Optimal Timing to Buy Options in Incomplete Markets

Mike Ludkovski

Department of Statistics & Applied Probability
University of California Santa Barbara

joint work with Tim Leung
http://www.pstat.ucsb.edu/faculty/ludkovski

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Making a Profit

- Classical theory: complete market

$\rightarrow$ unique no-arbitrage price for any derivative.

- More realistic: incomplete market.
- There is a range of prices consistent with no-arbitrage.
- A derivative has market price P; investor has their own model price $\tilde{P}$.
- The spread is a profit opportunity.
- Statistical arbitrage: buy at $P < \tilde{P}$, ... generate profit (on average) through hedging.

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Making a Profit (cont.)

- Step I: identify derivatives that seem to be "mispriced" by the market.
- Find a contract F such that $P_t < \tilde{P}_t$ – underpriced. Opportunity to buy and make a profit.
- But tomorrow, the spread might widen and can make even more profit.
- Step II: when to buy? $\rightarrow$ Timing option.
- Crucial factors: price dynamics, pricing measures, sources of risks/risk premia, & option payoff.

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Model Overview

- Market prices arise due to a spectrum of equivalent martingale measures (EMMs).
- Pricing measures can be parametrized by risk premia.
- Prevailing market measure & risk premium Q^{ϕ} vs. investor's $\tilde{Q}$.
- The investor wishes to buy the option so as to maximize $\tilde{P}_{\tau} - P_{\tau}$ over all (stopping) times τ.
- Link together literatures on EMMs in popular incomplete models and American options.
- No closed-form solutions, so focus on qualitative properties.
- Key: contract shape vis-a-vis risk premium spread.

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Outline

1. General incomplete market
   - Equivalent formulations/interpretations.
   - Delayed purchase premium
2. Defaultable stock model
   - Optimal stopping rule.
   - Default risk premium and option payoff.
   - Numerical examples - optimal purchase boundaries.
3. Stochastic volatility model
   - Optimal stopping rule.
   - Volatility risk premia and option payoff.
4. Link with Utility Pricing Approaches

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Optimal Purchase in a General Incomplete Market

- On $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$, model a risky asset S: $+$ve $\mathbb{F}$-locally bounded semimartingale.
- Universal filtration $\mathbb{F}$ – known to all participants (no insiders, etc.).
- Under the market pricing measure Q, the price of an European option F with maturity T is

$$P_t = \boldsymbol{E}^Q\{e^{-r(T-t)}F(S_T)|\,\mathcal{F}_t\}, \qquad 0 \le t \le T.$$

- The buyer prices the option under another EMM $\tilde{Q}$:

$$\tilde{P}_t = \boldsymbol{E}^{\tilde{Q}}\{e^{-r(T-t)}F(S_T)|\,\mathcal{F}_t\}, \qquad 0 \le t \le T.$$

Option Price Spread

- The buyer maximizes the expected discounted price spread:

$$J_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \boldsymbol{E}^{\tilde{Q}} \{ e^{-r(\tau - t)} (\tilde{P}_\tau - P_\tau) | \mathcal{F}_t \},$$

where $\mathcal{T}_{t,T}$ is the set of $\mathbb{F}$-stopping times taking values in $[t, T]$.

- J_t can be viewed as an American spread option.
- By iterated conditioning, J_t simplifies to

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$$= \tilde{P}_t - V_t,$$

where V_t is the buyer's minimized expected cost to buy the option:

$$V_t = \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_{t,T}} \boldsymbol{E}^{Q} \left\{ (Z_\tau / Z_t) \, e^{-r(\tau - t)} P_\tau \, | \, \mathcal{F}_t \right\}, \quad Z_t = \boldsymbol{E}^{Q} \{ \frac{d\tilde{Q}}{dQ} | \mathcal{F}_t \}.$$

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Basic properties

- Since $P_T = \tilde{P}_T = F(S_T)$, we have $J_T = 0$ and $J_t \geq 0$.
- If $\tilde{P}_u \leq P_u \; \forall u \geq t$, then $J_t = 0$ and $V_t = \tilde{P}_t$.
- Since t and T are candidate stopping times, we have $V_t \leq P_t \wedge \tilde{P}_t$.
- The optimal purchase time:

$$\tau_t^* = \inf\{\, t \leq u \leq T \;:\; V_u = P_u \,\} = \inf\{t \leq u \leq T : J_u = \tilde{P}_u - P_u\}.$$

- If $Q = \tilde{Q}$, then $V_t = P_t$ and the timing option is worthless.
- One of our goals: explicitly characterize when τ is trivial.
- Also, what factors delay/accelerate purchasing decisions?

