)^n$ that is increasing in each of its coordinates.
  - The (n-dimensional) Laplace transform of a n-dimensional subordinator is given by (here $u_i \geq 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^n u_i v_i$):

$$\mathbb{E}[e^{-\langle \mathbf{u}, \mathcal{T}_t \rangle}] = e^{-t\phi(\mathbf{u})}$$

  - The Laplace exponent given by the Lévy-Khintchine formula:

$$\phi(\mathbf{u}) = \langle \gamma, \mathbf{u} \rangle + \int_{\mathbb{R}_+^n} (1 - e^{-\langle \mathbf{u}, \mathbf{s} \rangle}) \nu(d\mathbf{s}),$$

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$

- Multivariate Time Change $\mathcal{T}$.
  - $\mathcal{T}$ is an n-dimensional subordinator: A *n-dimensional subordinator* is a Lévy process in $\mathbb{R}_+^n = [0, \infty)^n$ that is increasing in each of its coordinates.
  - The (n-dimensional) Laplace transform of a n-dimensional subordinator is given by (here $u_i \geq 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^n u_i v_i$):

$$\mathbb{E}[e^{-\langle \mathbf{u}, \mathcal{T}_t \rangle}] = e^{-t\phi(\mathbf{u})}$$

  - The Laplace exponent given by the Lévy-Khintchine formula:

$$\phi(\mathbf{u}) = \langle \gamma, \mathbf{u} \rangle + \int_{\mathbb{R}_+^n} (1 - e^{-\langle \mathbf{u}, \mathbf{s} \rangle}) \nu(d\mathbf{s}),$$

    where $\gamma \in \mathbb{R}_+^n$ is the drift and the Lévy measure ν is a σ-finite measure such that $\int_{\mathbb{R}_+^n} (\|\mathbf{s}\| \wedge 1) \nu(d\mathbf{s}) < \infty$.

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S^i_t of n firms under an EMM $\mathbb{Q}$:

$$S^i_t = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{\mathcal{T}^i_t} \equiv \begin{cases} e^{\rho_i t} X^i_{\mathcal{T}^i_t}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, \ldots, n.$$

- Multivariate Time Change $\mathcal{T}$.
  - $\mathcal{T}$ is an n-dimensional subordinator: A *n-dimensional subordinator* is a Lévy process in $\mathbb{R}^n_+ = [0, \infty)^n$ that is increasing in each of its coordinates.
  - The (n-dimensional) Laplace transform of a n-dimensional subordinator is given by (here $u_i \geq 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^n u_i v_i$):

$$\mathbb{E}[e^{-\langle \mathbf{u}, \mathcal{T}_t \rangle}] = e^{-t\phi(\mathbf{u})}$$

  - The Laplace exponent given by the Lévy-Khintchine formula:

$$\phi(\mathbf{u}) = \langle \gamma, \mathbf{u} \rangle + \int_{\mathbb{R}^n_+} (1 - e^{-\langle \mathbf{u}, \mathbf{s} \rangle}) \nu(d\mathbf{s}),$$

    where $\gamma \in \mathbb{R}^n_+$ is the drift and the Lévy measure ν is a σ-finite measure such that $\int_{\mathbb{R}^n_+} (\|\mathbf{s}\| \wedge 1) \nu(d\mathbf{s}) < \infty$.

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{T_t^i} \equiv \begin{cases} e^{\rho_i t} X^i_{T_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$

- Default Times τ_i.
  - We define the positive random variable τ_i as the time of default of the ith firm on its debt.

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$

- Default Times τ_i.
  - We define the positive random variable τ_i as the time of default of the ith firm on its debt.
  - The default times, τ_i, are constructed as follows:

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X_{\mathcal{T}_t^i}^i \equiv \begin{cases} e^{\rho_i t} X_{\mathcal{T}_t^i}^i, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$

- Default Times τ_i.
  - We define the positive random variable τ_i as the time of default of the ith firm on its debt.
  - The default times, τ_i, are constructed as follows:
    1. Let H_0^i be the first time that the diffusion X^i hits zero.

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, \ldots, n.$$

- Default Times τ_i.
  - We define the positive random variable τ_i as the time of default of the ith firm on its debt.
  - The default times, τ_i, are constructed as follows:
    1. Let H_0^i be the first time that the diffusion X^i hits zero.
    2. Let $\mathcal{E}_i$ be n independent exponential random variables with unit mean and independent of all X^i and $\mathcal{T}^i$

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$

- Default Times τ_i.
  - We define the positive random variable τ_i as the time of default of the ith firm on its debt.
  - The default times, τ_i, are constructed as follows:
    1. Let H_0^i be the first time that the diffusion X^i hits zero.
    2. Let $\mathcal{E}_i$ be n independent exponential random variables with unit mean and independent of all X^i and $\mathcal{T}^i$
    3. Define the X_t^i's lifetime (we assume that $\inf\{\emptyset\} = H_0$ by convention):

$$\zeta_i := \inf\{t \in [0, H_0^i] : \int_0^t k_i(X_u^i) du \geq \mathcal{E}_i\}.$$

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, \ldots, n.$$

- Default Times τ_i.
  - We define the positive random variable τ_i as the time of default of the ith firm on its debt.
  - The default times, τ_i, are constructed as follows:
    1. Let H_0^i be the first time that the diffusion X^i hits zero.
    2. Let $\mathcal{E}_i$ be n independent exponential random variables with unit mean and independent of all X^i and $\mathcal{T}^i$
    3. Define the X_t^i's lifetime (we assume that $\inf\{\emptyset\} = H_0$ by convention):
$$\zeta_i := \inf\{t \in [0, H_0^i] : \int_0^t k_i(X_u^i) du \geq \mathcal{E}_i\}.$$
    4. Then, time of default of the ith firm is defined by applying the time change $\mathcal{T}^i$:
$$\tau_i := \inf\{t \geq 0 : \mathcal{T}_t^i \geq \zeta_i\}.$$

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$

- Martingale Conditions.
  - Each single-name stock price process S^i is a non-negative martingale under the EMM $\mathbb{Q}$ if and only if,

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$

- Martingale Conditions.
  - Each single-name stock price process S^i is a non-negative martingale under the EMM $\mathbb{Q}$ if and only if,

    1. the constant μ_i in the drift of X^i satisfies the following condition:

$$\int_{[1,\infty)} e^{\mu_i s} \nu_i(ds) < \infty,$$

where ν_i is the Lévy measure of the one-dimensional subordinator $\mathcal{T}^i$ ($\nu_i(A) = \nu(\mathbb{R}_+ \times ... \times A \times ...\mathbb{R}_+)$ with A in the ith place, for any Borel set $A \subset \mathbb{R}_+$ bounded away from zero),

Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices S_t^i of n firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$

- Martingale Conditions.
  - Each single-name stock price process S^i is a non-negative martingale under the EMM $\mathbb{Q}$ if and only if,

    1. the constant μ_i in the drift of X^i satisfies the following condition:

    $$\int_{[1,\infty)} e^{\mu_i s} \nu_i(ds) < \infty,$$

    where ν_i is the Lévy measure of the one-dimensional subordinator $\mathcal{T}^i$ ($\nu_i(A) = \nu(\mathbb{R}_+ \times ... \times A \times ...\mathbb{R}_+)$ with A in the ith place, for any Borel set $A \subset \mathbb{R}_+$ bounded away from zero),

    2. the constant ρ_i is:

    $$\rho_i = r - q_i + \phi_i(-\mu_i),$$

    where $\phi_i(u)$ is the Laplace exponent of $\mathcal{T}^i$, $\phi_i(u) = \phi(0, ..., 0, u, 0, ..., 0)$ (u is in the ith place)

Credit-Equity Derivatives Pricing

- We are interested in pricing contingent claims written on multiple defaultable stocks.

Credit-Equity Derivatives Pricing

- We are interested in pricing contingent claims written on multiple defaultable stocks.

- In particular, the price of a European-style derivative expiring at time $t > 0$ with the payoff function $f(S_t^1, ..., S_t^n)$ is given by

$$e^{-rt}\mathbb{E}[f(S_t^1, ..., S_t^n)]$$

Credit-Equity Derivatives Pricing

- We are interested in pricing contingent claims written on multiple defaultable stocks.
- In particular, the price of a European-style derivative expiring at time $t > 0$ with the payoff function $f(S_t^1, ..., S_t^n)$ is given by

$$e^{-rt}\mathbb{E}[f(S_t^1, ..., S_t^n)]$$

- Recall: Each of the n firms may default by time t (and its stock becomes worthless).

  Therefore, at time t, the firm's stock price is either:

  - $S_t^i > 0$ (survival to time t, i.e., $\tau_i > t$) or,
  - $S_t^i = 0$ (default by time t, i.e., $\tau_i \leq t$).

