Using 3-dimensional Brownian bridges for valuation of barrier options

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Outline

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The model

Asset price model: $(Y_t)_{t\in[0,T]}$, one-dimensional diffusion.

$$\mathrm{d}Y_t = b(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t,\ t \in [0, T],\ Y_0 = y.$$

- Dynamics under the *pricing* probability $\mathbb{P}_y^{(b,\sigma)}$.
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Claim payoff: $P_T = G\big(T, (Y_t)_{t\in[0,T]}\big)$, paid at *maturity* T.

- Price of claim at time $t = 0$ is $P_0 := \mathbb{E}_y^{(b,\sigma)}\left[e^{-rT}P_T\right]$.

Barrier Options

Plain vanilla calls and puts. Most basic examples of claims:

- **Call:** $P_T = (Y_T - K)_+$. **Put:** $P_T = (K - Y_T)_+$.

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Barrier options. They become "in" or "out" depending on whether the asset price crosses a certain level ℓ. Define $m_T := \min_{t\in[0,T]} Y_t$ and $M_T := \max_{t\in[0,T]} Y_t$. Then, for example:

- Down-and-out put: $P_T = (K - Y_T)_+ \mathbb{I}_{\{m_T > \ell\}}, \ \ell < y \wedge K$.
- Up-and-in put: $P_T = (K - Y_T)_+ \mathbb{I}_{\{M_T > \ell\}}, K < \ell$.

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General "down" barrier option's payoff:
$g_i(Y_T)\mathbb{I}_{\{m_T>\ell\}} + g_o(Y_T)\mathbb{I}_{\{m_T\le\ell\}}$

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General "down" barrier option's payoff:
$g_i(Y_T)\mathbb{I}_{\{m_T>\ell\}} + g_o(Y_T)\mathbb{I}_{\{m_T\leq\ell\}} = (g_i - g_o)(Y_T)\mathbb{I}_{\{m_T>\ell\}} + g_o(Y_T)$.

- The second is just plain vanilla. We only consider then *down-and-out* options: $P_T = g(Y_T)\mathbb{I}_{\{m_T>\ell\}}$.

Prices and Hedges for Barrier Options

Pricing function for barrier options: In diffusion models, there exists a deterministic function $q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that:

- $P_t = q(T - t, Y_t)$.
- $\Delta_t = q'_y(T - t, Y_t)$: *hedging* strategy in complete models.

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Remarks:

- Even in incomplete markets, q'_y is an interesting quantity.
- Other derivatives can be interesting; for example:
  - *Gamma*: q''_{yy}.
  - *Rho*: derivative with respect to interest rate.

Finite differences

PDE approach: Solve numerically for $(T, y) \in (0, \infty) \times (\ell, \infty)$:

$$q'_T(T, y) + b(y)q'_y(T, y) + \frac{1}{2}\sigma^2(y)q''_{yy}(T, y) = rq(T, y).$$

Boundary conditions: $q(0, y) = g(y)$, $q(T, \ell) = 0$.

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Numerical issues: Near $(0, \ell)$, q and its derivatives are badly-behaved. This affects negatively numerical approximations. In fact, the following limits exist and depend on $w > 0$:

- $\lim_{T \downarrow 0} q\left(T, \ell + w\sqrt{T}\right)$;
- $\lim_{T \downarrow 0} \sqrt{T} q'_y\left(T, \ell + w\sqrt{T}\right)$.

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Analytical issues: If we replace $g(Y_T)$ by $G\left(T, (Y_t)_{t \in [0,T]}\right)$, the PDE approach does not work.

Monte Carlo approach to valuing barrier options

Monte-Carlo: Based on simulation. Use expected value representation for price. Use some differentiation method for delta.

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Approximate simulation. Via *discretization*; for example, Euler:

$$\hat{Y}_{t_i} = \hat{Y}_{t_{i-1}} + b(\hat{Y}_{t_{i-1}})h + \sigma(\hat{Y}_{t_{i-1}})\sqrt{h}Z_i,$$

where $h = t_i - t_{i-1}$ and $Z_1, \ldots, Z_n$ are i.i.d. standard normals.