Delayed Purchase Premium

- Can determine τ^* from the delayed purchase premium:

$$L_t := P_t - V_t = J_t - (\tilde{P}_t - P_t). \quad (\geq 0)$$

- The process $(e^{-rt} P_t Z_t)_{t \in [0,T]}$ satisfies

$$e^{-r\tau} P_\tau Z_\tau = e^{-rt} P_t Z_t + \int_t^\tau Z_{s-}\, d(e^{-rs} P_s) + \int_t^\tau e^{-rs} P_{s-}\, dZ_s + \int_t^\tau e^{-rs} d[P,Z]_s,$$

$$\begin{aligned} L_t &= P_t - \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_{t,T}} \boldsymbol{E}^Q \left\{ (Z_\tau / Z_t)\, e^{-r(\tau - t)} P_\tau \,\middle|\, \mathcal{F}_t \right\} \\ &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \boldsymbol{E}^Q \left\{ -(Z_t)^{-1} \int_t^\tau e^{-r(s-t)} d[P,Z]_s \,\middle|\, \mathcal{F}_t \right\}. \end{aligned}$$

- Hence, the quadratic covariation process $G_t := [P,Z]_t$ plays a vital role.
- Optimal purchase time: $\tau_t^* = \inf\{\, t \leq u \leq T \;:\; L_u = 0 \,\}$.

The τ-Optimal Pricing Measure Q^{τ^*}

- Denote the density processes associated with $\tilde{Q}$ and Q (with respect to $\mathbb{P}$) by

$$Z_t^b = \boldsymbol{E}\Big\{\frac{d\tilde{Q}}{d\mathbb{P}}\,|\,\mathcal{F}_t\Big\}, \quad \text{and} \quad Z_t^m = \boldsymbol{E}\Big\{\frac{dQ}{d\mathbb{P}}\,|\,\mathcal{F}_t\Big\}.$$

- Concatenate $\tilde{Q}$ and Q to form another measure Q^τ. Let $Z_t^\tau := \frac{dQ^\tau}{d\mathbb{P}}|_{\mathcal{F}_t}$ s.t.

$$Z_t^\tau := Z_t^b\,\mathbf{1}_{[0,\tau)}(t) + Z_t^m\frac{Z_\tau^b}{Z_\tau^m}\,\mathbf{1}_{[\tau,T]}(t), \qquad 0 \le t \le T.$$

- By change of measure, we obtain an alternative representation for V_t:

$$V_t = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \boldsymbol{E}^{\tilde{Q}}\left\{e^{-r(\tau-t)}P_\tau\,|\,\mathcal{F}_t\right\} = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \boldsymbol{E}\left\{\frac{Z_\tau^b}{Z_t^b}\frac{Z_T^m}{Z_\tau^m}\,e^{-r(T-t)}F(S_T)\,\Big|\,\mathcal{F}_t\right\}$$
$$= \operatorname*{ess\,inf}_{Q^\tau\in\mathcal{M}(Q,\tilde{Q})} \boldsymbol{E}^{Q^\tau}\{e^{-r(T-t)}F(S_T)\,|\,\mathcal{F}_t\}, \qquad \text{where } \mathcal{M}(Q,\tilde{Q}) = \{Q^\tau\}_{\tau\in\mathcal{T}}.$$

- Purchase at τ^* $\rightarrow$ *adopting* the market measure Q at τ^*.
- Timing flexibility $\rightarrow$ *expands* from one $\tilde{Q}$ to the collection $\mathcal{M}(Q,\tilde{Q})$.