Multivariate Subordination of Multiparameter Semigroups

- Thus we are interested on calculating expectations of the form

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\ldots,n\}}>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\right)\right]$$

▸ Multiparameter Semigroups

Multivariate Subordination of Multiparameter Semigroups

- Thus we are interested on calculating expectations of the form

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\ldots,n\}}>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\right)\right]$$

$$= \mathbb{E}\left[\mathbf{1}_{\{\tau_1>t\}} \cdots \mathbf{1}_{\{\tau_n>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\right)\right] \qquad \left(\begin{array}{c} \tau_{\{1,\ldots,n\}} \\ = \bigwedge_{i=1}^n \tau_i \end{array}\right)$$

Multiparameter Semigroups

Multivariate Subordination of Multiparameter Semigroups

- Thus we are interested on calculating expectations of the form

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\ldots,n\}}>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\right)\right]$$

$$= \mathbb{E}\left[\mathbf{1}_{\{\tau_1>t\}} \cdots \mathbf{1}_{\{\tau_n>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\right)\right] \qquad \left(\begin{array}{c}\tau_{\{1,\ldots,n\}} \\ = \bigwedge_{i=1}^n \tau_i\end{array}\right)$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1>\mathcal{T}^1_t\}} \cdots \mathbf{1}_{\{\zeta_n>\mathcal{T}^n_t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\right) \middle| \mathcal{T}_t\right]\right] \qquad \left(\begin{array}{c}\mathcal{T}_t \;\&\; X_t \\ \textit{are indep.}\end{array}\right)$$

Multiparameter Semigroups

Multivariate Subordination of Multiparameter Semigroups

- Thus we are interested on calculating expectations of the form

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\dots,n\}}>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right)\right]$$

$$= \mathbb{E}\left[\mathbf{1}_{\{\tau_1>t\}} \cdots \mathbf{1}_{\{\tau_n>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right)\right] \quad \left(\begin{array}{c} \tau_{\{1,\dots,n\}} \\ = \bigwedge_{i=1}^n \tau_i \end{array} \right)$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1>\mathcal{T}^1_t\}} \cdots \mathbf{1}_{\{\zeta_n>\mathcal{T}^n_t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right) \middle| \mathcal{T}_t\right]\right] \quad \left(\begin{array}{c} \mathcal{T}_t \,\&\, X_t \\ \textit{are indep.} \end{array} \right)$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1>\mathcal{T}^1_t\}} \cdots \mathbb{E}\left[\mathbf{1}_{\{\zeta_n>\mathcal{T}^n_t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right) \middle| \mathcal{T}_t\right] \cdots \middle| \mathcal{T}_t\right]\right] \quad \left(\begin{array}{c} X^{i}_t{}'s \\ \textit{are indep.} \end{array} \right)$$

▸ Multiparameter Semigroups

Multivariate Subordination of Multiparameter Semigroups

- Thus we are interested on calculating expectations of the form

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\ldots,n\}}>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\right)\right]$$

$$= \mathbb{E}\left[\mathbf{1}_{\{\tau_1>t\}} \cdots \mathbf{1}_{\{\tau_n>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\right)\right] \qquad \left(\begin{array}{c} \tau_{\{1,\ldots,n\}} \\ = \bigwedge_{i=1}^n \tau_i \end{array}\right)$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1>\mathcal{T}^1_t\}} \cdots \mathbf{1}_{\{\zeta_n>\mathcal{T}^n_t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\right) \middle| \mathcal{T}_t\right]\right] \qquad \left(\begin{array}{c} \mathcal{T}_t \text{ \& } X_t \\ \textit{are indep.} \end{array}\right)$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1>\mathcal{T}^1_t\}} \cdots \mathbb{E}\left[\mathbf{1}_{\{\zeta_n>\mathcal{T}^n_t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\right) \middle| \mathcal{T}_t\right] \cdots \middle| \mathcal{T}_t\right]\right] \qquad \left(\begin{array}{c} X^{i\,\prime}_t s \\ \textit{are indep.} \end{array}\right)$$

$$\underbrace{(\mathcal{P}_{\mathbf{s}} f)}_{\substack{\textit{Multi}- \\ \textit{parameter} \\ \textit{Semigroup}}}$$

Multiparameter Semigroups

Multivariate Subordination of Multiparameter Semigroups

- Thus we are interested on calculating expectations of the form

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\dots,n\}}>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right)\right]$$

$$= \mathbb{E}\left[\mathbf{1}_{\{\tau_1>t\}} \cdots \mathbf{1}_{\{\tau_n>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right)\right] \qquad \left(\begin{array}{c} \tau_{\{1,\dots,n\}} \\ = \bigwedge_{i=1}^n \tau_i \end{array}\right)$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1>\mathcal{T}^1_t\}} \cdots \mathbf{1}_{\{\zeta_n>\mathcal{T}^n_t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right) \middle| \mathcal{T}_t\right]\right] \qquad \left(\begin{array}{c} \mathcal{T}_t \,\&\, X_t \\ \textit{are indep.} \end{array}\right)$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1>\mathcal{T}^1_t\}} \cdots \mathbb{E}\left[\mathbf{1}_{\{\zeta_n>\mathcal{T}^n_t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right) \middle| \mathcal{T}_t\right] \cdots \middle| \mathcal{T}_t\right]\right] \qquad \left(\begin{array}{c} X^{i'}_t s \\ \textit{are indep.} \end{array}\right)$$

$$\int_{\mathbb{R}^n_+} \underbrace{(\mathcal{P}_{\mathbf{s}} f)}_{\substack{\textit{Multi}- \\ \textit{parameter} \\ \textit{Semigroup}}} \underbrace{\pi_t(d\mathbf{s})}_{\substack{\textit{Multi}- \\ \textit{Subord.} \\ \textit{transition} \\ \textit{kernel}}}$$

Multiparameter Semigroups

Multivariate Subordination of Multiparameter Semigroups

- Thus we are interested on calculating expectations of the form

$$
\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\{\tau_{\{1,2,\ldots,n\}}>t\}} f\big(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\big)\Big] \\
&= \mathbb{E}\Big[\mathbf{1}_{\{\tau_1>t\}} \cdots \mathbf{1}_{\{\tau_n>t\}} f\big(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\big)\Big] && \left(\begin{array}{c} \tau_{\{1,\ldots,n\}} \\ = \bigwedge_{i=1}^n \tau_i \end{array}\right) \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{\{\zeta_1>\mathcal{T}^1_t\}} \cdots \mathbf{1}_{\{\zeta_n>\mathcal{T}^n_t\}} f\big(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\big)\big|\mathcal{T}_t\Big]\Big] && \left(\begin{array}{c} \mathcal{T}_t \,\&\, X_t \\ \textit{are indep.} \end{array}\right) \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{\{\zeta_1>\mathcal{T}^1_t\}} \cdots \mathbb{E}\Big[\mathbf{1}_{\{\zeta_n>\mathcal{T}^n_t\}} f\big(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\big)\big|\mathcal{T}_t\Big] \cdots \big|\mathcal{T}_t\Big]\Big] && \left(\begin{array}{c} X^{i}_t{'s} \\ \textit{are indep.} \end{array}\right) \\
&= \underbrace{\int_{\mathbb{R}^n_+} \underbrace{(\mathcal{P}_s f)}_{\substack{\textit{Multi}-\\ \textit{parameter} \\ \textit{Semigroup}}} \underbrace{\pi_t(ds)}_{\substack{\textit{Multi}-\\ \textit{Subord.} \\ \textit{transition} \\ \textit{kernel}}}}_{\substack{\textit{Multivariate Subordination} \\ \textit{of} \\ \textit{Multiparameter Semigroups}}}
\end{aligned}
$$

▸ Multiparameter Semigroups

Multivariate Subordination of Multiparameter Semigroups

- Thus we are interested on calculating expectations of the form

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\dots,n\}}>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right)\right]$$

$$= \mathbb{E}\left[\mathbf{1}_{\{\tau_1>t\}} \cdots \mathbf{1}_{\{\tau_n>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right)\right] \qquad \left(\begin{array}{c} \tau_{\{1,\dots,n\}} \\ = \bigwedge_{i=1}^n \tau_i \end{array} \right)$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1>\mathcal{T}^1_t\}} \cdots \mathbf{1}_{\{\zeta_n>\mathcal{T}^n_t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right) \middle| \mathcal{T}_t\right]\right] \qquad \left(\begin{array}{c} \mathcal{T}_t \;\&\; X_t \\ \textit{are indep.} \end{array} \right)$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1>\mathcal{T}^1_t\}} \cdots \mathbb{E}\left[\mathbf{1}_{\{\zeta_n>\mathcal{T}^n_t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right) \middle| \mathcal{T}_t\right] \cdots \middle| \mathcal{T}_t\right]\right] \qquad \left(\begin{array}{c} X^{i}_t{}'s \\ \textit{are indep.} \end{array} \right)$$

$$= \underbrace{\int_{\mathbb{R}^n_+} \underbrace{(\mathcal{P}_s f)}_{\substack{Multi- \\ parameter \\ Semigroup}} \underbrace{\pi_t(ds)}_{\substack{Multi- \\ Subord. \\ transition \\ kernel}}}_{\substack{Multivariate\ Subordination \\ of \\ Multiparameter\ Semigroups}} = \underbrace{\mathcal{P}^{\phi}_t f}_{\substack{Subordinated \\ Semigroup \\ (one\ -\ parameter)}}$$

▸ Multiparameter Semigroups

Spectral Decomposition (I)

- We assume that all X^i are 1D diffusions (symmetric Markov processes) on $(0, \infty)$ such that:
  - the semigroups $\mathcal{P}^i$ defined in the Hilbert spaces $\mathcal{H}_i = L^2((0, \infty), m_i)$ endowed with the inner products $(f, g)_{m_i} = \int_{(0,\infty)} f(x)g(x)m_i(x)dx$ are symmetric with respect to the speed density $m(x)$, i.e.,

$$(\mathcal{P}^i_{t_i} f, g)_{m_i} = (f, \mathcal{P}^i_{t_i} g)_{m_i}, \quad \forall t_i \geq 0, \ \& \ i = 1, ..., n$$

Spectral Decomposition (I)

- We assume that all X^i are 1D diffusions (symmetric Markov processes) on $(0, \infty)$ such that:
  - the semigroups $\mathcal{P}^i$ defined in the Hilbert spaces $\mathcal{H}_i = L^2((0,\infty), m_i)$ endowed with the inner products $(f,g)_{m_i} = \int_{(0,\infty)} f(x)g(x)m_i(x)dx$ are symmetric with respect to the speed density $m(x)$, i.e.,

$$(\mathcal{P}^i_{t_i} f, g)_{m_i} = (f, \mathcal{P}^i_{t_i} g)_{m_i}, \quad \forall t_i \geq 0, \ \& \ i = 1, ..., n$$

  - Then $\mathbf{H} = L^2((0,\infty)^n, m)$ is defined on the product space $(0,\infty)^n = (0,\infty) \times ... \times (0,\infty)$ with the product speed density $m(\mathbf{x}) = m_1(x_1)...m_n(x_n)$ and the inner product

$$(f,g)_m = \int_{(0,\infty)^n} f(\mathbf{x})g(\mathbf{x})m(\mathbf{x})d\mathbf{x}$$

Spectral Decomposition (II)

