Meromorphic Lévy processes and their applications in Finance and Insurance

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Definitions

A Lévy process X is specified by the triple (μ, σ, Π), where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi(\mathrm{d}x)$ satisfies $\int_{\mathbb{R}} \min(1, x^2)\Pi(\mathrm{d}x) < \infty$.

If $\lambda = \Pi(\mathbb{R})$ is finite then X is a sum of BM and a compound Poisson process:

$$X_t = \sigma W_t + \mu t + \sum_{i=1}^{N(\lambda t)} \xi_i,$$

where $\mathbb{P}(\xi_i \in \mathrm{d}x) = \lambda^{-1}\Pi(\mathrm{d}x)$. The characteristic exponent $\Psi(z)$ is defined as

$$\mathbb{E}\left[e^{\mathrm{i}zX_t}\right] = e^{-t\Psi(z)}.$$

The Lévy-Khintchine representation for $\Psi(z)$ is

$$\Psi(z) = \frac{\sigma^2 z^2}{2} - \mathrm{i}\mu z - \int_{\mathbb{R}} \left(e^{\mathrm{i}zx} - 1 - \mathrm{i}zxh(x)\right)\Pi(\mathrm{d}x).$$

Examples: VG process

- The density of the jump measure is

$$\pi(x) = \frac{1}{k|x|} e^{Ax - B|x|}.$$

- The characteristic exponent is

$$\Psi(z) = \mathrm{i}z\gamma - \frac{1}{k} \ln\left(1 + \frac{\sigma^2 k}{2} z^2 - \mathrm{i}\theta k z\right).$$

Examples: generalized tempered stable process (KoBoL, CGMY)

- The density of the jump measure is

$$\pi(x) = \frac{c}{x^{1+\alpha}} e^{-\lambda x} \mathbb{I}(x>0) + \frac{\hat{c}}{|x|^{1+\hat{\alpha}}} e^{\hat{\lambda} x} \mathbb{I}(x<0).$$

- The characteristic exponent is

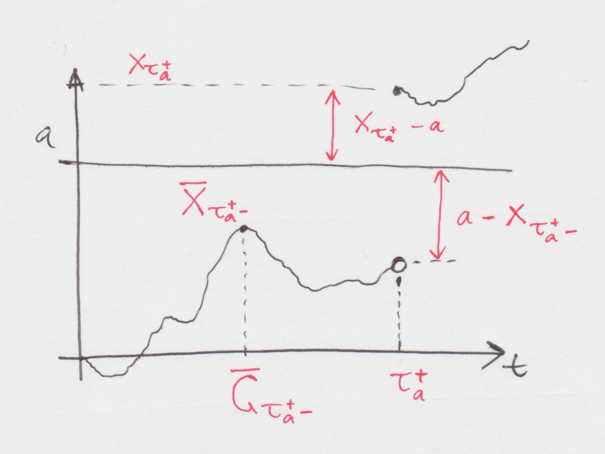
$$\begin{aligned} \Psi(u) = \mathrm{i}z\gamma \quad &+ \quad \Gamma(-\alpha)\lambda^{\alpha} c \left[\left(1 - \frac{\mathrm{i}z}{\lambda}\right)^{\alpha} - 1 + \frac{\mathrm{i}z}{\alpha} \right] \\ &+ \quad \Gamma(-\hat{\alpha})\hat{\lambda}^{\hat{\alpha}} \hat{c} \left[\left(1 + \frac{\mathrm{i}z}{\hat{\lambda}}\right)^{\hat{\alpha}} - 1 - \frac{\mathrm{i}z}{\hat{\alpha}} \right]. \end{aligned}$$

Functionals of a Lévy process: one sided exit

- supremum $\overline{X}_t$ and infimum $\underline{X}_t$
- first passage times $\tau_a^+ = \inf\{t \geq 0 : X_t > a\}$ and τ_b^-
- overshoot $X_{\tau_a^+} - a$ and undershoot $a - X_{\tau_a^+ -}$
- last maximum/minimum before the first passage time: $\overline{X}_{\tau_a^+ -}$ and $\underline{X}_{\tau_b^- -}$
- last time the maximum/minimum was achieved before the first passage time: $\overline{G}_{\tau_a^+ -}$ and $\underline{G}_{\tau_b^- -}$, where

$$\begin{aligned} \overline{G}_t &= \sup\{s < t : X_s = \overline{X}_s\} \\ \underline{G}_t &= \sup\{s < t : X_s = \underline{X}_s\} \end{aligned}$$

Functionals of a Lévy process: one sided exit



Functionals of a Lévy process: two sided exit

- exit time from a finite interval $[0, a]$: $\tau_a^+ \wedge \tau_0^-$ and location of X at this time
- entrance time into a finite interval $\tau_{[0,a]} = \inf\{t \geq 0 : X_t \in (0, a)\}$ and location of X at this time
- process reflected at supremum $Y_t = \overline{X}_t - X_t$
- first passage time of the reflected process $\sigma_a = \inf\{t \geq 0 : Y_t > a\}$
- overshoot and undershoot of the reflected process at the first passage: $Y_{\sigma_a} - a$ and $a - Y_{\sigma_a -}$

Applications: Math finance

- Up-and-out barrier option

$$\mathbb{E}_x\left[f(X_t)\mathbb{I}(\tau_a^+ > t)\right]$$

- Rebate barrier option

$$\mathbb{E}_x\left[f(X_t)\mathbb{I}(\tau_a^+ > t)\right] + \mathbb{E}_x\left[g(X_{\tau_a^+})\mathbb{I}(\tau_a^+ < t)\right]$$