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Buying Options on Defaultable Stock

- The pre-default dynamics of stock price S is

$$dS_t = (\mu + \hat{\lambda}_t) S_t \, dt + \sigma S_t \, d\hat{W}_t - S_{t-} \, dN_t, \qquad S_0 = s > 0,$$

  with $\mu, \sigma > 0$.
- $\hat{W}$ is a BM under $\mathbb{P}$ and $\hat{\lambda}$ is the $\mathcal{F}^S$-adapted default intensity process.
- At default time $\tau^{\hat{\lambda}}$, S drops to zero permanently.

$$\tau^{\hat{\lambda}} = \inf\{ t : \int_0^t \hat{\lambda}_s \, ds > E \}, \quad E \sim Exp(1), \quad E \perp \mathcal{F}^{\hat{W}}; \quad N_t = 1_{\{t \geq \tau^{\hat{\lambda}}\}}.$$

- Denote $\mathcal{F}_t = \mathcal{F}^S_t \vee \sigma(E)$; the compensated $(\mathbb{P}, \mathbb{F})$-martingale is $\hat{M}_t = N_t - \int_0^t \hat{\lambda}_s \, ds$.
- Focus on Markovian local intensities $\hat{\lambda}_t = \hat{\lambda}(t, S_t)$.
- Similar models include Merton ('76), Carr-Linetsky ('06), Linetsky ('06), etc.

The Buyer's Optimal Stopping Problem

- The set of EMMs $\{Q^{\phi,\alpha}\}$ is parametrized through the RN density

$$Z_t^{\phi,\alpha} := \frac{dQ^{\phi,\alpha}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{E}(-\phi\hat{W})_t\,\mathcal{E}(\alpha\hat{M})_t,$$

where the default risk premium α is a $+$ve bounded $\mathcal{F}_t$-predictable process, and ϕ is the market price of risk satisfying

$$\phi_t = \frac{\mu - r - \hat{\lambda}_t(\alpha_t - 1)}{\sigma}.$$

- By Girsanov Theorem, the evolution of S under any EMM $Q^{\phi,\alpha}$ is

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^{\phi,\alpha} - S_{t-}\,dM_t^{\phi,\alpha}, \qquad S_0 = s > 0,$$

where $W_t^{\phi,\alpha} = \hat{W}_t + \int_0^t \phi_u\,du$ and $M_t^{\phi,\alpha} = N_t - \int_0^t \alpha_s\hat{\lambda}_s\,ds$.

- $\{Q^{\phi,\alpha}\}$ is parametrized by α only, and $Q^{\phi,\alpha}$-default intensity is $\lambda^{\alpha_t} = \alpha_t\hat{\lambda}_t$.

The Optimal Timing Rule

- Pricing measures are $\tilde{Q} = Q^{\tilde{\phi},\tilde{\alpha}}$ (buyer) & $Q = Q^{\phi,\alpha}$ (market).
- Market price $P(t, S_t) := \boldsymbol{E}^{Q}\{e^{-r(T-t)}F(S_T) \mid S_t\}$. The buyer solves

$$V(t,s) := \inf_{\tau \in \mathcal{T}_{t,T}} \boldsymbol{E}^{\tilde{Q}}\{e^{-r(\tau-t)}P(\tau, S_\tau) \mid S_t = s\}$$

- The delayed purchase premium is

$$L(t,s) = \sup_{\tau \in \mathcal{T}_{t,T}} \boldsymbol{E}^{\tilde{Q}}\left\{ -\int_t^\tau e^{-r(u-t)}G(u, S_u)\,du \mid S_t = s \right\}, \text{ with}$$

$$G(t,s) = (\tilde{\lambda}(t,s) - \lambda(t,s))(s\frac{\partial P}{\partial s}(t,s) + P(t,0) - P(t,s)).$$

Theorem

If $G(t,s) \le 0\ \forall (t,s)$, then $\tau^* = T$ and $L(t,s) = P(t,s) - \tilde{P}(t,s)$.
If $G(t,s) \ge 0\ \forall (t,s)$, then $\tau^* = t$ is optimal for $V(t,s)$, and $L(t,s) = 0$.

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If $G(t,s) \geq 0 \; \forall (t,s)$, then $\tau^* = t$ is optimal for $V(t,s)$, and $L(t,s) = 0$.