Approximating the payoff. Set $\hat{m}_T := \min_{i \in \{0,\ldots,n\}} \hat{Y}_{t_i}$ and $\hat{P}_T := g(\hat{Y}_T)\mathbb{I}_{\{\hat{m}_T > \ell\}}$.

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- Discrete minimum as continuous-time proxy: $\mathcal{O}(\sqrt{h})$.

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Partial remedy: In Euler scheme, we regard $(\hat{Y}_t)_{t \in [t_{i-1}, t_i]}$ given $\hat{Y}_{t_{i-1}}$ as Brownian motion, so we can use better estimators for $\hat{m}_T$.

Monte Carlo approach to hedging barrier options I

There are several ways of trying to estimate q'_y for barrier options.

1. **Finite differences:** For "small" ϵ, use the estimator:

$$\frac{\hat{q}_N(T, y+\epsilon) - \hat{q}_N(T, y-\epsilon)}{2\epsilon}.$$

- Bias is $\mathcal{O}(\epsilon^2)$.
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2. Pathwise differentiation: Write Y^y for the (strong) solution of the SDE with $Y_0 = y$. Can we then write:

$$\frac{\partial}{\partial y}\mathbb{E}\left[g(Y_T^y)\mathbb{I}_{\{m_T^y > \ell\}}\right] \stackrel{?}{=} \mathbb{E}\Big[\frac{\partial}{\partial y} g(Y_T^y)\mathbb{I}_{\{m_T^y > \ell\}}\Big]$$

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NO! The indicator is not differentiable (not even continuous).

Monte Carlo approach to hedging barrier options II

3. Likelihood ratio differentiation: Differentiate the (approximate) *density* φ of $(\hat{Y}_{t_1}, \ldots, \hat{Y}_{t_n})$ with respect to y:

$$
\begin{aligned}
q'_y(T, y) &\approx \int_{\mathbb{R}^n} g(y_n) \frac{\partial}{\partial y} \varphi(y_1, \ldots, y_n; y) \mathrm{d}y_1 \ldots \mathrm{d}y_n \\
&= \mathbb{E}_y \left[g(\hat{Y}_T) \frac{\partial}{\partial y} \log \varphi(\hat{Y}_{t_1}, \ldots, \hat{Y}_{t_n}; y) \right]
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- Estimator is biased — we are using *approximate* likelihood.
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4. Malliavin calculus: Efforts have been made, but estimators are complicated and not very efficient.

Aim of the present work

1. Find (as) unbiased (as possible) estimators for price and delta. We do this by transforming the problem:

- First we make $\sigma = 1$.
- Next we make $b = 0$. (Now we have a Brownian motion.)
- Then we pass to a 3-d Bessel process and eliminate $\mathbb{I}_{\{m_T > \ell\}}$.
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2. Enhance price and hedge estimators for small maturities. We want the variance to be *very small* when T is small.

3. Use previous methods for first-passage-time density estimation. The wish is to do better then the usual kernel estimation of densities via CDF estimation.

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- $X := H(Y)$, $x := H(y)$, with $H(y) := \int_\ell^y (1/\sigma(u))\mathrm{d}u$.
- $a := \left(b/\sigma - \sigma'/2\right) \circ H^{-1}$

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No loss of generality: For $f := g \circ H^{-1}$, define

$$p(T,x) := \mathbb{E}_x^{(a,1)}[f(X_T)\mathbb{I}_{\{m_T>0\}}].$$

- $q(T,y) = p\big(T, H(y)\big)$
- $q'_y(T,y) = p'_x\big(T, H(y)\big)/\sigma(y)$

We focus on p from now on.