- Double barrier option

$$\mathbb{E}_x\left[f(X_t)\mathbb{I}(\tau_a^+ \wedge \tau_b^- > t)\right]$$

Applications: Actuarial Mathematics

- Ruin probability (here $\tau = \tau_0^-$)

$$\mathbb{P}_x(\tau < t)$$

- Expected discounted penalty function

$$\mathbb{E}_x\left[e^{-q\tau} w\left(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}\right) \mathbb{I}_{\{\tau<\infty\}}\right]$$

- Finite time expected discounted penalty function

$$\mathbb{E}_x\left[e^{-q\tau} w\left(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}\right) \mathbb{I}_{\{\tau<t\}}\right]$$

Reducing the complexity

Many of these functionals are related to the distribution of the extrema of the process.

- First passage time VS supremum:

$$\mathbb{P}(\tau_a^+ < t) = \mathbb{P}(\overline{X}_t > a).$$

- First passage time and overshoot VS supremum:

$$\mathbb{E}\left[e^{-q\tau_a^+ - z(X_{\tau_a^+} - a)}\right] = \frac{\mathbb{E}\left[e^{-z(\overline{X}_{\mathrm{e}(q)} - a)}\mathbb{I}(\overline{X}_{\mathrm{e}(q)} > a)\right]}{\mathbb{E}\left[e^{-z\overline{X}_{\mathrm{e}(q)}}\right]}.$$

Here and everywhere else $\mathrm{e}(q) \sim Exp(q)$ and independent of the process X.

Outline

Review of the Wiener-Hopf factorization

Define

$$\phi_q^+(z) = \mathbb{E}\left[e^{\mathrm{i}z\overline{X}_{\mathrm{e}(q)}}\right], \quad \phi_q^-(z) = \mathbb{E}\left[e^{\mathrm{i}z\underline{X}_{\mathrm{e}(q)}}\right].$$

Theorem

- *Random variables* $\overline{X}_{\mathrm{e}(q)}$ *and* $X_{\mathrm{e}(q)} - \overline{X}_{\mathrm{e}(q)}$ *are independent.*
- $X_{\mathrm{e}(q)} - \overline{X}_{\mathrm{e}(q)} \stackrel{d}{=} \underline{X}_{\mathrm{e}(q)}$.
- *Random variable* $\overline{X}_{\mathrm{e}(q)}$ *[*$\underline{X}_{\mathrm{e}(q)}$*] is infinitely divisible, positive [negative] and has zero drift.*

For $z \in \mathbb{R}$ *we have*

$$\frac{q}{q + \Psi(z)} = \phi_q^+(z)\phi_q^-(z).$$

Review of the Wiener-Hopf factorization

Wiener-Hopf factors can be given as

$$\phi_q^+(z) = \exp\left[\frac{z}{2\pi\mathrm{i}}\int\limits_{\mathbb{R}} \ln\left(\frac{q}{q+\Psi(u)}\right)\frac{\mathrm{d}u}{u(u-z)}\right], \quad z \in \mathbb{C}^+.$$

A.L. Lewis and E. Mordecki.
Wiener-Hopf factorization for Lévy processes having positive jumps with rational transforms.
J. Appl. Probab., 45(1):118–134., 2008.

Pricing up-and-out barrier option

First compute the price of an option with random maturity $T = \mathrm{e}(q)$:

$$\begin{aligned}
\mathbb{E}\left[f(X_{\mathrm{e}(q)})\mathbb{I}(\overline{X}_{\mathrm{e}(q)} < a)\right] &= \mathbb{E}\left[f(\overline{X}_{\mathrm{e}(q)} + X_{\mathrm{e}(q)} - \overline{X}_{\mathrm{e}(q)})\mathbb{I}(\overline{X}_{\mathrm{e}(q)} < a)\right] \\
&= \mathbb{E}\left[f(S+I)\mathbb{I}(S<a)\right] \\
&= \int\limits_{\mathbb{R}^-} \mathbb{P}(I \in \mathrm{d}y) \int\limits_0^a f(x+y)\mathbb{P}(S \in \mathrm{d}x),
\end{aligned}$$

where $S \stackrel{d}{=} \overline{X}_{\mathrm{e}(q)}$ and $I \stackrel{d}{=} \underline{X}_{\mathrm{e}(q)}$. The price with the deterministic maturity can be found as an inverse Laplace transform in the q-variable of the above expression.

Outline

Classical Poisson risk model

The risk process is $X_t = X_0 + \mu t - \sum_{i=1}^{N(at)} \xi_i$, where $\xi_i \sim Exp(\rho)$. The Laplace exponent is defined as $\psi(z) = \ln \mathbb{E}\left[e^{zX_1}\right] = -\Psi(-\mathrm{i}z)$, and by the Lévy-Khinchine formula we have

$$\psi(z) = \frac{\sigma^2 z^2}{2} + \mu z + \int\limits_{\mathbb{R}} \left(e^{zx} - 1 - zxh(x)\right) \Pi(\mathrm{d}x).$$

In our case $\Pi(\mathrm{d}x) = \mathbb{I}(x<0) a\rho \exp(\rho x) \mathrm{d}x$, therefore

$$\psi(z) = \mu z - \frac{az}{\rho + z}.$$

For $q > 0$ equation $\psi(z) = q$ has two solutions $-\zeta$ and Φ, where $0 < \zeta < \rho$ and $\Phi > 0$ and

$$\zeta = \frac{a + q - \mu\rho - \sqrt{(a + q - \mu\rho)^2 + 4q\mu\rho}}{2\mu}.$$