Outline of Proof

- Consider the super/sub-martingality of $(e^{-rt}P(t,S_t)Z_t)_t$, with $Z_t := \frac{d\tilde{Q}}{dQ}\Big|_{\mathcal{F}_t}$.
- Recall that $\hat{P}_t = e^{-rt}P(t,S_t)$ and Z_t are both Q-martingales.
- Using Ito's formula, compute the dynamics of $e^{-rt}P(t,S_t)Z_t$ under Q:

$$\begin{aligned} d(Z_t\hat{P}_t) &= \hat{P}_t\,dZ_t + Z_t\,d\hat{P}_t + d\hat{P}_t\,dZ_t \\ &= \hat{P}_t\,dZ_t + Z_t\,d\hat{P}_t + Z_t(\frac{\tilde{\lambda}_t}{\lambda_t} - 1)(\hat{P}(t,0) - \hat{P}(t,S_{t-}))\,dM_t^Q \\ &\quad + Z_t(\tilde{\lambda}_t - \lambda_t)\big(S_t\frac{\partial\hat{P}}{\partial s}(t,S_t) + \hat{P}(t,0) - \hat{P}(t,S_{t-})\big)\,dt. \end{aligned}$$

- The drift of $d(Z_t\hat{P}_t)$ is the last dt term.
- Hence, the condition $G(t,s) \le 0$ (resp. $G(t,s) \ge 0$) implies that $Z\hat{P}$ is a Q-supermartingale (resp. Q-submartingale), and thus $\tau^* = T$ (resp. $\tau^* = 0$).

Price Convexity & Purchase Timing

- Recall: $G(t,s) = (\tilde{\lambda}(t,s) - \lambda(t,s))\big(s\frac{\partial P}{\partial s}(t,s) + P(t,0) - P(t,s)\big)$.

Corollary

Suppose $s \mapsto P(t,s)$ is convex for each $t \in [0,T]$ (i.e. gamma $P_{ss}(t,s) \geq 0$).
If $\tilde{\lambda}(t,s) \leq \lambda(t,s)\ \forall (t,s)$, then it is optimal to never buy the option, i.e. $\tau^* = T$.
If $\tilde{\lambda}(t,s) \geq \lambda(t,s)\ \forall (t,s)$, then it is optimal to buy the option now.

Example

Take $\lambda(t,s) = \lambda$, then the market Call and Put prices are

$$C(t,s) = C^{BS}(t,s;r+\lambda,\sigma,K,T), \quad P(t,s) = P^{BS}(t,s;r+\lambda,\sigma,K,T) + Ke^{-r(T-t)}(1-e^{-\lambda(T-t)}).$$

Calls and Puts are convex in s and admit the same drift function (P-C parity):
$G(t,s) = (\tilde{\lambda}(t,s) - \lambda)Ke^{-(r+\lambda)(T-t)}\Phi(d_2)$.

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Buying a European Call or Put

Numerically solve the variational inequality for $V(t,s)$ using implicit PSOR method:

$$\min\left(\frac{\partial V}{\partial t} + \mathcal{L}_{\tilde{\lambda}}\, V + \tilde{\lambda}(t,s)V(t,0), P - V\right) = 0, \qquad V(T,s) = F(s).$$

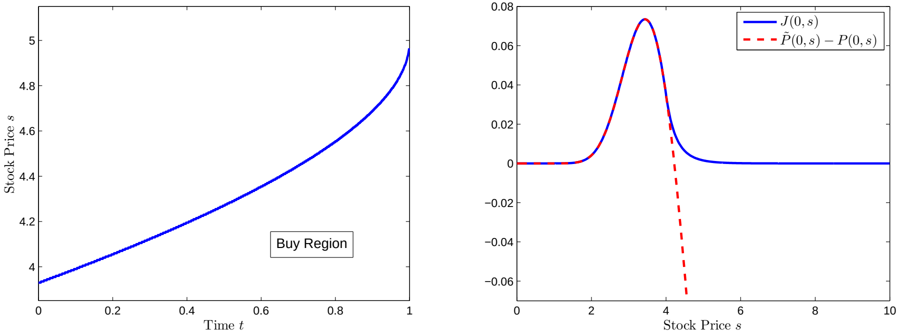

Figure: Parameters: $\lambda(t,s) = 0.2$, $\tilde{\lambda}(t,s) = 0.2e^{-0.2(s-K)}$, $r = 5\%$, $\sigma = 20\%$, $T = 1$, $K = 5$. *Right*: $J(t,s) = \tilde{P}(t,s) - V(t,s) = [\tilde{P}(t,s) - P(t,s)] + L(t,s)$.