Eliminating the drift

Girsanov's theorem: $p(T,y) = \mathbb{E}_x^{(0,1)}[Z_T f(X_T)\mathbb{I}_{\{m_T>0\}}]$, where

$$Z_T := \exp\left(\int_0^T a(X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^T a^2(X_s)\mathrm{d}s\right).$$

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Stochastic to Lebesgue: Set $\gamma := (a' + a^2)/2$. By Itô:

$$\int_0^T a(X_s)\mathrm{d}X_s = \int_x^{X_T} a(v)\mathrm{d}v - \frac{1}{2}\int_0^T a'(X_s)\mathrm{d}s.$$

Therefore, $\quad Z_T = \exp\left(\int_x^{X_T} a(v)\mathrm{d}v - \int_0^T \gamma(X_s)\mathrm{d}s\right).$

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$$\text{Therefore,} \quad Z_T = \exp\left(\int_x^{X_T} a(v)\mathrm{d}v - \int_0^T \gamma(X_s)\mathrm{d}s\right).$$

Putting everything together: $p(T,x)$ is equal to

$$\mathbb{E}_x^{(0,1)}\left[\exp\left(\int_x^{X_T} a(v)\mathrm{d}v - \int_0^T \gamma(X_s)\mathrm{d}s\right) f(X_T)\mathbb{I}_{\{m_T>0\}}\right].$$

Eliminating the indicator

From Brownian Motion (BM) to 3-d Bessel (BES^3). With τ_0 being the first passage time of X to zero, define $\mathbb{P}_x^{\mathsf{BES}^3}$ via

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\mathsf{BES}^3}}{\mathrm{d}\mathbb{P}_x^{(0,1)}}\right|_{\mathcal{F}_T} := \frac{X_{\tau_0 \wedge T}}{x}.$$

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New representation for price:

$$p(T,x) \quad = \quad x\mathbb{E}_x^{\mathrm{BES}^3}\left[\exp\left(\int_x^{X_T} a(v)\mathrm{d}v - \int_0^T \gamma(X_s)\mathrm{d}s\right)\frac{f(X_T)}{X_T}\right]$$

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where $A(x) = \exp(-\int_0^x a(v)\mathrm{d}v)$ and $f_A(x) = f(x)/A(x)$.

Issues. . . to be taken care of

1. We can use the MC method already from the representation

$$p(T,x) = xA(x)\mathbb{E}_x^{\mathrm{BES}^3}\left[\exp\Big(-\int_0^T \gamma(X_s)\mathrm{d}s\Big)\frac{f_A(X_T)}{X_T}\right].$$

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$$p(T,x) = \mathbb{E}_0^{\mathrm{BM}^3}\left[xA(x)\exp\Big(-\int_0^T \gamma(|x\mathrm{e}^1 + W_s|)\mathrm{d}s\Big)\frac{f_A(|x\mathrm{e}^1 + W_T|)}{|x\mathrm{e}^1 + W_T|}\right]$$

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and differentiating w.r.t. x inside the expectation.

- Could be that f is not differentiable. . .
- . . . but even if it is, this delta estimator has infinite variance.

Steps for Monte-Carlo simulation

1. **Simulation of X_T.** With ξ a 3-d standard normal,

$$X_T \stackrel{d}{=} \sqrt{T}|z\mathrm{e}_1 + \xi|, \quad \text{where } z = x/\sqrt{T}.$$

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2. **Simulation of $(X_s)_{s\in[0,T]}$ given X_T.** With β a BB^3 — standard Brownian bridge, independent of ξ:

$$\Big((X_s)_{s\in[0,T]} \mid X_T = \sqrt{T}\xi\Big) \stackrel{d}{=} \left(\sqrt{T}\left|z\mathrm{e}_1 + \frac{s}{T}\xi + \beta_{s/T}\right|\right)_{s\in[0,T]}.$$

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- With $z = x/\sqrt{T}$ and some change of variables, we get that $\left(\int_0^T \gamma(X_s)\mathrm{d}s \mid X_T = \sqrt{T}\xi\right)$ has the distribution of

$$T\int_0^1 \gamma\big(\sqrt{T}|z\mathrm{e}_1 + u\xi + \beta_u|\big)\mathrm{d}u$$

Monte-Carlo estimation for price

A final transformation. Set $\pi(T, z) := p(T, z\sqrt{T})$.