Classical Poisson risk model

We have a factorization

$$\frac{q}{q-\psi(z)} = \frac{1+\frac{z}{\rho}}{1+\frac{z}{\zeta}} \times \frac{1}{1-\frac{z}{\Phi}}$$

Thus the positive/negative Wiener-Hopf factors are

$$\begin{aligned}
\phi_q^+(-\mathrm{i}z) &= \mathbb{E}\left[e^{z\overline{X}_{\mathrm{e}(q)}}\right] = \frac{1}{1-\frac{z}{\Phi}}, \\
\phi_q^-(-\mathrm{i}z) &= \mathbb{E}\left[e^{z\underline{X}_{\mathrm{e}(q)}}\right] = \frac{1+\frac{z}{\rho}}{1+\frac{z}{\zeta}} = \frac{\zeta}{\rho} + \left(1-\frac{\zeta}{\rho}\right)\frac{\zeta}{\zeta+z}.
\end{aligned}$$

We have $\overline{X}_{\mathrm{e}(q)} \sim Exp(\Phi)$ and

$$\mathbb{P}(\underline{X}_{\mathrm{e}(q)} \in \mathrm{d}x) = \frac{\zeta}{\rho}\delta_0(\mathrm{d}x) + \left(1-\frac{\zeta}{\rho}\right)\zeta e^{-\zeta x}\mathrm{d}x.$$

Classical Poisson risk model

Therefore

$$\mathbb{P}(-\underline{X}_{\mathrm{e}(q)} > x) = \left(1 - \frac{\zeta}{\rho}\right) e^{-\zeta x}$$

Using $\mathbb{P}(-\underline{X}_{\mathrm{e}(q)} > x) = \mathbb{P}_x(\tau_0^- < \mathrm{e}(q))$ and

$$\mathbb{P}_x(\tau_0^- < \mathrm{e}(q)) = q \int\limits_0^{\infty} e^{-qt} \mathbb{P}_x(\tau_0^- < t) \mathrm{d}t$$

we conclude that

$$\mathbb{P}_x(\tau_0^- < t) = \frac{1}{2\pi \mathrm{i}} \int\limits_{c+\mathrm{i}\mathbb{R}} \left(1 - \frac{\zeta(q)}{\rho}\right) e^{-\zeta(q)x + qt} \frac{\mathrm{d}q}{q}$$

where

$$\zeta = \frac{a + q - \mu\rho - \sqrt{(a+q-\mu\rho)^2 + 4q\mu\rho}}{2\mu}.$$

Kou model: double exponential jump diffusion process

X is a Lévy process with jumps defined by

$$\pi(x) = \mathbb{I}_{\{x>0\}} a_1 \rho_1 e^{-\rho_1 x} + \mathbb{I}_{\{x<0\}} \hat{a}_1 \hat{\rho}_1 e^{\hat{\rho}_1 x}.$$

Then the Laplace exponent is

$$\psi(z) = \frac{\sigma^2}{2} z^2 + \mu z + \frac{a_1 z}{\rho_1 - z} - \frac{\hat{a}_1 z}{\hat{\rho}_1 + z}.$$

Thus equation $\psi(z) = q$ is a *fourth degree polynomial equation*, and we have explicit solutions and exact WH factorization.

Hyper-exponential jumps

The jump measure is a “mixture” of exponential distributions:

$$\pi(x) = \mathbb{I}_{\{x>0\}} \sum_{i=1}^{N} a_i \rho_i e^{-\rho_i x} + \mathbb{I}_{\{x<0\}} \sum_{i=1}^{\hat{N}} \hat{a}_i \hat{\rho}_i e^{\hat{\rho}_i x},$$

where all the coefficients are positive.
Consider the Laplace exponent

$$\psi(z) = \frac{\sigma^2}{2} z^2 + \mu z + z \sum_{i=1}^{N} \frac{a_i}{\rho_i - z} - z \sum_{i=1}^{\hat{N}} \frac{\hat{a}_i}{\hat{\rho}_i + z}.$$

Hyper-exponential jumps

Assume $\sigma > 0$. Then equation $\psi(z) = q$ has

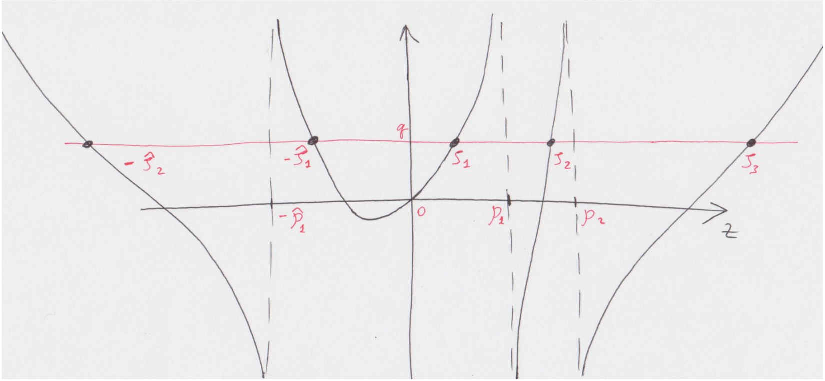
- $N + 1$ positive solutions ζ_i,
- $\hat{N} + 1$ negative solutions $-\hat{\zeta}_i$.
- These solutions interlace with the poles of $\psi(z)$:

$$... - \rho_2 < -\zeta_2 < -\rho_1 < -\zeta_1 < 0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 < ...$$