Further Remarks

- If $G(t, s) < 0$ then should wait.
- So the purchase boundary $s^*(t)$ must satisfy $G(t, s^*(t)) > 0$.
- e.g. for a Call, must have $\tilde{\lambda}(t, s^*(t)) - \lambda(t, s^*(t)) > 0$: the market is underestimating the default intensity in the buy region.
- Near expiry, $\tilde{\lambda}(t, s^*(t)) = \lambda(t, s^*(t))$ in the limit $t \to T$.
- **Comparison principle**: If $G_1(t, s) \leq G_2(t, s)\ \forall (t, s)$, then $L_1(t, s) \geq L_2(t, s)$, so $\tau_1^* \geq \tau_2^*$ a.s. (bigger G means earlier purchase).

Digital Call Purchase Timing

Consider $F(s) = 1_{\{s>K\}}$ (not convex) w/constant default intensities, the drift function is

$$G(t,s) = (\tilde{\lambda} - \lambda) e^{-(r+\lambda)(T-t)} \left(\phi(d_2) \frac{1}{\sigma\sqrt{T-t}} - \Phi(d_2) \right),$$

which changes sign, with $\lim_{s\to 0} G(t,s) = 0$ and $\lim_{s\to\infty} G(t,s) = (\lambda - \tilde{\lambda})$.

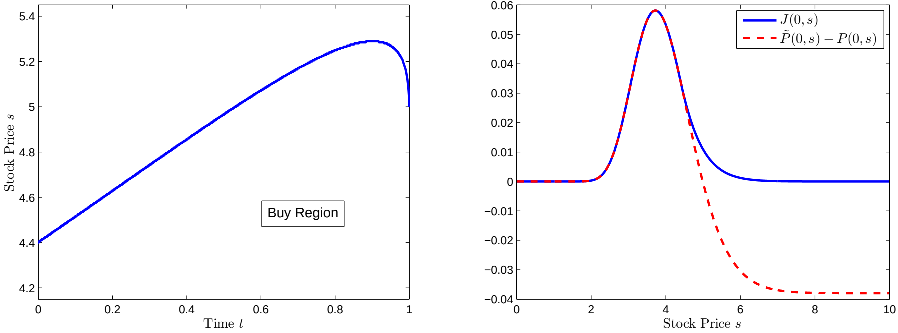

Figure: $\lambda(t,s) = 0.2$, $\tilde{\lambda}(t,s) = 0.25$, $r = 0.05$, $\sigma = 0.2$, $T = 1$ and $K = 5$.

American Put Purchase Timing

The buyer's American option price: $\tilde{P}^A(t,s) = \sup_{\nu \in \mathcal{T}_{t,T}} \boldsymbol{E}^{\tilde{Q}}_{t,s}\left\{ e^{-r(\nu - t)} F(S_\nu) \right\}$. The buyer solves: $J^A(t,s) = \sup_{\tau \in \mathcal{T}_{t,T}} \boldsymbol{E}^{\tilde{Q}}_{t,s}\left\{ e^{-r(\tau - t)} (\tilde{P}^A(\tau, S_\tau) - P^A(\tau, S_\tau)) \right\}$.