- In the special case when each infinitesimal generator $\mathcal{G}_i$ has a purely discrete spectrum with eigenvalues $\{-\lambda_k^i\}_{k=1}^{\infty}$ and the corresponding eigenfunctions $\varphi_k^i(x_i)$,

$$\mathcal{G}_i \varphi_k^i(x_i) = -\lambda_k^i \varphi_k^i(x_i),$$

Spectral Decomposition (II)

- In the special case when each infinitesimal generator $\mathcal{G}_i$ has a purely discrete spectrum with eigenvalues $\{-\lambda_k^i\}_{k=1}^{\infty}$ and the corresponding eigenfunctions $\varphi_k^i(x_i)$,

$$\mathcal{G}_i \varphi_k^i(x_i) = -\lambda_k^i \varphi_k^i(x_i),$$

- the spectral representation of the multi-parameter semigroup takes the form of the eigenfunction expansion:

$$\mathcal{P}_{\mathbf{t}} f = \sum_{\mathbf{k}\in\mathbb{N}^n} e^{-\langle \lambda, \mathbf{t}\rangle} c_{\mathbf{k}}^f \varphi_{\mathbf{k}}, \quad f \in \mathbf{H}, \quad \mathbf{t} = (t_1, ..., t_n) \geq \mathbf{0},$$

where $\sum_{\mathbf{k}\in\mathbb{N}^n} = \sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty}$, $\mathbb{N} = \{1, 2, ...\}$,

Spectral Decomposition (II)

- In the special case when each infinitesimal generator $\mathcal{G}_i$ has a purely discrete spectrum with eigenvalues $\{-\lambda_k^i\}_{k=1}^{\infty}$ and the corresponding eigenfunctions $\varphi_k^i(x_i)$,

$$\mathcal{G}_i \varphi_k^i(x_i) = -\lambda_k^i \varphi_k^i(x_i),$$

- the spectral representation of the multi-parameter semigroup takes the form of the eigenfunction expansion:

$$\mathcal{P}_{\mathbf{t}} f = \sum_{\mathbf{k} \in \mathbb{N}^n} e^{-\langle \lambda, \mathbf{t} \rangle} c_{\mathbf{k}}^f \varphi_{\mathbf{k}}, \quad f \in \mathbf{H}, \quad \mathbf{t} = (t_1, ..., t_n) \geq \mathbf{0},$$

where $\sum_{\mathbf{k} \in \mathbb{N}^n} = \sum_{k_1=1}^{\infty} \ldots \sum_{k_n=1}^{\infty}$, $\mathbb{N} = \{1, 2, \ldots\}$,
the eigenvalues and eigenfunctions are

$$\lambda = (\lambda_{k_1}^1, ..., \lambda_{k_n}^n)$$

$$\varphi_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{n} \varphi_{k_i}^i(x_i), \quad x_i \in (0, \infty), \quad \mathbf{x} = (x_1, ..., x_n) \in (0, \infty)^n, \quad \mathbf{k} \in \mathbb{N}^n,$$

and the expansion coefficients are

$$c_{\mathbf{k}}^f = (f, \varphi_{\mathbf{k}})_m, \quad \mathbf{k} \in \mathbb{N}^n.$$

Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^{\phi}$

- Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^{\phi}$

- Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

$$\mathcal{P}_t^{\phi} f$$

Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^{\phi}$

- Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

$$\mathcal{P}_t^{\phi} f \quad = \mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\ldots,n\}} > t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \ldots, X^n_{\mathcal{T}^n_t}\right)\right]$$

Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^{\phi}$

- Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

$$\begin{aligned}\mathcal{P}_t^{\phi} f &= \mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\ldots,n\}}>t\}} f\left(X^1_{\mathcal{T}_t^1}, X^2_{\mathcal{T}_t^2}, \ldots, X^n_{\mathcal{T}_t^n}\right)\right] \\ &= \int_{\mathbb{R}_+^n} \mathcal{P}_{\mathbf{s}} f \pi_t(d\mathbf{s}) \qquad \left(\begin{array}{c}\textit{Multivariate subordination} \\ \textit{of the} \\ \text{n}-\textit{parameter semigroup}\end{array}\right)\end{aligned}$$

Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^{\phi}$

- Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

$$
\begin{aligned}
\mathcal{P}_t^{\phi} f &= \mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\ldots,n\}}>t\}} f\left(X^1_{\mathcal{T}_t^1}, X^2_{\mathcal{T}_t^2}, \ldots, X^n_{\mathcal{T}_t^n}\right)\right] \\
&= \int_{\mathbb{R}_+^n} \mathcal{P}_{\mathbf{s}} f \, \pi_t(d\mathbf{s}) && \left(\begin{array}{c} \textit{Multivariate subordination} \\ \textit{of the} \\ \text{n}-\textit{parameter semigroup} \end{array}\right) \\
&= \int_{\mathbb{R}_+^n} \left(\sum_{\mathbf{k}\in\mathbb{N}^n} e^{-\langle\lambda,\mathbf{s}\rangle} c_{\mathbf{k}}^f \varphi_{\mathbf{k}}\right) \pi_t(d\mathbf{s}) && \left(\begin{array}{c} \textit{Spectral representation} \\ \textit{of the} \\ \text{n}-\textit{parameter semigroup} \end{array}\right)
\end{aligned}
$$

Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^{\phi}$

- Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

$$
\begin{aligned}
\mathcal{P}_t^{\phi} f &= \mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\dots,n\}}>t\}} f\left(X^1_{\mathcal{T}_t^1}, X^2_{\mathcal{T}_t^2}, \dots, X^n_{\mathcal{T}_t^n}\right)\right] \\
&= \int_{\mathbb{R}_+^n} \mathcal{P}_{\mathbf{s}} f \,\pi_t(d\mathbf{s}) && \begin{pmatrix} \textit{Multivariate subordination} \\ \textit{of the} \\ \text{n}-\textit{parameter semigroup} \end{pmatrix} \\
&= \int_{\mathbb{R}_+^n} \left(\sum_{\mathbf{k}\in\mathbb{N}^n} e^{-\langle \lambda, \mathbf{s}\rangle} c_{\mathbf{k}}^f \varphi_{\mathbf{k}}\right) \pi_t(d\mathbf{s}) && \begin{pmatrix} \textit{Spectral representation} \\ \textit{of the} \\ \text{n}-\textit{parameter semigroup} \end{pmatrix} \\
&= \sum_{\mathbf{k}\in\mathbb{N}^n} \left(\int_{\mathbb{R}_+^n} e^{-\langle \lambda, \mathbf{s}\rangle} \pi_t(d\mathbf{s})\right) c_{\mathbf{k}}^f \varphi_{\mathbf{k}} && \begin{pmatrix} \textit{Laplace transform} \\ \textit{of the} \\ \text{n}-\textit{dimensional subordinator} \end{pmatrix}
\end{aligned}
$$

Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^{\phi}$

- Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

$$\mathcal{P}_t^{\phi} f = \mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2,\dots,n\}}>t\}} f\left(X^1_{\mathcal{T}^1_t}, X^2_{\mathcal{T}^2_t}, \dots, X^n_{\mathcal{T}^n_t}\right)\right]$$

$$= \int_{\mathbb{R}^n_+} \mathcal{P}_{\mathbf{s}} f \pi_t(d\mathbf{s}) \qquad \left(\begin{array}{c}\textit{Multivariate subordination}\\ \textit{of the}\\ \text{n}-\textit{parameter semigroup}\end{array}\right)$$

$$= \int_{\mathbb{R}^n_+} \left(\sum_{\mathbf{k}\in\mathbb{N}^n} e^{-\langle \lambda, \mathbf{s}\rangle} c^f_{\mathbf{k}} \varphi_{\mathbf{k}}\right) \pi_t(d\mathbf{s}) \qquad \left(\begin{array}{c}\textit{Spectral representation}\\ \textit{of the}\\ \text{n}-\textit{parameter semigroup}\end{array}\right)$$

$$= \sum_{\mathbf{k}\in\mathbb{N}^n} \left(\int_{\mathbb{R}^n_+} e^{-\langle \lambda, \mathbf{s}\rangle} \pi_t(d\mathbf{s})\right) c^f_{\mathbf{k}} \varphi_{\mathbf{k}} \qquad \left(\begin{array}{c}\textit{Laplace transform}\\ \textit{of the}\\ \text{n}-\textit{dimensional subordinator}\end{array}\right)$$

$$= \sum_{\mathbf{k}\in\mathbb{N}^n} e^{-\phi(\lambda^1_{k_1}, \dots, \lambda^n_{k_n})t} c^f_{\mathbf{k}} \varphi_{\mathbf{k}} \qquad \left(\begin{array}{c}\textit{Levy} - \textit{Khintchine}\\ \textit{exponent}\end{array}\right)$$

Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^{\phi}$

- Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

$$
\begin{aligned}
\mathcal{P}_t^{\phi} f &= \mathbb{E}\big[\mathbf{1}_{\{\tau_{\{1,2,\dots,n\}}>t\}} f\big(X^1_{\mathcal{T}_t^1}, X^2_{\mathcal{T}_t^2}, \dots, X^n_{\mathcal{T}_t^n}\big)\big] \\
&= \int_{\mathbb{R}_+^n} \mathcal{P}_{\mathbf{s}} f \,\pi_t(d\mathbf{s}) && \left(\begin{array}{c}\textit{Multivariate subordination} \\ \textit{of the} \\ \text{n} - \textit{parameter semigroup}\end{array}\right) \\
&= \int_{\mathbb{R}_+^n} \left(\sum_{\mathbf{k}\in\mathbb{N}^n} e^{-\langle \lambda, \mathbf{s}\rangle} c_{\mathbf{k}}^f \varphi_{\mathbf{k}}\right) \pi_t(d\mathbf{s}) && \left(\begin{array}{c}\textit{Spectral representation} \\ \textit{of the} \\ \text{n} - \textit{parameter semigroup}\end{array}\right) \\
&= \sum_{\mathbf{k}\in\mathbb{N}^n} \left(\int_{\mathbb{R}_+^n} e^{-\langle \lambda, \mathbf{s}\rangle} \pi_t(d\mathbf{s})\right) c_{\mathbf{k}}^f \varphi_{\mathbf{k}} && \left(\begin{array}{c}\textit{Laplace transform} \\ \textit{of the} \\ \text{n} - \textit{dimensional subordinator}\end{array}\right) \\
&= \sum_{\mathbf{k}\in\mathbb{N}^n} e^{-\phi(\lambda^1_{k_1}, \dots, \lambda^n_{k_n})t} c_{\mathbf{k}}^f \varphi_{\mathbf{k}} && \left(\begin{array}{c}\textit{Levy} - \textit{Khintchine} \\ \textit{exponent}\end{array}\right)
\end{aligned}
$$

- Remark: When $n = 1$ the modeling framework is reduced to the Credit-Equity Model of Mendoza-Arriaga et al. (2009).