Example

Solving $\psi(z) = q$ for $N = 2$ and $\hat{N} = 1$:

$$\psi(z) = \frac{\sigma^2}{2} z^2 + \mu z + z \left[\frac{a_1}{\rho_1 - z} + \frac{a_2}{\rho_2 - z} \right] - \frac{\hat{a}_1 z}{\hat{\rho}_1 + z}.$$



Hyper-exponential jumps

Define functions

$$f^+(z) = \frac{1}{1+\frac{z}{\zeta_1}} \prod_{i=1}^{N} \frac{1+\frac{z}{\rho_i}}{1+\frac{z}{\zeta_{i+1}}}, \quad f^-(z) = \frac{1}{1+\frac{z}{\hat{\zeta}_1}} \prod_{i=1}^{\hat{N}} \frac{1+\frac{z}{\hat{\rho}_i}}{1+\frac{z}{\hat{\zeta}_{i+1}}}.$$

Then $f^+(-z)f^-(z) = q/(q-\psi(z))$. Partial fractions decomposition gives us

$$f^+(z) = \mathbb{E}\left[e^{-z\overline{X}_{\mathrm{e}(q)}}\right] = \sum_{i=1}^{N+1} c_i \frac{\zeta_i}{z+\zeta_i}.$$

Thus the distribution of $\overline{X}_{\mathrm{e}(q)}$ is a mixture of exponentials

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(\overline{X}_{\mathrm{e}(q)} \le x) = \sum_{i=1}^{N+1} c_i \zeta_i e^{-\zeta_i x},$$

where $c_i > 0$ and $\sum c_i = 1$.

Completely monotone jumps

Definition

A function $f(x)$ is called completely monotone if $(-1)^n f^{(n)}(x) > 0$ for all $n = 0, 1, 2, \ldots$.

Theorem

The jump density of a process X is completely monotone if and only if $\overline{X}_{\mathrm{e}(q)}$ and $\underline{X}_{\mathrm{e}(q)}$ are mixtures of exponentials.

L.C.G. Rogers.
Wiener-Hopf factorization of diffusions and Lévy processes.
Proc. London Math. Soc., 47(3):177–191, 1983.

Distribution of $X_{\mathrm{e}(q)}$

Theorem

Distribution of $X_{\mathrm{e}(q)}$ is a mixture of exponentials:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(X_{\mathrm{e}(q)} \le x) = \mathbb{I}(x>0)\sum_{i=1}^{N} b_i \zeta_i e^{-\zeta_i x} + \mathbb{I}(x<0)\sum_{i=1}^{\hat{N}} \hat{b}_i \hat{\zeta}_i e^{\hat{\zeta}_i x}$$

Proof.

Again, use the partial fraction decomposition and the interlacing property:

$$\mathbb{E}\left[e^{zX_{\mathrm{e}(q)}}\right] = \frac{q}{q-\psi(z)} = \sum_{i=1}^{N} \frac{b_i\zeta_i}{\zeta_i - z} + \sum_{i=1}^{\hat{N}} \frac{\hat{b}_i\hat{\zeta}_i}{\hat{\zeta}_i + z}$$

∎

Outline

Meromorphic Lévy processes

Assumption

Series $\sum_{n\geq 1} a_n \rho_n^{-2}$ *and* $\sum_{n\geq 1} \hat{a}_n \hat{\rho}_n^{-2}$ *converge.*

Next we define the function $\pi(x)$ as

$$\pi(x) = \mathbb{I}(x>0) \sum_{n\geq 1} a_n \rho_n e^{-\rho_n x} + \mathbb{I}(x<0) \sum_{n\geq 1} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x}.$$

Assumption 1 implies that $\int_{\mathbb{R}} x^2 \pi(x) \mathrm{d}x < \infty$.

Meromorphic Lévy processes: solutions to $\psi(z) = q$

The Laplace exponent $\psi(z)$ is a meromorphic function having a partial fraction decomposition

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + z^2 \sum_{n \geq 1} \frac{a_n}{\rho_n(\rho_n - z)} + z^2 \sum_{n \geq 1} \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)}.$$

Proposition

Assume that $q > 0$. Equation $\psi(z) = q$ has solutions $\{\zeta_n, -\hat{\zeta}_n\}_{n \geq 1}$, where $\{\zeta_n\}_{n \geq 1}$ and $\{\hat{\zeta}_n\}_{n \geq 1}$ are sequences of positive numbers which satisfy the following interlacing property

$$\begin{aligned} &0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \dots \\ &0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 < \dots \end{aligned} \tag{1}$$

Meromorphic Lévy processes: W-H factors

Theorem

Assume that $q>0$. Then for $\mathrm{Re}(z)>0$

$$\mathbb{E}\left[e^{-z\overline{X}_{\mathrm{e}(q)}}\right]=\prod_{n\ge 1}\frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}}.$$

The distribution of $\overline{X}_{\mathrm{e}(q)}$ can be identified as an infinite mixture of exponential distributions

$$\mathbb{P}(\overline{X}_{\mathrm{e}(q)}=0)=c_0, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(\overline{X}_{\mathrm{e}(q)}\le x)=\sum_{n\ge 1}c_n\zeta_n e^{-\zeta_n x}, \quad x>0,$$

where $c_n\ge 0$, satisfy $\sum_{n\ge 0}c_n=1$, and can be computed as

$$c_0=\lim_{n\to+\infty}\prod_{k=1}^{n}\frac{\zeta_k}{\rho_k}, \quad c_n=\left(1-\frac{\zeta_n}{\rho_n}\right)\prod_{k\ge 1}\frac{1-\frac{\zeta_n}{\rho_k}}{1-\frac{\zeta_n}{\zeta_k}}.$$