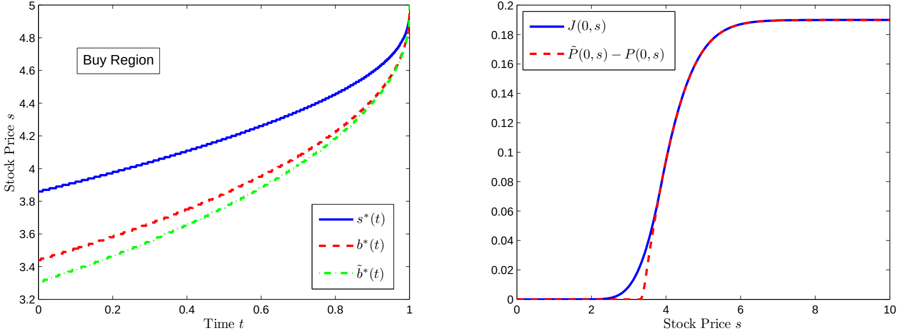

Figure: Parameters: $\lambda(t,s) = 0.2$, $\tilde{\lambda}(t,s) = 0.25$, $r = 0.05$, $\sigma = 0.2$, $T = 1$ and $K = 5$. *Left panel*: Solid line shows the purchase boundary $s^*(t)$; dashed line shows the market exercise boundary $b^*(t)$ and the dash-dotted line shows the buyer's exercise boundary $\tilde{b}^*(t)$.

Buying Options under Stochastic Volatility

- Consider a general stochastic volatility model under an EMM Q^{ϕ}:

$$\begin{cases} dS_t = S_t \left(r\, dt + \sigma(Y_t)\, dW_t^{\phi} \right), \\ dY_t = \left[b(t, Y_t) - \rho c(t, Y_t) \dfrac{\mu(t, Y_t) - r}{\sigma(Y_t)} - \hat{\rho} c(t, Y_t) \phi_t \right] dt + c(t, Y_t) \left(\rho dW_t^{\phi} + \hat{\rho} d\hat{W}_t^{\phi} \right), \end{cases}$$

where $W_t^{\phi} = W_t + \int_0^t \frac{\mu(s, Y_s) - r}{\sigma(Y_s)} ds$, $\hat{W}_t^{\phi} = \hat{W}_t + \int_0^t \phi_s\, ds$ are indep. Q^{ϕ}-BMs.

- Buyer's vol. risk premium: $\tilde{\phi}_t = \tilde{\phi}(t, S_t, Y_t)$, and market's $\phi_t = \phi(t, S_t, Y_t)$.
- Market price $P(t, s, y) = \boldsymbol{E}^{Q} \left\{ e^{-r(T-t)} F(S_T) \,|\, S_t = s, Y_t = y \right\}$.
- The buyer faces the optimal stopping problem

$$V(t, s, y) = \inf_{\tau \in \mathcal{T}_{t,T}} \boldsymbol{E}^{\tilde{Q}} \left\{ e^{-r(\tau - t)} P(\tau, S_\tau, Y_\tau) | S_t = s, Y_t = y \right\}.$$

Buying Options under Stochastic Volatility

Theorem

Let

$$G(t,s,y) := \frac{\partial P}{\partial y}(t,s,y)\big(\tilde{\phi}(t,s,y) - \phi(t,s,y)\big).$$

If $G(t,s,y) \le 0 \ \forall (t,s,y)$, *then* $\tau^* = T$ *and* $L(t,s,y) = P(t,s,y) - \tilde{P}(t,s,y)$.
If $G(t,s,y) \ge 0 \ \forall (t,s,y)$, *then* $\tau^* = 0$ *(buy now) and* $L(t,s,y) = 0$.

In general, the optimal purchase time $\tau^* = \inf\{t \le T : L(t, S_t, Y_t) = 0\}$, where

$$\begin{aligned} L(t,s,y) &= P(t,s,y) - V(t,s,y) \\ &= \sup_{\tau \in \mathcal{T}_{t,T}} \boldsymbol{E}^{\tilde{Q}}\left\{ -\int_t^\tau e^{-r(u-t)} \hat{\rho} c(u, Y_u) G(u, S_u, Y_u)\, du \,\middle|\, S_t = s, Y_t = y \right\}. \end{aligned}$$

Corollaries: Optimal Purchase Timing

Corollary

Assume $P(t,s,y)$ is convex in $s \in \mathbb{R}_+$ $\forall (t,y)$ and $\sigma'(y) > 0$.

If $\tilde{\phi}(t,s,y) \le \phi(t,s,y)$ $\forall (t,s,y)$, then it is optimal to never buy the option.
If $\tilde{\phi}(t,s,y) \ge \phi(t,s,y)$ $\forall (t,s,y)$, then it is optimal to purchase the option immediately.