Two Firms Illustration: *the JDCEV process*

- Recall: we model the joint risk-neutral dynamics of stock prices S_t^i of 2 firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, 2$$

Two Firms Illustration: *the JDCEV process*

- Recall: we model the joint risk-neutral dynamics of stock prices S_t^i of 2 firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, 2$$

- Let $X_t^i \quad i = 1, 2$ be Jump-to-Default Extended Constant Elasticity of Variance (JDCEV) processes of Carr & Linetsky (2006):

$$\underline{\underline{dX_t = [\mu + k(X_t)]X_t\,dt + \sigma(X_t)X_t\,dB_t}}, \quad X_0 = x > 0$$

$$\underline{\sigma(X) = aX^\beta} \qquad\qquad \underline{k(X) = b + c\,\sigma^2(X)}$$

CEV Volatility
(Power function of X)

Killing Rate
(Affine function of Variance)

Two Firms Illustration: *the JDCEV process*

- Recall: we model the joint risk-neutral dynamics of stock prices S_t^i of 2 firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases} , \quad i = 1, 2$$

- Let $X_t^i \quad i = 1, 2$ be Jump-to-Default Extended Constant Elasticity of Variance (JDCEV) processes of Carr & Linetsky (2006):

$$\underline{\underline{dX_t = [\mu + k(X_t)]X_t\, dt + \sigma(X_t)X_t\, dB_t}}, \quad X_0 = x > 0$$

$\underline{\sigma(X) = aX^{\beta}}$ $\qquad\qquad$ $\underline{k(X) = b + c\,\sigma^2(X)}$

CEV Volatility
(Power function of X)

Killing Rate
(Affine function of Variance)

$a > 0 \quad \Rightarrow$ volatility scale parameter (fixing ATM volatility)
$\beta < 0 \quad \Rightarrow$ volatility elasticity parameter
$b \geq 0 \quad \Rightarrow$ constant default intensity
$c \geq 0 \quad \Rightarrow$ sensitivity of the default intensity to variance

Two Firms Illustration: *the JDCEV process*

- Recall: we model the joint risk-neutral dynamics of stock prices S_t^i of 2 firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, 2$$

- Let $X_t^i \quad i = 1, 2$ be Jump-to-Default Extended Constant Elasticity of Variance (JDCEV) processes of Carr & Linetsky (2006):

$$\underline{\underline{dX_t = [\mu + k(X_t)]X_t\,dt + \sigma(X_t)X_t\,dB_t}}, \quad X_0 = x > 0$$

$$\underline{\sigma(X) = aX^\beta} \qquad\qquad \underline{k(X) = b + c\,\sigma^2(X)}$$

CEV Volatility (Power function of X)

Killing Rate (Affine function of Variance)

$a > 0 \quad \Rightarrow$ volatility scale parameter (fixing ATM volatility)
$\beta < 0 \quad \Rightarrow$ volatility elasticity parameter
$b \geq 0 \quad \Rightarrow$ constant default intensity
$c \geq 0 \quad \Rightarrow$ sensitivity of the default intensity to variance

For $c = 0$ and $b = 0$ the JDCEV reduces to the standard CEV process

Two Firms Illustration: *the JDCEV process*

- Recall: we model the joint risk-neutral dynamics of stock prices S_t^i of 2 firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} X^i{}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases} , \quad i = 1, 2$$

- Let $X_t^i \quad i = 1, 2$ be Jump-to-Default Extended Constant Elasticity of Variance (JDCEV) processes of Carr & Linetsky (2006):

$$\underline{\underline{dX_t = [\mu + k(X_t)]X_t\, dt + \sigma(X_t)X_t\, dB_t}}, \quad X_0 = x > 0$$

$\underline{\sigma(X) = aX^{\beta}}$

CEV Volatility
(Power function of X)

$\underline{k(X) = b + c\,\sigma^2(X)}$

Killing Rate
(Affine function of Variance)

$a > 0$ $\Rightarrow$ volatility scale parameter (fixing ATM volatility)
$\beta < 0$ $\Rightarrow$ volatility elasticity parameter
$b \geq 0$ $\Rightarrow$ constant default intensity
$c \geq 0$ $\Rightarrow$ sensitivity of the default intensity to variance

The model is consistent with:

leverage effect $\Rrightarrow$ $S \Downarrow \rightarrow \sigma(S) \Uparrow$

Two Firms Illustration: *the JDCEV process*

- Recall: we model the joint risk-neutral dynamics of stock prices S_t^i of 2 firms under an EMM $\mathbb{Q}$:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} {X^i}_{\mathcal{T}_t^i} \equiv \begin{cases} e^{\rho_i t} {X^i}_{\mathcal{T}_t^i}, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, 2$$

- Let $X_t^i \quad i = 1, 2$ be Jump-to-Default Extended Constant Elasticity of Variance (JDCEV) processes of Carr & Linetsky (2006):

$$\underline{\underline{dX_t = [\mu + k(X_t)]X_t\, dt + \sigma(X_t)X_t\, dB_t}}, \quad X_0 = x > 0$$

$$\underline{\sigma(X) = aX^{\beta}} \qquad\qquad \underline{k(X) = b + c\,\sigma^2(X)}$$

CEV Volatility (Power function of X) | Killing Rate (Affine function of Variance)

$a > 0 \quad \Rightarrow$ volatility scale parameter (fixing ATM volatility)
$\beta < 0 \quad \Rightarrow$ volatility elasticity parameter
$b \geq 0 \quad \Rightarrow$ constant default intensity
$c \geq 0 \quad \Rightarrow$ sensitivity of the default intensity to variance

The model is consistent with:

leverage effect $\Rightarrow S \Downarrow \rightarrow \sigma(S) \Uparrow$
stock volatility–credit spreads *linkage* $\Rightarrow \sigma(S) \Uparrow \leftrightarrow k(S) \Uparrow$

JDCEV Eigenvalues and Eigenfunctions

- When $\mu + b \neq 0$, the spectrum is purely discrete. When $\mu + b < 0$, the eigenvalues and eigenfunctions are:

$$\lambda_n = \omega(n-1) + \lambda_1, \quad \varphi_n(x) = A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!|\mu+b|}{\Gamma(\nu+n)}}\, x\, L^{\nu}_{n-1}(Ax^{-2\beta}), \quad n = 1, 2, \ldots,$$

where $L^{\nu}_{n}(x)$ are the generalized Laguerre polynomials.

JDCEV Eigenvalues and Eigenfunctions

- When $\mu + b \neq 0$, the spectrum is purely discrete. When $\mu + b < 0$, the eigenvalues and eigenfunctions are:

$$\lambda_n = \omega(n-1) + \lambda_1, \quad \varphi_n(x) = A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!|\mu+b|}{\Gamma(\nu+n)}}\, x\, L_{n-1}^{\nu}(Ax^{-2\beta}), \quad n = 1, 2, ...,$$

where $L_n^{\nu}(x)$ are the generalized Laguerre polynomials.

- The principal eigenvalue λ_1, A, ν and ω are,

$$\lambda_1 := |\mu|, \quad A := \frac{|\mu+b|}{a^2|\beta|}, \quad \nu := \frac{1+2c}{2|\beta|}, \quad \omega := 2|\beta(\mu+b)|, ,$$

JDCEV Eigenvalues and Eigenfunctions

- When $\mu + b \neq 0$, the spectrum is purely discrete. When $\mu + b < 0$, the eigenvalues and eigenfunctions are:

$$\lambda_n = \omega(n-1) + \lambda_1, \quad \varphi_n(x) = A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!|\mu+b|}{\Gamma(\nu+n)}}\, x\, L_{n-1}^{\nu}(Ax^{-2\beta}), \quad n = 1, 2, \ldots,$$

where $L_n^{\nu}(x)$ are the generalized Laguerre polynomials.