Outline

Definition of the β-family

Definition

We define the β-family of Lévy processes by the generating triple (μ, σ, π), where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and the density of the Lévy measure is

$$\pi(x) = c_1 \frac{e^{-\alpha_1 \beta_1 x}}{(1 - e^{-\beta_1 x})^{\lambda_1}} \mathbf{I}_{\{x>0\}} + c_2 \frac{e^{\alpha_2 \beta_2 x}}{(1 - e^{\beta_2 x})^{\lambda_2}} \mathbf{I}_{\{x<0\}}$$

and parameters satisfy $\alpha_i > 0$, $\beta_i > 0$, $c_i \geq 0$ and $\lambda_i \in (0, 3)$.

Lévy processes similar to the β-family

The generalized tempered stable family

$$\pi(x) = c_+ \frac{e^{-\alpha_+ x}}{x^{\lambda_+}} \mathbf{I}_{\{x>0\}} + c_- \frac{e^{\alpha_- x}}{|x|^{\lambda_-}} \mathbf{I}_{\{x<0\}}.$$

can be obtained as the limit as $\beta \to 0^+$ if we let

$$c_1 = c_+ \beta^{\lambda_+}, \quad c_2 = c_- \beta^{\lambda_-}, \quad \alpha_1 = \alpha_+ \beta^{-1}, \quad \alpha_2 = \alpha_- \beta^{-1}, \quad \beta_1 = \beta_2 = \beta$$

Particular cases:

- $\lambda_1 = \lambda_2 \longrightarrow$ tempered stable, or KoBoL processes
- $c_1 = c_2$, $\lambda_1 = \lambda_2$ and $\beta_1 = \beta_2 \longrightarrow$ CGMY processes

Computing the characteristic exponent

Theorem

If $\lambda_i \in (0,3) \setminus \{1,2\}$ *then*

$$
\begin{aligned}
\Psi(z) &= \frac{\sigma^2 z^2}{2} + \mathrm{i}\rho z + \gamma \\
&- \frac{c_1}{\beta_1}\mathrm{B}\left(\alpha_1 - \frac{\mathrm{i}z}{\beta_1}; 1-\lambda_1\right) - \frac{c_2}{\beta_2}\mathrm{B}\left(\alpha_2 + \frac{\mathrm{i}z}{\beta_2}; 1-\lambda_2\right).
\end{aligned}
$$

Here $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function.

Definition of the θ-family: $\chi = 3/2$

Definition

Define the Lévy measure

$$\pi(x) = \mathbb{I}_{\{x>0\}} 2c_1\beta_1 \sum_{n\ge1} n^2 e^{-(\alpha_1+\beta_1 n^2)x} + \mathbb{I}_{\{x<0\}} 2c_2\beta_2 \sum_{n\ge1} n^2 e^{(\alpha_2+\beta_2 n^2)x}$$

Then $\pi(x) \sim c|x|^{-\frac{3}{2}}$ as $x \to 0$ and we have a process with jumps of infinite activity/bounded variation.

The characteristic exponent can be computed as follows

$$\begin{aligned} \Psi(z) &= \frac{1}{2}\sigma^2 z^2 - \mathrm{i}\rho z + c_1\pi\sqrt{(\alpha_1 - \mathrm{i}z)\beta_1^{-1}} \coth\left(\pi\sqrt{(\alpha_1 - \mathrm{i}z)\beta_1^{-1}}\right) \\ &+ c_2\pi\sqrt{(\alpha_2 + \mathrm{i}z)\beta_2^{-1}} \coth\left(\pi\sqrt{(\alpha_2 + \mathrm{i}z)\beta_2^{-1}}\right) - \gamma. \end{aligned}$$

Outline

Notation: partial fraction decomposition

Define the coefficients $\mathrm{a}_n(\rho,\zeta)$ and $\mathrm{b}_n(\zeta,\rho)$ as the coefficients in the partial fraction decomposition

$$\begin{aligned}
\prod_{n\ge1}\frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}} &= \mathrm{a}_0(\rho,\zeta)+\sum_{n\ge1}\mathrm{a}_n(\rho,\zeta)\frac{\zeta_n}{\zeta_n+z},\\
\prod_{n\ge1}\frac{1+\frac{z}{\zeta_n}}{1+\frac{z}{\rho_n}} &= 1+z\mathrm{b}_0(\zeta,\rho)+\sum_{n\ge1}\mathrm{b}_n(\zeta,\rho)\left[1-\frac{\rho_n}{\rho_n+z}\right].
\end{aligned}$$

Vector/matrix notation

Everything will depend on the coefficients $\{\mathrm{a}_n(\rho,\zeta), \mathrm{a}_n(\hat{\rho},\hat{\zeta})\}_{n\geq 0}$ and $\{\mathrm{b}_n(\zeta,\rho), \mathrm{b}_n(\hat{\zeta},\hat{\rho})\}_{n\geq 0}$. We define for convenience a column vector

$$\bar{\mathrm{a}}(\rho,\zeta) = [\mathrm{a}_0(\rho,\zeta), \mathrm{a}_1(\rho,\zeta), \mathrm{a}_2(\rho,\zeta), ...]^T$$

and similarly for $\mathrm{a}(\hat{\rho},\hat{\zeta})$, $\mathrm{b}(\zeta,\rho)$ and $\mathrm{b}(\hat{\zeta},\hat{\rho})$. Next, given a sequence of positive numbers $\zeta = \{\zeta_n\}_{n\geq 1}$, we define the column vector $\bar{\mathrm{v}}(\zeta,x)$ as a vector of distributions

$$\bar{\mathrm{v}}(\zeta,x) = \left[\delta_0(x), \zeta_1 e^{-\zeta_1 x}, \zeta_2 e^{-\zeta_2 x}, \dots\right]^T,$$

where $\delta_0(x)$ is the Dirac delta function at $x=0$.