- $p'_x(T, \sqrt{T}z) = \pi'_z(T, z)/\sqrt{T}$.

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Notation: Under $\mathbb{P}^{(0,\mathrm{BB}^3)}$, the pair (ξ,β) consists of two independent elements: $\xi \sim \mathcal{N}_3(0, Id)$ and $\beta \sim \mathrm{BB}^3$.

The representation for the price.

$$\pi(T,z) = \mathbb{E}^{(0,\mathrm{BB}^3)}\left[\frac{f_A(\sqrt{T}|z\mathrm{e}_1+\xi|)}{|z\mathrm{e}_1+\xi|} H^0(T,z;\,\xi,\beta)\right]$$

where

$$H^0(T,z;\,\xi,\beta) := A(\sqrt{T}z)\exp\left(-T\int_0^1 \gamma\big(\sqrt{T}|z\mathrm{e}_1+u\xi+\beta_u|\big)\mathrm{d}u\right).$$

Monte-Carlo estimation for delta

Idea: Write the price as

$$\pi(T,z) = \mathbb{E}^{(z,\,\mathrm{BB}^3)}\left[\frac{f_A(\sqrt{T}|\xi|)}{|\xi|} H^0(T,z;\,\xi - z\mathrm{e}_1,\beta)\right]$$

$$= \mathbb{E}^{(0,\,\mathrm{BB}^3)}\left[\left(\frac{\mathrm{d}\mathbb{P}^{(z,\,\mathrm{BB}^3)}}{\mathrm{d}\mathbb{P}^{(0,\,\mathrm{BB}^3)}}\right)\frac{f_A(\sqrt{T}|\xi|)}{|\xi|} H^0(T,z;\,\xi - z\mathrm{e}_1,\beta)\right]$$

where $\mathbb{P}^{(z,\,\mathrm{BB}^3)}$ is a new probability with

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The representation for the delta. Differentiate, take then converse steps (pass to $\mathbb{P}^{(z,\,\mathrm{BB}^3)}$ and then back to $\mathbb{P}^{(0,\,\mathrm{BB}^3)}$):

$$\pi'_z(T,z) = \mathbb{E}^{(0,\,\mathrm{BB}^3)}\left[\frac{f_A(\sqrt{T}|z\mathrm{e}_1+\xi|)}{|z\mathrm{e}_1+\xi|}H^1(T,z;\,\xi,\beta)\right], \quad \text{where } \ldots$$

Some facts to keep in mind.

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   - Exact simulation of all stochastic quantities can be performed.
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1. **Bias** is not an issue, since:
   - Exact simulation of all stochastic quantities can be performed.
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2. **Variance for long maturities** does not depend on the discretization steps we choose.

3. **Variance for short maturities close to barrier**. We have:
   - Estimators for $\pi(T, z)$ and $\pi'_z(T, z)$ have $\mathcal{O}(1)$ variance.
   - Therefore, the estimator for:
     - $q(T, \ell + w\sqrt{T})$ has $\mathcal{O}(1)$ variance.
     - $q'_y(T, \ell + w\sqrt{T})$ has $\mathcal{O}(1/T)$ variance.

We need to improve on those.

Outline

General remarks on control variates

Control variates. Suppose we can jointly simulate (κ, λ).

- $\mathbb{E}[\lambda]$ is *not* known, but $\mathbb{E}[\kappa]$ *is* known.
- With a sample $(\kappa_j, \lambda_j)_{j=1,\dots,N}$, regress λ on $\kappa - \mathbb{E}[\kappa]$. Use the intercept from the regression as an estimator for $\mathbb{E}[\lambda]$.
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Improving efficiency. Now consider (κ_T, λ_T) processes.