- The principal eigenvalue λ_1, A, ν and ω are,

$$\lambda_1 := |\mu|, \quad A := \frac{|\mu+b|}{a^2|\beta|}, \quad \nu := \frac{1+2c}{2|\beta|}, \quad \omega := 2|\beta(\mu+b)|,,$$

- The *speed density* is defined as,

$$m(x) = \frac{2}{a^2} x^{2c-2-2\beta} e^{-Ax^{-2\beta}}$$

Ex. Joint Survival Probability

- Then the joint survival probability for two firms by time $t > 0$ is given by the eigenfunction expansion ($\mathbf{x} = (x_1, x_2) = (S_0^1, S_0^2)$):

$$\mathbb{Q}(\tau_{\{1,2\}} > t) = \mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}\right]$$

Ex. Joint Survival Probability

- Then the joint survival probability for two firms by time $t > 0$ is given by the eigenfunction expansion ($\mathbf{x} = (x_1, x_2) = (S_0^1, S_0^2)$):

$$\mathbb{Q}(\tau_{\{1,2\}} > t) = \mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}\right]$$

$$= \sum_{n_1, n_2 = 1}^{\infty} e^{-\phi(\lambda_{n_1}^1, \lambda_{n_2}^2)t} c_{n_1}^1 c_{n_2}^2 \varphi_{n_1}^1(x_1)\varphi_{n_2}^2(x_2)$$

Ex. Joint Survival Probability

- Then the joint survival probability for two firms by time $t > 0$ is given by the eigenfunction expansion ($\mathbf{x} = (x_1, x_2) = (S_0^1, S_0^2)$):

$$\mathbb{Q}(\tau_{\{1,2\}} > t) = \mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}\right]$$

$$= \sum_{n_1, n_2 = 1}^{\infty} e^{-\phi(\lambda_{n_1}^1, \lambda_{n_2}^2)t} c_{n_1}^1 c_{n_2}^2 \varphi_{n_1}^1(x_1) \varphi_{n_2}^2(x_2)$$

- Similarly, the single-name survival probabilities are given by the eigenfunction expansions:

$$\mathbb{Q}(\tau_k > t) = \sum_{n=1}^{\infty} e^{-\phi_k(\lambda_n^k)t} c_n^k \varphi_n^k(x_k), \quad k = 1, 2.$$

Ex. Joint Survival Probability

- Then the joint survival probability for two firms by time $t > 0$ is given by the eigenfunction expansion ($\mathbf{x} = (x_1, x_2) = (S_0^1, S_0^2)$):

$$\mathbb{Q}(\tau_{\{1,2\}} > t) = \mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}\right]$$

$$= \sum_{n_1, n_2 = 1}^{\infty} e^{-\phi(\lambda_{n_1}^1, \lambda_{n_2}^2)t} c_{n_1}^1 c_{n_2}^2 \varphi_{n_1}^1(x_1) \varphi_{n_2}^2(x_2)$$

- Similarly, the single-name survival probabilities are given by the eigenfunction expansions:

$$\mathbb{Q}(\tau_k > t) = \sum_{n=1}^{\infty} e^{-\phi_k(\lambda_n^k)t} c_n^k \varphi_n^k(x_k), \quad k = 1, 2.$$

The expansion coefficients are given by:

$$c_n^k = (\varphi_n, 1)_m = \frac{A_k^{\frac{1-2c_k}{4|\beta_k|}} (1/(2|\beta_k|))_{n-1} \Gamma(c_k/|\beta_k| + 1)}{\sqrt{(n-1)! |\mu_k + b_k| \Gamma(\nu_k + n)}}, \quad k = 1, 2, \quad n = 1, 2, ...,$$

where $(z)_n = z(z-1)...(z-n-1)$ is the Pochhammer symbol.

Ex. Joint Survival Probability

- Then the joint survival probability for two firms by time $t > 0$ is given by the eigenfunction expansion ($\mathbf{x} = (x_1, x_2) = (S_0^1, S_0^2)$):

$$\mathbb{Q}(\tau_{\{1,2\}} > t) = \mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}\right]$$

$$= \sum_{n_1, n_2 = 1}^{\infty} e^{-\phi(\lambda_{n_1}^1, \lambda_{n_2}^2)t} c_{n_1}^1 c_{n_2}^2 \varphi_{n_1}^1(x_1)\varphi_{n_2}^2(x_2)$$

- Similarly, the single-name survival probabilities are given by the eigenfunction expansions:

$$\mathbb{Q}(\tau_k > t) = \sum_{n=1}^{\infty} e^{-\phi_k(\lambda_n^k)t} c_n^k \varphi_n^k(x_k), \quad k = 1, 2.$$

$\phi(u, v)$ is the Laplace exponent of the two-dimensional subordinator $(\mathcal{T}^1, \mathcal{T}^2)^\top$

Ex. Joint Survival Probability

- Then the joint survival probability for two firms by time $t > 0$ is given by the eigenfunction expansion ($\mathbf{x} = (x_1, x_2) = (S_0^1, S_0^2)$):

$$\mathbb{Q}(\tau_{\{1,2\}} > t) = \mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}\right]$$

$$= \sum_{n_1,n_2=1}^{\infty} e^{-\phi(\lambda_{n_1}^1, \lambda_{n_2}^2)t} c_{n_1}^1 c_{n_2}^2 \varphi_{n_1}^1(x_1)\varphi_{n_2}^2(x_2)$$

- Similarly, the single-name survival probabilities are given by the eigenfunction expansions:

$$\mathbb{Q}(\tau_k > t) = \sum_{n=1}^{\infty} e^{-\phi_k(\lambda_n^k)t} c_n^k \varphi_n^k(x_k), \quad k = 1, 2.$$

$\phi(u, v)$ is the Laplace exponent of the two-dimensional subordinator $(\mathcal{T}^1, \mathcal{T}^2)^\top$

$\phi_1(u) := \phi(u, 0)$, and $\phi_2(u) := \phi(0, u)$ are the Laplace exponents of the marginal one-dimensional subordinators $\mathcal{T}^k$, $k \in \{1, 2\}$, respectively.

Default Correlation

- The default correlation has the form:

$$\operatorname{Corr}\left(\mathbf{1}_{\{\tau_1>t\}},\mathbf{1}_{\{\tau_2>t\}}\right) = \frac{\mathbb{Q}\big(\tau_{\{1,2\}}>t\big)-\mathbb{Q}(\tau_1>t)\mathbb{Q}(\tau_2>t)}{\prod_{k=1}^{2}\sqrt{\mathbb{Q}(\tau_k>t)(1-\mathbb{Q}(\tau_k>t))}}$$

Default Correlation

- The default correlation has the form:

$$
\begin{aligned}
\mathrm{Corr}\left(\mathbf{1}_{\{\tau_1>t\}},\mathbf{1}_{\{\tau_2>t\}}\right) &= \frac{\mathbb{Q}\big(\tau_{\{1,2\}}>t\big)-\mathbb{Q}(\tau_1>t)\mathbb{Q}(\tau_2>t)}{\prod_{k=1}^{2}\sqrt{\mathbb{Q}(\tau_k>t)(1-\mathbb{Q}(\tau_k>t))}} \\
&= \frac{\sum_{\mathbf{n}\in\mathbb{N}_1^2}\left(e^{-\phi(\lambda_{n_1}^1,\lambda_{n_2}^2)t}-e^{-\left(\phi_1(\lambda_{n_1}^1)+\phi_2(\lambda_{n_2}^2)\right)t}\right)c_{\mathbf{n}}\varphi_{\mathbf{n}}(\mathbf{x})}{\prod_{k=1}^{2}\sqrt{\mathbb{Q}(\tau_k>t)(1-\mathbb{Q}(\tau_k>t))}}
\end{aligned}
$$

Default Correlation

- The default correlation has the form:

$$\mathrm{Corr}\left(\mathbf{1}_{\{\tau_1>t\}},\mathbf{1}_{\{\tau_2>t\}}\right) = \frac{\mathbb{Q}\big(\tau_{\{1,2\}}>t\big)-\mathbb{Q}(\tau_1>t)\mathbb{Q}(\tau_2>t)}{\prod_{k=1}^{2}\sqrt{\mathbb{Q}(\tau_k>t)(1-\mathbb{Q}(\tau_k>t))}}$$

$$= \frac{\sum_{\mathbf{n}\in\mathbb{N}_1^2}\left(e^{-\phi(\lambda^1_{n_1},\lambda^2_{n_2})t}-e^{-\left(\phi_1(\lambda^1_{n_1})+\phi_2(\lambda^2_{n_2})\right)t}\right)c_{\mathbf{n}}\varphi_{\mathbf{n}}(\mathbf{x})}{\prod_{k=1}^{2}\sqrt{\mathbb{Q}(\tau_k>t)(1-\mathbb{Q}(\tau_k>t))}}$$

- From this expression we observe that:
  - it is zero if and only if $\phi(u_1,u_2)=\phi(u_1,0)+\phi(0,u_2)$,,

Default Correlation

- The default correlation has the form:

$$\mathrm{Corr}\left(\mathbf{1}_{\{\tau_1>t\}},\mathbf{1}_{\{\tau_2>t\}}\right) = \frac{\mathbb{Q}\big(\tau_{\{1,2\}}>t\big)-\mathbb{Q}(\tau_1>t)\mathbb{Q}(\tau_2>t)}{\prod_{k=1}^{2}\sqrt{\mathbb{Q}(\tau_k>t)(1-\mathbb{Q}(\tau_k>t))}}$$

$$= \frac{\sum_{\mathbf{n}\in\mathbb{N}_1^2}\left(e^{-\phi(\lambda^1_{n_1},\lambda^2_{n_2})t}-e^{-\left(\phi_1(\lambda^1_{n_1})+\phi_2(\lambda^2_{n_2})\right)t}\right)c_{\mathbf{n}}\varphi_{\mathbf{n}}(\mathbf{x})}{\prod_{k=1}^{2}\sqrt{\mathbb{Q}(\tau_k>t)(1-\mathbb{Q}(\tau_k>t))}}$$

- From this expression we observe that:
  - it is zero if and only if $\phi(u_1,u_2)=\phi(u_1,0)+\phi(0,u_2)$,,

$\Rightarrow$ That is, the coordinates $\mathcal{T}^1$ and $\mathcal{T}^2$ of the two-dimensional subordinator are independent.

Numerical Illustration

- We consider the two-name defaultable stock model.

Numerical Illustration

- We consider the two-name defaultable stock model.
- For this example the two diffusion processes X are taken to be JDCEV with the same set of parameters are,

$X_0 = x$	a	b	c	q	β	μ	r
50	10	0.01	0.5	0	-1	-0.3	0.05

Table: *JDCEV parameter values.*

Numerical Illustration

- We consider the two-name defaultable stock model.
- For this example the two diffusion processes X are taken to be JDCEV with the same set of parameters are,

$X_0 = x$	a	b	c	q	β	μ	r
50	10	0.01	0.5	0	-1	-0.3	0.05

Table: *JDCEV parameter values.*

- The volatility scale parameter a in the local volatility function $\sigma(x) = ax^{\beta}$ is selected so that $\sigma(50) = 0.2$.