Distribution of extrema

Corollary

(i) *For* $x \geq 0$

$$\begin{aligned} \mathbb{P}(\overline{X}_{\mathrm{e}(q)} \in \mathrm{d}x) &= \bar{\mathrm{a}}(\rho, \zeta)^T \times \bar{\mathrm{v}}(\zeta, x)\mathrm{d}x \\ \mathbb{P}(-\underline{X}_{\mathrm{e}(q)} \in \mathrm{d}x) &= \bar{\mathrm{a}}(\hat{\rho}, \hat{\zeta})^T \times \bar{\mathrm{v}}(\hat{\zeta}, x)\mathrm{d}x. \end{aligned}$$

(ii) $\mathrm{a}_0(\rho, \zeta)$ (equiv. $\mathrm{a}_0(\hat{\rho}, \hat{\zeta})$) *is nonzero if and only if* 0 *is irregular for* $(0, \infty)$ (equiv. $(-\infty, 0)$).

(iii) $\mathrm{b}_0(\zeta, \rho)$ (equiv. $\mathrm{b}_0(\hat{\zeta}, \hat{\rho})$) *is nonzero if and only if the process* X_t *creeps upwards.* (equiv. downwards)

Distribution of extrema: notation

Expression in vector/matrix form

$$\mathbb{P}(\overline{X}_{\mathrm{e}(q)} \in \mathrm{d}x) = \bar{\mathrm{a}}(\rho, \zeta)^T \times \bar{\mathrm{v}}(\zeta, x)\mathrm{d}x$$

is equivalent to

$$\mathbb{P}(\overline{X}_{\mathrm{e}(q)} = 0) = \mathrm{a}_0(\rho, \zeta)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(\overline{X}_{\mathrm{e}(q)} < x) = \sum_{n \geq 1} \mathrm{a}_n(\rho, \zeta)\zeta_n e^{-\zeta_n x}$$

Joint distribution of the fpt and the overshoot

Define $\tau_a^+ = \inf\{t > 0 : X_t > a\}$.

Theorem

Define a matrix $\mathbf{A} = \{a_{i,j}\}_{i,j\geq 0}$ *as*

$$a_{i,j} = \begin{cases} 0 & \text{if } i = 0,\ j \geq 0 \\ \mathrm{a}_i(\rho,\zeta)\mathrm{b}_0(\zeta,\rho) & \text{if } i \geq 1,\ j = 0 \\ \dfrac{\mathrm{a}_i(\rho,\zeta)\mathrm{b}_j(\zeta,\rho)}{\rho_j - \zeta_i} & \text{if } i \geq 1,\ j \geq 1 \end{cases}$$

Then for $c > 0$ *and* $y \geq 0$ *we have*

$$\mathbb{E}\left[e^{-q\tau_c^+}\mathbb{I}\left(X_{\tau_c^+} - c \in \mathrm{d}y\right)\right] = \bar{\mathrm{v}}(\zeta,c)^T \times \mathbf{A} \times \bar{\mathrm{v}}(\rho,y)\mathrm{d}y.$$

Outline

Two-sided exit problem

Theorem

Let $a > 0$ and define a matrix $\mathbf{B} = \mathbf{B}(\hat\rho, \zeta, a) = \{b_{i,j}\}_{i,j\geq 0}$ with

$$b_{i,j} = \begin{cases} \zeta_j e^{-a\zeta_j} & \text{if } i = 0,\ j \geq 1 \\ 0 & \text{if } i \geq 0,\ j = 0 \\ \dfrac{\hat\rho_i \zeta_j}{\hat\rho_i + \zeta_j} e^{-a\zeta_j} & \text{if } i \geq 1,\ j \geq 1 \end{cases}$$

and similarly $\hat{\mathbf{B}} = \mathbf{B}(\rho, \hat\zeta, a)$. There exist matrices $\mathbf{C}_1$, $\mathbf{C}_2$ and $\hat{\mathbf{C}}_1$, $\hat{\mathbf{C}}_2$ such that for $x \in (0, a)$ we have

$$\mathbb{E}_x \left[e^{-q\tau_a^+} \mathbb{I}\left(X_{\tau_a^+} \in \mathrm{d}y\ ;\ \tau_a^+ < \tau_0^- \right) \right] = \left[\bar{\mathrm{v}}(\zeta, a-x)^T \times \mathbf{C}_1 + \bar{\mathrm{v}}(\hat\zeta, x)^T \times \mathbf{C}_2 \right] \times \bar{\mathrm{v}}(\rho, y-a)\mathrm{d}y$$

Two-sided exit problem

These matrices satisfy the following system of linear equations

$$\begin{cases} \mathbf{C}_1 & = \mathbf{A} - \hat{\mathbf{C}}_2 \mathbf{B} \mathbf{A} \\ \hat{\mathbf{C}}_2 & = -\mathbf{C}_1 \hat{\mathbf{B}} \hat{\mathbf{A}} \end{cases} \qquad \begin{cases} \hat{\mathbf{C}}_1 & = \hat{\mathbf{A}} - \mathbf{C}_2 \hat{\mathbf{B}} \hat{\mathbf{A}} \\ \mathbf{C}_2 & = -\hat{\mathbf{C}}_1 \mathbf{B} \mathbf{A} \end{cases}$$

This system of linear equations can be solved iteratively with exponential convergence.