- Want to estimate $\mathbb{E}[\lambda_T]$ for small T; $\mathbb{E}[\kappa_T]$ is known.

<u>Lemma:</u> If $\lambda_T = \kappa_T + \mathcal{O}(h_T)$ for $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$, then

$$\sqrt{1-\rho^2_{\kappa_T,\lambda_T}} = \mathcal{O}(h_T)$$

Back to our problem

Idea. Remember that for $k = 0, 1$ we have:

$$\pi(T, z) = \mathbb{E}^{(0,\,\mathrm{BB}^3)}\left[\frac{f_A(\sqrt{T}|z\mathrm{e}_1 + \xi|)}{|z\mathrm{e}_1 + \xi|} H^0(T, z;\, \xi, \beta)\right]$$

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\end{aligned}
$$

- Expand the above quantities around $T = 0$.
- If the expectations of the expansions above are computable, we are in business.
- This will also give limits of price and delta for short maturities and near the barrier.

The expansions for $T \approx 0$

For $k = 0, 1$, we can write:

$$\frac{f_A(\sqrt{T}|z\mathrm{e}_1 + \xi|)}{|z\mathrm{e}_1 + \xi|} H^k(T, z;\, \xi, \beta) = \sum_{i=0}^{2} \eta_i^k(z;\xi)\, T^{i/2} + \mathcal{O}(\sqrt{T^3})$$

Remarks.

- None of the η's above involves β.
- Both η_0^k's do *not* involve a (or γ).
- $\mathbb{E}^{(0,\,\mathrm{BB}^3)}[\eta_i^k(z;\xi)]$ has closed form for $k = 0, 1$, $i = 0, 1, 2$.
- We can go further in the expansion. Alas, the η_3^k's involve β and their expectations are not straightforward.

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Original problem. We have managed to find estimators...

- ...of $q(T,\, \ell + \sqrt{T}w)$ with $\mathcal{O}(T^3)$ variance.
- ...of $q_y'(T,\, \ell + \sqrt{T}w)$ with $\mathcal{O}(T^2)$ variance.

Limiting behavior of price and delta for short maturities

Limits close to the barrier for short maturities:

$$
\begin{aligned}
\lim_{T\downarrow 0} q\left(T,\, \ell + \sqrt{T} z\sigma(\ell)\right) &= g(\ell)\big(2\Phi(z) - 1\big), \\
\lim_{T\downarrow 0} \sqrt{T} q'_y\left(T,\, \ell + \sqrt{T} z\sigma(\ell)\right) &= \frac{2g(\ell)}{\sigma(\ell)}\left(\Phi(-z)(1+z) + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\right)
\end{aligned}
$$

- Φ: standard normal CDF.

Outline

Other sensitivities

Gamma. We can also represent:

$$\pi''_{zz}(T,z) = \mathbb{E}^{(0,\,\mathrm{BB}^3)}\left[\frac{f_A(\sqrt{T}|z\mathrm{e}_1+\xi|)}{|z\mathrm{e}_1+\xi|}H^2(T,z;\,\xi,\beta)\right].$$

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Rho. If underlying is traded ($b(y)=ry$), r appears in a (and γ):

$$a(y) = \frac{rH^{-1}(y)}{\sigma\circ H^{-1}(y)} - \frac{\sigma'\circ H^{-1}(y)}{2}.$$

With this in mind, we can carry the previous steps.

Smooth path-dependency; Non-homogeneity

1. **More complicated payoffs**. We consider:

$$P_T \ := \ G\big(T, (Y_t)_{t\in[0,T]}\big)\mathbb{I}_{\{m_T > \ell\}}.$$

- **Price representation:** Exactly same as before.
- **Delta representation:** G must be differentiable. Apart from extra work to differentiate G, no further problems.