Numerical Illustration

- The two-dimensional subordinator $\mathcal{T}$ is constructed from three independent Inverse Gaussian processes subordinators $\mathcal{S}_t^i$, $i = 1, 2, 3$, as follows:

$$\mathcal{T}_t^k = \mathcal{S}_t^k + \mathcal{S}_t^3, \quad k = 1, 2.$$

	γ	Y	η	C
$\mathcal{S}_t^1$	0	0.5	1	0.7
$\mathcal{S}_t^2$	0	0.5	1	0.7
$\mathcal{S}_t^3$	0	0.5	0.001	0.025

Table: *IG parameter values.*

Numerical Illustration

- The two-dimensional subordinator $\mathcal{T}$ is constructed from three independent Inverse Gaussian processes subordinators $\mathcal{S}_t^i$, $i = 1, 2, 3$, as follows:

$$\mathcal{T}_t^k = \mathcal{S}_t^k + \mathcal{S}_t^3, \quad k = 1, 2.$$

	γ	Y	η	C
$\mathcal{S}_t^1$	0	0.5	1	0.7
$\mathcal{S}_t^2$	0	0.5	1	0.7
$\mathcal{S}_t^3$	0	0.5	0.001	0.025

Table: *IG parameter values.*

- In this specification $\mathcal{S}^1$ and $\mathcal{S}^2$ are two idiosyncratic components that influence only the first stock and the second stock, respectively, and

Numerical Illustration

- The two-dimensional subordinator $\mathcal{T}$ is constructed from three independent Inverse Gaussian processes subordinators $\mathcal{S}_t^i$, $i = 1, 2, 3$, as follows:

$$\mathcal{T}_t^k = \mathcal{S}_t^k + \mathcal{S}_t^3, \quad k = 1, 2.$$

	γ	Y	η	C
$\mathcal{S}_t^1$	0	0.5	1	0.7
$\mathcal{S}_t^2$	0	0.5	1	0.7
$\mathcal{S}_t^3$	0	0.5	0.001	0.025

Table: *IG parameter values.*

- In this specification $\mathcal{S}^1$ and $\mathcal{S}^2$ are two idiosyncratic components that influence only the first stock and the second stock, respectively, and
- $\mathcal{S}_t^3$ is the systematic component common to both stocks.

Numerical Illustration

- The two-dimensional subordinator $\mathcal{T}$ is constructed from three independent Inverse Gaussian processes subordinators $\mathcal{S}_t^i$, $i = 1, 2, 3$, as follows:

$$\mathcal{T}_t^k = \mathcal{S}_t^k + \mathcal{S}_t^3, \quad k = 1, 2.$$

	γ	Y	η	C
$\mathcal{S}_t^1$	0	0.5	1	0.7
$\mathcal{S}_t^2$	0	0.5	1	0.7
$\mathcal{S}_t^3$	0	0.5	0.001	0.025

Table: *IG parameter values.*

- In this specification $\mathcal{S}^1$ and $\mathcal{S}^2$ are two idiosyncratic components that influence only the first stock and the second stock, respectively, and
- $\mathcal{S}_t^3$ is the systematic component common to both stocks.
- The parameter η is the decay parameter (damping parameter), which controls large size jumps $\Rrightarrow$ $\mathcal{S}_t^3$ exhibits larger jumps.

Numerical Illustration

- The two-dimensional subordinator $\mathcal{T}$ is constructed from three independent Inverse Gaussian processes subordinators $\mathcal{S}_t^i$, $i = 1, 2, 3$, as follows:

$$\mathcal{T}_t^k = \mathcal{S}_t^k + \mathcal{S}_t^3, \quad k = 1, 2.$$

	γ	Y	η	C
$\mathcal{S}_t^1$	0	0.5	1	0.7
$\mathcal{S}_t^2$	0	0.5	1	0.7
$\mathcal{S}_t^3$	0	0.5	0.001	0.025

Table: *IG parameter values.*

- In this specification $\mathcal{S}^1$ and $\mathcal{S}^2$ are two idiosyncratic components that influence only the first stock and the second stock, respectively, and
- $\mathcal{S}_t^3$ is the systematic component common to both stocks.
- The parameter η is the decay parameter (damping parameter), which controls large size jumps $\Rrightarrow$ $\mathcal{S}_t^3$ exhibits larger jumps.
- Since the drift is zero ($\gamma = \mathbf{0}$) then the time changed processes $X^i_{\mathcal{T}_t^i}$ are pure jump processes

Numerical Illustration: Survival Probability

- As the sock price falls, the firm's survival probability decreases

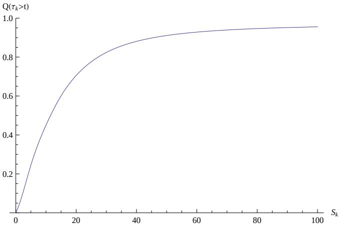
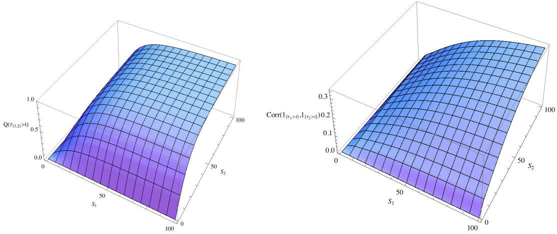


Figure: *Single-name survival probability $\mathbb{Q}(\tau > t)$ for $t = 1$ year as a function of the stock price $S_0 = x$.*

Numerical Illustration: Joint Survival Probability & Default Correlation

- As the stock prices fall, the joint survival probability also decreases which, in turn, causes the default correlation to decrease

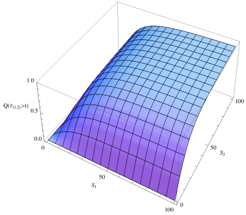


(a) Joint survival probability. (b) Correlation of default indicators.

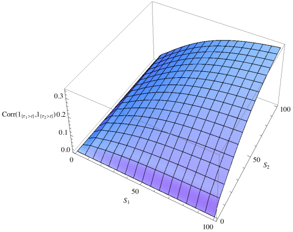
Figure: *Joint survival probability* $\mathbb{Q}(\tau_{\{1,2\}} > t)$ *and default correlation* $Corr(\mathbf{1}_{\{\tau_1>t\}}, \mathbf{1}_{\{\tau_2>t\}})$ *for* $t = 1$ *year as functions of stock prices* S_0^1 *and* S_0^2.

Numerical Illustration: Joint Survival Probability & Default Correlation

- As the stock prices fall, the joint survival probability also decreases which, in turn, causes the default correlation to decrease



(a) Joint survival probability.



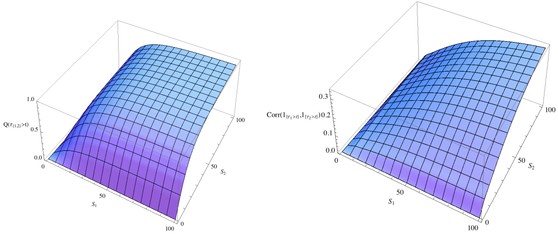
(b) Correlation of default indicators.

Figure: *Joint survival probability* $\mathbb{Q}(\tau_{\{1,2\}} > t)$ *and default correlation* $Corr(\mathbf{1}_{\{\tau_1>t\}}, \mathbf{1}_{\{\tau_2>t\}})$ *for* $t = 1$ *year as functions of stock prices* S_0^1 *and* S_0^2.

- When the stock price is relatively high, the default can only be triggered by a large catastrophic jump to zero $\Rightarrow$ the systematic component $\mathcal{S}^3$ governs large jumps.

Numerical Illustration: Joint Survival Probability & Default Correlation

- As the stock prices fall, the joint survival probability also decreases which, in turn, causes the default correlation to decrease



(a) Joint survival probability. (b) Correlation of default indicators.

Figure: *Joint survival probability* $\mathbb{Q}(\tau_{\{1,2\}} > t)$ *and default correlation* $Corr(\mathbf{1}_{\{\tau_1 > t\}}, \mathbf{1}_{\{\tau_2 > t\}})$ *for* $t = 1$ *year as functions of stock prices* S_0^1 *and* S_0^2.

- When the stock price is relatively high, the default can only be triggered by a large catastrophic jump to zero $\Rrightarrow$ the systematic component $\mathcal{S}^3$ governs large jumps.
- When the stock price is low, a smaller jump is enough to trigger default $\Rrightarrow$ the idiosyncratic components $\mathcal{S}^1$ and $\mathcal{S}^2$ primarily govern small jumps.

Two Firms Basket Put Option

Basket Option: Analytical Solutions

- Consider a basket put option on the portfolio of two stocks with the payoff at time t

$$f(S_t^1, S_t^2) = (K - w_1 S_t^1 - w_2 S_t^2)^+$$

Two Firms Basket Put Option

▸ Basket Option: Analytical Solutions

- Consider a basket put option on the portfolio of two stocks with the payoff at time t

$$f(S_t^1, S_t^2) = (K - w_1 S_t^1 - w_2 S_t^2)^+$$

- We observe six contingent claims:

Two Firms Basket Put Option

▸ Basket Option: Analytical Solutions

- Consider a basket put option on the portfolio of two stocks with the payoff at time t

$$f(S_t^1, S_t^2) = (K - w_1 S_t^1 - w_2 S_t^2)^+$$

- We observe six contingent claims:
  - One basket put that delivers the payoff if and only if both firms survive to maturity

$$\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}(K - w_1 S_t^1 + w_2 S_t^2)^+$$

Two Firms Basket Put Option

▸ Basket Option: Analytical Solutions

- Consider a basket put option on the portfolio of two stocks with the payoff at time t

$$f(S_t^1, S_t^2) = (K - w_1 S_t^1 - w_2 S_t^2)^+$$

- We observe six contingent claims:
  - One basket put that delivers the payoff if and only if both firms survive to maturity

$$\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}(K - w_1 S_t^1 + w_2 S_t^2)^+$$

  - Two single-name puts that deliver the payoffs if and only if both firms survive to maturity

$$\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}(K - w_k S_t^k)^+, \quad k = 1, 2$$