Outline

Parameters

We use a process from the β-family with parameters

$$(\sigma, \mu, \alpha_1, \beta_1, \lambda_1, c_1, \alpha_2, \beta_2, \lambda_2, c_2) = (\sigma, \mu, 1, 1.5, 1.5, 1, 1, 1.5, 1.5, 1)$$

Here $\mu = \mathbb{E}[X_1]$ and σ is the Gaussian coefficient, the other parameters define the density of a Lévy measure, which has exponentially decaying tails and $O(|x|^{-3/2})$ singularity at $x = 0$, thus this process has jumps of infinite activity but finite varation. We define the following four parameter sets

Set 1: $\sigma = 0.5, \mu = 1$	Set 2: $\sigma = 0.5, \mu = -1$
Set 3: $\sigma = 0, \mu = 1$	Set 4: $\sigma = 0, \mu = -1$

Double-sided exit problem

(i) density of the overshoot if the exit happens at the upper boundary

$$f_1(x,y) = \frac{\mathrm{d}}{\mathrm{d}y} \mathbb{E}_x \left[e^{-q\tau_1^+} \mathbb{I}\left(X_{\tau_1^+} \le y \; ; \; \tau_1^+ < \tau_0^- \right) \right]$$

(ii) probability of exiting from the interval $[0,1]$ at the upper boundary

$$f_2(x) = \mathbb{E}_x \left[e^{-q\tau_1^+} \mathbb{I}\left(\tau_1^+ < \tau_0^- \right) \right]$$

(iii) probability of exiting the interval $[0,1]$ by creeping across the upper boundary

$$f_3(x) = \mathbb{E}_x \left[e^{-q\tau_1^+} \mathbb{I}\left(X_{\tau_1^+} = 1 \; ; \; \tau_1^+ < \tau_0^- \right) \right]$$

Details of the alrgorithm

- Truncate coefficients $\mathrm{a}_i(\rho,\zeta)$ and $\mathrm{a}_i(\hat{\rho},\hat{\zeta})$ at $i=200$; coefficients $\mathrm{b}_j(\zeta,\rho)$ and $\mathrm{b}_j(\hat{\zeta},\hat{\rho})$ at $j=100$.
- In order to compute coefficients $\mathrm{a}_i(\rho,\zeta)$, $\mathrm{a}_i(\hat{\rho},\hat{\zeta})$, $\mathrm{b}_j(\zeta,\rho)$ and $\mathrm{b}_j(\hat{\zeta},\hat{\rho})$ we truncate the corresponding infinite products at $k=400$
- All the computations depend on precomputing $\{\zeta_n,\hat{\zeta}_n\}$ for $n=1,2,..,400$ (solving $\psi(z)=q$).
- The code was written in Fortran and the computations were performed on a standard laptop (Intel Core 2 Duo 2.5 GHz processor and 3 GB of RAM).
- Time to produce the three graphs for each parameter set: 0.15 sec.

Double sided exit: $\sigma > 0$ and positive drift

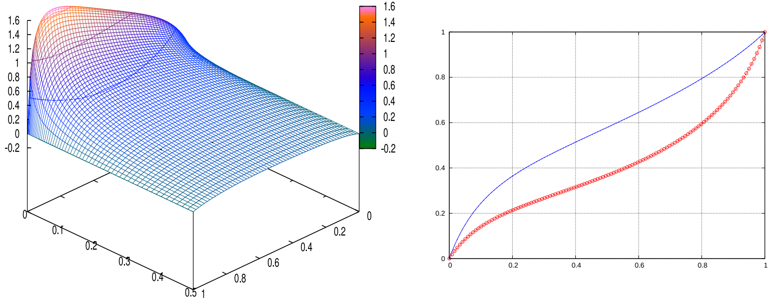
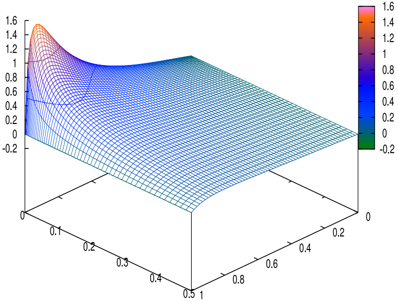


Figure: Unbounded variation case ($\sigma = 0.5$): computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 1, positive drift $\mu = 1$

Double sided exit: $\sigma > 0$ and negative drift



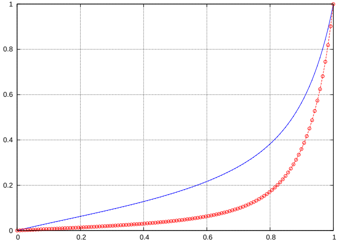


Figure: Unbounded variation case ($\sigma = 0.5$): computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 2, negative drift $\mu = -1$.