Idea: show that $\frac{\partial P}{\partial y}(t,s,y) \ge 0$ (Romano-Touzi ('97)).

Examples:

- For convex payoffs, don't buy at (t,s,y) if $\tilde{\phi}(t,s,y) \le \phi(t,s,y)$.
- Again by Put-Call Parity, the buyer's optimal purchase strategy for the European Call and European Put are identical.
- *Heston model and q-optimal measures.* HHHS'07 show that $q \mapsto \phi^q(t,s,y)$ is increasing. So if investor has $\tilde{Q} = Q^{(q_1)}$ and market has $Q = Q^{(q_2)}$ then the solution is trivial.

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Rolling Long-Dated Options

- Long-dated T-Put is not traded in the market, so buy and hold one with shorter maturity T_1.
- At the roll-over date $\tau \le T_1$, simultaneously buy a Put expiring at T and selling the Put expiring at T_1.
- Minimize the net cost at the roll date τ: $c^Q_\tau(T) - c^Q_\tau(T_1)$.
- Payoff has complicated non-convex shape...

Risk Averse Buyers

- So far the buyer is risk-neutral and we worked under $\tilde{Q}$.
- Can consider a risk-averse buyer who works under $\mathbb{P}$.
- Buyer's model price $\equiv$ indifference price of F.
- This is one way to justify the discrepancy between pricing measures involved and the choice of buyer's measure.
- Tractable framework with exponential utility $U(x) = -e^{-\gamma x}$, $\gamma > 0$.
- Related to static-dynamic hedging, see Leung-Sircar ('09).

Exponential Utility

- Buying problem is:

$$J_t(X_t;\alpha F) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\theta\in\Theta_{t,\tau}} \boldsymbol{E}\{V_\tau(X^\theta_\tau - \alpha P_\tau;\alpha F)\,|\mathcal{F}_t\},$$

  where the Merton optimal investment value function is

$$V_t(X_t;\alpha F) := \operatorname*{ess\,sup}_{\theta\in\Theta_{t,T}} \boldsymbol{E}\left\{U(X^\theta_T + \alpha F(S_T))\,|\,\mathcal{F}_t\right\}.$$

- Denote by h_t the indifference price of the contract αF.
- For exp. utility, duality interpretation of h_t through entropic penalties.
- Delayed purchase premium L_t: $J_t(X_t;\alpha F) =: V_t(X_t + L_t - \alpha P_t;\alpha F)$.
- Based on Leung-Sircar (2009),

$$J_t(X_t;\alpha F) = U(X_t)\cdot\exp\left(-\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{Q\in\mathbb{P}_f(P)} \left(\gamma \boldsymbol{E}^Q_t\{h_\tau - \alpha P_\tau\} + H^\tau_t(Q|P) + \boldsymbol{E}^Q_t\{H^T_\tau(Q^E|P)\}\right)\right)$$

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- Q^E is the minimal entropy martingale measure.

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Modified Problem

- As $\gamma \to 0$, recover $L_t = \alpha \cdot \left(\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \boldsymbol{E}_t^{Q^E} \{ h_\tau^E - P_\tau \} - (h_t^E - P_t) \right)$.
- Total value of purchasing the option is:

$$f_t = \underbrace{h_t}_{\text{indifference price for holding the option}} - \underbrace{\alpha P_t}_{\text{cost of the option}} + \underbrace{L_t}_{\text{delayed purchase premium}} .$$

- Conditional relative entropic penalty $\equiv$ quadratic penalty on the risk premium.
- e.g. classical non-traded asset: option on Y; trade in S (corr. ρ).

$$L_t = \sup_{t \le \tau \le T} \inf_{\phi} \boldsymbol{E}_{t,y}^{\phi} \left\{ \int_t^\tau \frac{1}{2\gamma} (\phi_s - \phi^*(s, Y_s))^2 + \sqrt{1-\rho^2} c(s, Y_s) P_y(s, Y_s)(\phi_s - \psi_s) \, ds \right\}$$

- ϕ^* is the optimal measure in the dual representation of $h(t, y)$; ψ is the market risk premium.
- Can again explicitly derive the drift function $G(t, s, y)$.
- Work in progress.

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