Two Firms Basket Put Option

Basket Option: Analytical Solutions

- Consider a basket put option on the portfolio of two stocks with the payoff at time t

$$f(S_t^1, S_t^2) = (K - w_1 S_t^1 - w_2 S_t^2)^+$$

- We observe six contingent claims:
  - One basket put that delivers the payoff if and only if both firms survive to maturity

$$\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}(K - w_1 S_t^1 + w_2 S_t^2)^+$$

  - Two single-name puts that deliver the payoffs if and only if both firms survive to maturity

$$\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}(K - w_k S_t^k)^+, \quad k = 1, 2$$

  - Two single-name puts that deliver the payoffs if and only if the firm whose stock the put is written on survives to maturity.

$$\mathbf{1}_{\{\tau_k > t\}}(K - w_k S_t^k)^+, \quad k = 1, 2$$

Two Firms Basket Put Option

▸ Basket Option: Analytical Solutions

- Consider a basket put option on the portfolio of two stocks with the payoff at time t

$$f(S_t^1, S_t^2) = (K - w_1 S_t^1 - w_2 S_t^2)^+$$

- We observe six contingent claims:
  - One basket put that delivers the payoff if and only if both firms survive to maturity

$$\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}(K - w_1 S_t^1 + w_2 S_t^2)^+$$

  - Two single-name puts that deliver the payoffs if and only if both firms survive to maturity

$$\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}(K - w_k S_t^k)^+, \quad k = 1, 2$$

  - Two single-name puts that deliver the payoffs if and only if the firm whose stock the put is written on survives to maturity.

$$\mathbf{1}_{\{\tau_k > t\}}(K - w_k S_t^k)^+, \quad k = 1, 2$$

  - An embedded multi-name credit derivative

$$K(\mathbf{1}_{\{\tau_{\{1,2\}} > t\}} + 1 - \mathbf{1}_{\{\tau_1 > t\}} - \mathbf{1}_{\{\tau_2 > t\}}) = K\mathbf{1}_{\{\tau_1 \vee \tau_2 \le t\}}$$

Two Firms Basket Put Option

Basket Option: Analytical Solutions

- Consider a basket put option on the portfolio of two stocks with the payoff at time t

$$f(S_t^1, S_t^2) = (K - w_1 S_t^1 - w_2 S_t^2)^+$$

- We observe six contingent claims:
  - One basket put that delivers the payoff if and only if both firms survive to maturity

$$\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}(K - w_1 S_t^1 + w_2 S_t^2)^+$$

  - Two single-name puts that deliver the payoffs if and only if both firms survive to maturity

$$\mathbf{1}_{\{\tau_{\{1,2\}} > t\}}(K - w_k S_t^k)^+, \quad k = 1, 2$$

  - Two single-name puts that deliver the payoffs if and only if the firm whose stock the put is written on survives to maturity.

$$\mathbf{1}_{\{\tau_k > t\}}(K - w_k S_t^k)^+, \quad k = 1, 2$$

  - An embedded multi-name credit derivative

$$K(\mathbf{1}_{\{\tau_{\{1,2\}} > t\}} + 1 - \mathbf{1}_{\{\tau_1 > t\}} - \mathbf{1}_{\{\tau_2 > t\}}) = K\mathbf{1}_{\{\tau_1 \vee \tau_2 \leq t\}}$$

  - **We obtained explicit analytical solutions for all these claims.** Solutions

Numerical Illustration: Joint Survival Probability & Default Correlation

- The price of a European-style basket put option on the equally-weighted portfolio of two stocks ($w_1 = w_2 = 1$) with one year to maturity ($t = 1$) and with the strike price $K = 100$ as a function of the initial stock prices S_0^1 and S_0^2.

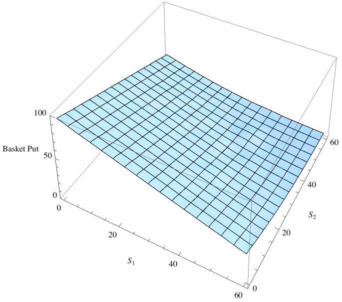


Figure: *Two-name basket put prices for the range of initial stock prices S_0^1 and S_0^2 from zero to \$60 for one year time to maturity and $K = 100$.*

Numerical Illustration: Joint Survival Probability & Default Correlation

- The price of a European-style basket put option on the equally-weighted portfolio of two stocks ($w_1 = w_2 = 1$) with one year to maturity ($t = 1$) and with the strike price $K = 100$ as a function of the initial stock prices S_0^1 and S_0^2.

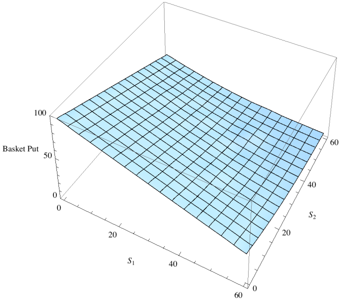


Figure: *Two-name basket put prices for the range of initial stock prices S_0^1 and S_0^2 from zero to \$60 for one year time to maturity and $K = 100$.*

- When both firms are in default, $(S_0^1, S_0^2) = (0, 0)$, the price of the basket put is equal to the discounted strike K.

Numerical Illustration: Joint Survival Probability & Default Correlation

- The price of a European-style basket put option on the equally-weighted portfolio of two stocks ($w_1 = w_2 = 1$) with one year to maturity ($t = 1$) and with the strike price $K = 100$ as a function of the initial stock prices S_0^1 and S_0^2.

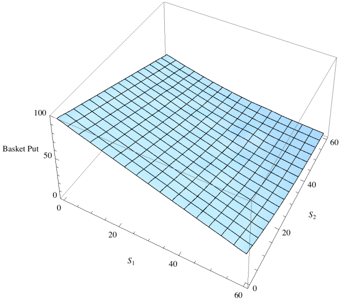


Figure: *Two-name basket put prices for the range of initial stock prices S_0^1 and S_0^2 from zero to \$60 for one year time to maturity and $K = 100$.*

- When both firms are in default, $(S_0^1, S_0^2) = (0, 0)$, the price of the basket put is equal to the discounted strike K.
- When one of the two firms is in default, the basket put reduces to the single-name European-style put on the surviving stock with the strike K.

Conclusion

- We propose a modeling framework based on multi-variate subordination of diffusion processes.

Conclusion

- We propose a modeling framework based on multi-variate subordination of diffusion processes.

  1. We start with n independent jump-to-default extended diffusions for n stocks.
  2. Then we time change each one with a coordinate of a n-dimensional Subordinator

Conclusion

- We propose a modeling framework based on multi-variate subordination of diffusion processes.

  1. We start with n independent jump-to-default extended diffusions for n stocks.
  2. Then we time change each one with a coordinate of a n-dimensional Subordinator

  $\Rrightarrow$ the result is multi-name credit-equity model with dependent jumps and jumps-to-default for all stocks.

Conclusion

- We propose a modeling framework based on multi-variate subordination of diffusion processes.

  1. We start with n independent jump-to-default extended diffusions for n stocks.
  2. Then we time change each one with a coordinate of a n-dimensional Subordinator

  $\Rrightarrow$ the result is multi-name credit-equity model with dependent jumps and jumps-to-default for all stocks.

- The dependence among jumps is governed by the Lévy measure of the n-dimensional subordinator.

Conclusion

- We propose a modeling framework based on multi-variate subordination of diffusion processes.

  1. We start with n independent jump-to-default extended diffusions for n stocks.
  2. Then we time change each one with a coordinate of a n-dimensional Subordinator

  $\Rrightarrow$ the result is multi-name credit-equity model with dependent jumps and jumps-to-default for all stocks.

- The dependence among jumps is governed by the Lévy measure of the n-dimensional subordinator.

- The semigroup theory provides powerful analytical and computational tools for securities pricing.

Conclusion

- We propose a modeling framework based on multi-variate subordination of diffusion processes.

  1. We start with n independent jump-to-default extended diffusions for n stocks.
  2. Then we time change each one with a coordinate of a n-dimensional Subordinator

  $\Rrightarrow$ the result is multi-name credit-equity model with dependent jumps and jumps-to-default for all stocks.

- The dependence among jumps is governed by the Lévy measure of the n-dimensional subordinator.

- The semigroup theory provides powerful analytical and computational tools for securities pricing.

- Thank you!

Multiparameter Semigroup

Return

- If $\{\mathcal{P}_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}_{+}^{n}\}$ is a *n*-parameter strongly continuous semigroup on a Banach space **B**, then:

Multiparameter Semigroup

Return

- If $\{\mathcal{P}_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}_+^n\}$ is a *n*-parameter strongly continuous semigroup on a Banach space **B**, then:

$\Rightarrow$ it is the product of *n* one-parameter strongly continuous semigroups $\{\mathcal{P}_t^i, t \geq 0\}$ on **B** with infinitesimal generators $\mathcal{G}_i$ with domains $\mathrm{Dom}(\mathcal{G}_i) \subset \mathbf{B}$.

Multiparameter Semigroup

Return

- If $\{\mathcal{P}_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}_+^n\}$ is a *n*-parameter strongly continuous semigroup on a Banach space **B**, then:

$\Rightarrow$ it is the product of *n* one-parameter strongly continuous semigroups $\{\mathcal{P}_t^i, t \geq 0\}$ on **B** with infinitesimal generators $\mathcal{G}_i$ with domains $\mathrm{Dom}(\mathcal{G}_i) \subset \mathbf{B}$.

- That is, for $\mathbf{t} = (t_1, ..., t_n)$ we have:

$$\mathcal{P}_{\mathbf{t}} = \prod_{i=1}^{n} \mathcal{P}_{t_i}^i$$

and the semigroup operators $\mathcal{P}_{t_i}^i$ commute with each other, $t_i \geq 0$, $i = 1, ..., n$.

Multiparameter Semigroup

Return

- If $\{\mathcal{P}_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}_{+}^{n}\}$ is a *n*-parameter strongly continuous semigroup on a Banach space **B**, then:

$\Rightarrow$ it is the product of *n* one-parameter strongly continuous semigroups $\{\mathcal{P}_t^i, t \geq 0\}$ on **B** with infinitesimal generators $\mathcal{G}_i$ with domains $\mathrm{Dom}(\mathcal{G}_i) \subset \mathbf{B}$.