Double sided exit: bounded variation and positive drift

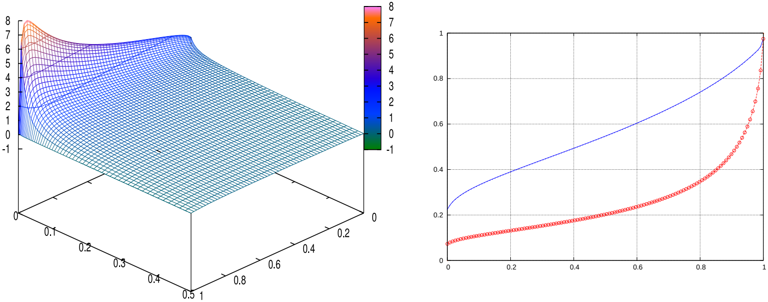


Figure: Bounded variation case ($\sigma = 0$): computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 3, positive drift $\mu = 1$.

Double sided exit: bounded variation and negative drift

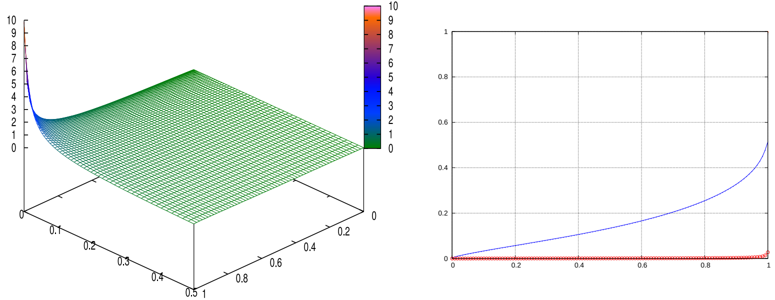


Figure: Bounded variation case ($\sigma = 0$): computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 4, positive drift $\mu = -1$.

Outline

Definitions

We consider a very general setup that generalizes the classical Poisson risk model:

$$X_t := x + Y_t \ , \qquad t \geqslant 0 \ ,$$

where $x \geqslant 0$ is the initial surplus and Y is a spectrally negative Lévy process.

Classical Poisson risk model:

$$Y_t = ct - \sum_{i=1}^{N(\lambda t)} \xi_i.$$

Definitions

Denote $\tau = \tau_0^- = \inf(t > 0 \,:\, X_t < 0)$.

Definition

The Generalized Expected Discounted Penalty Function ϕ associated with the risk process X is given by

$$\phi(x;q) := \mathbb{E}_x \left[e^{-q\tau} w \left(-X_\tau, X_{\tau-}, \underline{X}_{\tau-} \right) \mathbb{I}_{\{\tau<\infty\}} \right],$$

where $q \geq 0$ and w is a bounded measurable function on $\mathbb{R}_+^3$ satisfying $w(0,0,0) = w_0 > 0$.

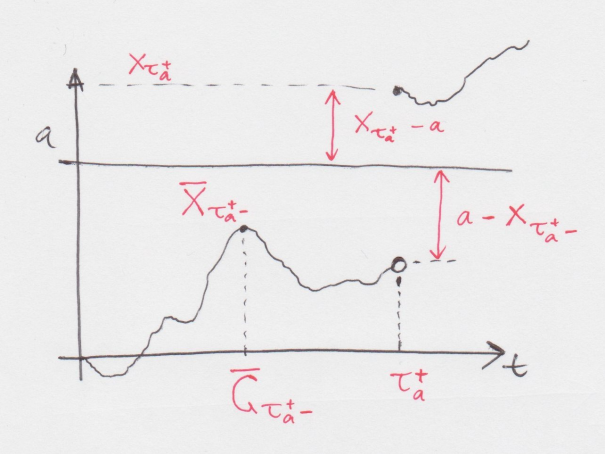
Gerber-Shiu measure

For $q \geq 0$ equation $\psi(z) = q$ has one positive solution Φ and negative solutions $-\zeta_n$.

For $q \geq 0$, $x > 0$, $y > 0$, $z > 0$ and $u \in (0, z \wedge x)$

$$\mathbb{E}_x \left[e^{-q\tau} \mathbb{I}(-X_\tau \in \mathrm{d}y \; ; \; X_{\tau-} \in \mathrm{d}z \; ; \; \underline{X}_{\tau-} \in \mathrm{d}u \; ; \; \tau < \infty) \right]$$
$$= \left[\frac{\Phi}{q} \sum_{m,n \geq 1} c_n \zeta_n a_m e^{-\zeta_n x - \rho_m y - (\Phi + \rho_m) z + (\Phi + \zeta_n) u} \right] \mathrm{d}y \mathrm{d}z \mathrm{d}u \, .$$

Functionals of a Lévy process: one sided exit



Numerical examples

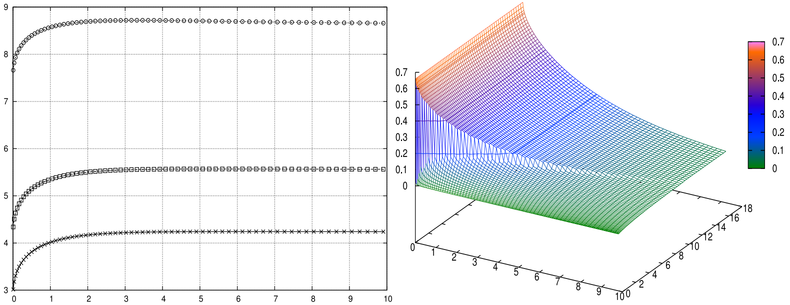


Figure: Computing the VaR of the deficit at ruin $-X_\tau$ and the distribution of the last minimum before ruin $f(x,u) = \mathbb{E}_x[e^{-q\tau}\mathbb{I}(\underline{X}_{\tau-} \le u)]$.

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