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2. **Non-homogeneous diffusions**.

$$\mathrm{d}Y_t = b(t, Y_t)\mathrm{d}t + \sigma(t, Y_t)\mathrm{d}W_t.$$

- H, a (and γ) are now functions of (t, y).
- Results in more complicated integration, but doable.

Outline

First-passage time density estimation — the problem

Model: A diffusion Y with $Y_0 = y$ and dynamics

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First passage time to $\ell < y$: the stopping time defined as

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Problem: Figure out the density $\phi^{(b,\sigma)}_{y\to\ell}$ of τ_ℓ under $\mathbb{P}^{(b,\sigma)}_y$:

$$\phi^{(b,\sigma)}_{y\to\ell}(T) \;=\; \frac{\mathrm{d}\,\mathbb{P}^{(b,\sigma)}_y\left[\tau_\ell \leq T\right]}{\mathrm{d}T}.$$

First-passage time density estimation — usual approach

CDF estimation: Simulate N discretized paths of Y.

- For each path $i = 1, \dots, N$, simulate $\hat{Y}_{t_1}, \dots,$ until the first k such that $\hat{Y}_{t_k} \leq \ell$ and set $\hat{\tau}^i$ equal to t_k.
- Consider the (biased, in general) estimator of the CDF:

$$\hat{F}_N(T) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{\{\hat{\tau}^i \leq T\}}.$$

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Density estimation. $\phi = F'$; use a *kernel* estimator to get $\hat{\phi}_N$ from $\hat{F}_N$. Even if $\hat{F}_N$ is unbiased, we *do not* get

$$\hat{\phi}_N = \phi + \mathcal{O}(N^{-1/2})$$

Can we do better?

Aim: Find an estimator of ϕ with $\hat{\phi}_N = \phi + \mathcal{O}(N^{-1/2})$.

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Representation with respect to Brownian bridge: Following the previous steps, with slight twists, we get with $x = H(y)$:

$$\frac{\phi^{(b,\sigma)}_{y\to\ell}(T)}{\phi^{(0,1)}_{y\to\ell}(T)} = xA(x)\,\mathbb{E}^{\mathrm{BB}^3}\Big[\exp\Big(-T\int_0^T \gamma(|xu\mathrm{e}_1 + \sqrt{T}\beta_u|)\mathrm{d}u\Big)\Big].$$

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- The representation and the CLT give an estimator such that
$$\widehat{\left(\phi^{(b,\sigma)}_{y\to\ell}\right)}_N(T) = \phi^{(b,\sigma)}_{y\to\ell}(T) + \mathcal{O}(N^{-1/2}).$$
- The convergence holds *uniformly* for (T, y) in compact sets.

Delving slightly deeper

The "rate" function: Write $\phi^{(b,\sigma)}_{y\to\ell}(T) = \exp\big(-T\lambda^{(b,\sigma)}_{y\to\ell}(T)\big)$, i.e,

$$\lambda^{(b,\sigma)}_{y\to\ell}(T) := -\frac{1}{T}\log\left(\phi^{(b,\sigma)}_{y\to\ell}(T)\right)$$

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Connection with 2nd order ODEs. For ergodic diffusions, $\lim_{T\to\infty}\lambda^{(b,\sigma)}_{y\to\ell}(T) =: \lambda_1$ exists, does *not* depend on y, and is the first eigenvalue of a certain Dirichlet problem on $[\ell,\infty)$.

- This fact can be used to improve the estimator for the density.
- Other direction: MC can help to numerically compute λ_1.

Concluding discussion

- We propose a MC simulation method for approximating the price and sensitivities of barrier options in diffusion models.
- The method has very low bias because all stochastic quantities involved can be simulated *exactly*.
- The variance does not depend on dicretization.
- Via asymptotic expansions near maturity we obtain control variates that dramatically reduce variance, especially for short maturities (but that should be used for any maturity).
- We apply the method for estimation of the *density* of diffusion first-passage times, where we are able to beat the typical non-parametric rate of convergence.

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THE END.