  - That is, for $\mathbf{t} = (t_1, ..., t_n)$ we have:

$$\mathcal{P}_{\mathbf{t}} = \prod_{i=1}^{n} \mathcal{P}_{t_i}^i$$

and the semigroup operators $\mathcal{P}_{t_i}^i$ commute with each other, $t_i \geq 0$, $i = 1, ..., n$.

- In our case, the expectation operators associated with the Markov processes X^i define the corresponding semigroups $\{\mathcal{P}_{t_i}^i, t_i \geq 0\}$,

$$\mathcal{P}_{t_i}^i f(x_i) := \mathbb{E}_{x_i}[\mathbf{1}_{\{\zeta_i > t_i\}} f(X_{t_i}^i)], \quad x_i \in E_i, \quad t_i \geq 0,$$

in Banach spaces of bounded Borel measurable functions on E_i.

Return

Return

Two Firms Basket Put Option ▸ Return

- The embedded multi-name credit derivative with the notional amount equal to the strike price K and paid at maturity if both firms default

$$e^{-rt}\mathbb{E}[K\mathbf{1}_{\{\tau_1 \vee \tau_2 \le t\}}] = e^{-rt}K(1 + \mathbb{Q}(\tau_{\{1,2\}} > t) - \mathbb{Q}(\tau_1 > t) - \mathbb{Q}(\tau_2 > t))$$

where the joint survival probability $\mathbb{Q}(\tau_{\{1,2\}} > t)$ and marginal survival probabilities $\mathbb{Q}(\tau_k > t)$, $k = 1, 2$; were given earlier.

Two Firms Basket Put Option ▸ Return

- The basket put that delivers the payoff if and only if both firms survive to maturity

$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}(K - w_1 S_t^1 + w_2 S_t^2)^+\right]$$

Two Firms Basket Put Option ▸ Return

- The basket put that delivers the payoff if and only if both firms survive to maturity

$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}(K - w_1 S_t^1 + w_2 S_t^2)^+\right]$$

$$= e^{-r\,t}\sum_{n_1,n_2=1}^{\infty} \overbrace{e^{-\phi(\lambda_{n_1}^1,\lambda_{n_2}^2)\,t}}^{\text{2D Lévy Exp.}} c_{n_1,n_2}(K)\varphi_{n_1}^1(x_1)\varphi_{n_1}^2(x_2)$$

Two Firms Basket Put Option ▸ Return

- The basket put that delivers the payoff if and only if both firms survive to maturity

$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}(K - w_1 S_t^1 + w_2 S_t^2)^+\right]$$

$$= e^{-r\,t}\sum_{n_1,n_2=1}^{\infty} e^{-\phi(\lambda^1_{n_1},\lambda^2_{n_2})\,t}\,\underbrace{c_{n_1,n_2}(K)}\,\varphi^1_{n_1}(x_1)\varphi^2_{n_1}(x_2)$$

- Where the expansion coefficient $c_{n_1,n_2}(K)$ is given by,

$$c_{n_1,n_2}(K) = \left((K - w_1 x_1 - w_2 x_2)^+, \varphi_{\mathbf{n}}(\mathbf{x})\right)_{\mathbf{m}}$$

$$= \int_{\mathbb{R}^2_+} (K - w_1 x_1 - w_2 x_2)^+ \varphi^1_{n_1}(x_1)\varphi^2_{n_2}(x_2) m_1(x_1) m_2(x_2) dx_1 dx_2$$

Two Firms Basket Put Option ▸ Return

- The basket put that delivers the payoff if and only if both firms survive to maturity

$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}(K - w_1 S_t^1 + w_2 S_t^2)^+\right]$$

$$= e^{-r\,t}\sum_{n_1,n_2=1}^{\infty} e^{-\phi(\lambda^1_{n_1},\lambda^2_{n_2})\,t}\,\underbrace{c_{n_1,n_2}(K)}\,\varphi^1_{n_1}(x_1)\varphi^2_{n_1}(x_2)$$

- Where the expansion coefficient $c_{n_1,n_2}(K)$ is given by,

$$c_{n_1,n_2}(K) = \left((K - w_1x_1 - w_2x_2)^+, \varphi_{\mathbf{n}}(\mathbf{x})\right)_{\mathbf{m}}$$

$$= \int_{\mathbb{R}^2_+}(K - w_1x_1 - w_2x_2)^+\varphi^1_{n_1}(x_1)\varphi^2_{n_2}(x_2)m_1(x_1)m_2(x_2)dx_1dx_2$$

$$= K\prod_{k=1}^{2}\left(\sqrt{\frac{\Gamma(\nu_k+n_k)}{\Gamma(n_k)|\mu_k+b_k|}}\frac{2|\beta_k|A_k^{\frac{\nu_k}{2}+1}\tilde{K}_k^{2c_k-2\beta_k}}{\Gamma(\nu_k+1)}\right)$$

$$\times\sum_{p_1,p_2=0}^{\infty}\frac{(-1)^{p_1+p_2}\,(\nu_1+n_1)_{p_1}\,(\nu_2+n_2)_{p_2}}{(\nu_1+1)_{p_1}\;p_1!\,(\nu_2+1)_{p_2}\;p_2!}\left(A_1\tilde{K}_1^{-2\beta_1}\right)^{p_1} d\left(A_2\tilde{K}_2^{-2\beta_2}\right)^{p_2}$$

$$\times\frac{\Gamma\left(2c_1-2\beta_1(p_1+1)\right)\Gamma\left(2c_2-2\beta_2(p_2+1)\right)}{\Gamma\left(2c_1-2\beta_1(p_1+1)+2c_2-2\beta_2(p_2+1)+2\right)}.$$

where $\tilde{K}_k = e^{-\rho_k t}K/w_k$.

Two Firms Basket Put Option ▸ Return

- The single-name put on the stock S^k that delivers the payoff if and only if the firm survives to maturity:

$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_k>t\}}(K-w_kS_t^k)^+\right]=e^{-rt}\sum_{n=1}^{\infty}\overbrace{e^{-\phi_k(\lambda_n^k)\,t}}^{\text{1D Lévy Exp.}}p_n^k(K)\varphi_n^k(x_k),$$

Two Firms Basket Put Option

Return

- The single-name put on the stock S^k that delivers the payoff if and only if the firm survives to maturity:

$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_k>t\}}(K-w_kS_t^k)^+\right]=e^{-rt}\sum_{n=1}^{\infty}e^{-\phi_k(\lambda_n^k)\,t}\,\underbrace{p_n^k(K)}\,\varphi_n^k(x_k),$$

- Where the expansion coefficient $p_n^k(K)$ is given as,

$$p_n^k(K)=\Big((K-w_kx_k)^+,\varphi_n^k(x_k)\Big)_{m_k}$$

$$=\int_{\mathbb{R}_+}(K-w_kx_k)^+\varphi_n^k(x_k)m_k(x_k)dx_k$$

Two Firms Basket Put Option ▸ Return

- The single-name put on the stock S^k that delivers the payoff if and only if the firm survives to maturity:

$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_k>t\}}(K-w_kS_t^k)^+\right]=e^{-rt}\sum_{n=1}^{\infty}e^{-\phi_k(\lambda_n^k)\,t}\,\underbrace{p_n^k(K)}\varphi_n^k(x_k),$$

- Where the expansion coefficient $p_n^k(K)$ is given as,

$$p_n^k(K)=\left((K-w_kx_k)^+,\varphi_n^k(x_k)\right)_{m_k}$$

$$=\int_{\mathbb{R}_+}(K-w_kx_k)^+\varphi_n^k(x_k)m_k(x_k)dx_k$$

$$=K\sqrt{\frac{\Gamma(\nu_k+n)}{\Gamma(n)|\mu_k+b_k|}}\frac{A_k^{\frac{\nu_k}{2}+1}\tilde{K}_k^{2(c_k-\beta_k)}}{\Gamma(\nu_k+1)}\times$$

$$\left\{\frac{1}{(1+c_k/|\beta_k|)}\,{}_2F_2\left(\begin{array}{cc}\nu_k+n, & \nu_k+1-\frac{1}{2|\beta_k|}\\ \nu_k+1, & \nu_k+2-\frac{1}{2|\beta_k|}\end{array};-A_k\tilde{K}_k^{-2\beta_k}\right)\right.$$

$$\left.-\frac{1}{(\nu_k+1)}\,{}_1F_1\left(\begin{array}{c}\nu_k+n\\ \nu_k+2\end{array};-A_k\tilde{K}_k^{-2\beta_k}\right)\right\},$$

where ${}_1F_1$ and ${}_2F_2$ are the Kummer confluent hypergeometric function and the generalized hypergeometric function, respectively; and $\tilde{K}_k=e^{-\rho_kt}K/w_k$.

Two Firms Basket Put Option

Return

- The single-name put on the stock S^1 that delivers the payoff if and only if both firms survive:

$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}(K-w_1S_t^1)^+\right]=e^{-rt}\sum_{n_1,n_2=1}^{\infty}\overbrace{e^{-\phi(\lambda_{n_1}^1,\lambda_{n_2}^2)\,t}}^{\text{2D Lévy Exp.}}p_{n_1}^1(K)c_{n_2}^2\varphi_{n_1}^1(x_1)\varphi_{n_2}^2(x_2),$$

Two Firms Basket Put Option

▸ Return

- The single-name put on the stock S^1 that delivers the payoff if and only if both firms survive:

$$e^{-rt}\mathbb{E}\left[\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}(K-w_1S_t^1)^+\right] = e^{-rt}\sum_{n_1,n_2=1}^{\infty} e^{-\phi(\lambda^1_{n_1},\lambda^2_{n_2})\,t}\,\underbrace{p^1_{n_1}(K)c^2_{n_2}}\,\varphi^1_{n_1}(x_1)\varphi^2_{n_2}(x_2),$$

- where c_n^2 are the coefficients of the expansion for the survival probability of the second stock and,
- $p_n^1(K)$ are the expansion coefficients for the single-name put on the first stock.

Return

Return