Pricing and Hedging in Affine Models with Possibility of Default

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Outline

1. Model
2. Pricing
3. Hedging

Motivation

Main risk factors for equity derivatives

- stock returns
- volatility
- default risk of the underlying
- interest rates

Assets needed for hedging all sources of risk

- stock
- vanilla options
- corporate bonds (or CDS)
- government bonds (or money market account)

Ingredients

- State space $D = \mathbb{R}_+^m \times \mathbb{R}^n$, $N = m + n$
- Time-homogeneous Markov process $(X_t, \mathbb{P}_x)_{t \geq 0, x \in D}$ on D
- $\mathbb{P}_x$ = pricing measure given that $X_0 = x \in D$
- $\mathcal{I} = \{1, \ldots, m\}$, $\mathcal{J} = \{m+1, \ldots, N\}$
- Standard Poisson Process $(N_t)_{t \geq 0}$ independent of X
- $\langle \cdot, \cdot \rangle$: Euclidean scalar product on $\mathbb{C}^N$, i.e.

$$\forall x, y \in \mathbb{C}^N : \langle x, y \rangle = \sum_{i=1}^{N} x_i y_i.$$

Stock price, interest rates and default

- Stock price:

$$S_t = \exp(s_t + R_t + \Lambda_t) 1_{\{t<\tau\}}$$

- Excess return: $s_t = e + \langle \varepsilon, X_t \rangle$
- Interest rates: $r_t = d + \langle \delta, X_{t,\mathcal{I}} \rangle$ $(d, \delta) \in \mathbb{R}_+^{m+1}$, $R_t = \int_0^t r_s ds$
- Default intensity: $\lambda_t = c + \langle \gamma, X_{t,\mathcal{I}} \rangle$, $(c, \gamma) \in \mathbb{R}_+^{m+1}$ $\Lambda_t = \int_0^t \lambda_s ds$,
- Default time: $\tau = \inf\{t > 0 : N_{\Lambda_t} = 1\}$

Affine Markov processes

- Affine term structure models: Vasicek (1977), Cox–Ingersoll–Ross (1985), Duffie–Kan (1996), Dai–Singleton (2000)
- Affine models of stochastic volatility: Stein–Stein (1991), Heston (1993)
- Reduced form affine models of credit default: Lando (1998)
- Unified pricing model: Carr–Schoutens (2008)
- General theory: Duffie–Pan–Singleton (2000), Duffie–Filipovic–Schachermayer (2003), Keller-Ressel–Teichmann–Schachermayer (2009)

Regular Affine Processes

Definition

$(X_t, \mathbb{P}_x)_{t \geq 0, x \in D}$ is *regular affine* if there exist functions $\phi(t,u)$ and $\psi(t,u)$ such that

$$\mathbb{E}_x \left[\exp \left(\langle u, X_t \rangle \right) \right] = \exp \left(\phi(t,u) + \langle \psi(t,u), x \rangle \right), \; u \in i\mathbb{R}^m \times i\mathbb{R}^n$$

and $X_s \to X_t$ in probability as $s \to t$.

Infinitesimal generator

X is a Feller process with infinitesimal generator

$$
\begin{aligned}
\mathcal{G}f(x) &= \sum_{k,l=1}^{N} \left(a_{kl} + \langle \alpha_{kl}^{\mathcal{I}}, x_{\mathcal{I}} \rangle\right) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} \\
&+ \langle b + \beta x, \nabla f(x) \rangle - (c + \langle \gamma, x_{\mathcal{I}} \rangle) \\
&+ \int_{D \setminus \{0\}} \left(f(x+\xi) - f(x) - \langle \nabla_{\mathcal{J}} f(x), \chi_{\mathcal{J}}(\xi) \rangle\right) \nu(d\xi) \\
&+ \sum_{i=1}^{m} \int_{D \setminus \{0\}} \left(f(x+\xi) - f(x)\right. \\
&\quad \left. - \left\langle \nabla_{\mathcal{J} \cup \{i\}} f(x), \chi_{\mathcal{J} \cup \{i\}} f(\xi) \right\rangle\right) x_i \mu_i(d\xi).
\end{aligned}
$$

Additional technical assumption

$$\int_{D\setminus Q} e^{\langle q,\xi\rangle}\nu(d\xi) < \infty \quad \text{and} \quad \int_{D\setminus Q} e^{\langle q,\xi\rangle}\mu_i(d\xi) < \infty, \quad i \in \mathcal{I},$$

where $Q = \{\xi \in D : |\xi_k| \leq 1,\ k = 1, \ldots, N\}$.

Generalized Riccati equations

For all $u \in i\mathbb{R}^m \times i\mathbb{R}^n$ the functions ϕ and ψ are solutions to a coupled system of generalized Riccati equations involving the coefficients $a, \alpha, b, \beta, c, \gamma, \nu, \mu$

Example: Heston model with stochastic interest rates and jump to default

$$
\begin{aligned}
dX_t^1 =& \kappa_1 \left(\theta_1 - X_t^1\right) dt + \eta_1 \sqrt{X_t^1} dW_t^1 \\
dX_t^2 =& \kappa_2 \left(\theta_2 - X_t^2\right) dt + \eta_2 \sqrt{X_t^2} dW_t^2 \\
dX_t^3 =& -\frac{1}{2} X_t^1 dt + \sqrt{X_t^1} dW_t^3
\end{aligned}
$$

with correlation matrix of the Brownian motions

$$
\begin{pmatrix}
1 & 0 & \rho \\
0 & 1 & 0 \\
\rho & 0 & 1
\end{pmatrix}
$$

s_t, r_t, λ_t

- Excess log returns: $s_t = X_t^3$
- Interest rates: $r_t = d + \delta_1 X_t^1 + \delta_2 X_t^2$
- Default intensity: $\lambda_t = c + \gamma_1 X_t^1 + \gamma_2 X_t^2$

Outline

Goal

- We want to price payoffs of the form $\varphi(S_t)$
- Government bonds: $\varphi \equiv 1$
- Corporate bonds: $\varphi(x) = 1_{\{x>0\}}$
- Call options: $\varphi(x) = (x - K)^+$
- Power payoffs $\varphi(x) = x^p 1_{\{x>0\}}$
- ...

Fast Fourier transform methods

For stock price dynamics such that $S_t > 0$ there exists pricing literature (Carr–Madan (1999) with extensions by Lee (2003) and many others) that gives formulas for European option prices in terms of the *discounted characteristic function*

$$\mathbb{E}_x \left[\exp \left(-R_t + iz \log S_t \right) \right]$$

Generalized discounted characteristic functions

Define

$$h_{t,x}(z) := \mathbb{E}_x \left[\exp\left(-R_t + z \log S_t\right) 1_{\{\tau > t\}} \right].$$

for all $z \in U_{t,x}$, where

$$U_{t,x} := \{ z \in \mathbb{C} : h_{t,x}(\mathrm{Re}(z)) < \infty \}.$$

One can show that $U_{t,x}$ is an open vertical strip, an open vertical half-space or all of $\mathbb{C}$.

Extension of the state space

- $\hat{D} = \mathbb{R}^{m+2} \times \mathbb{R}^n \cup \{\Delta\}$
-
$$Y_t = \begin{cases} (X_t, R_t, \Lambda_t) & \text{if } t < \tau \\ \Delta & \text{otherwise} \end{cases}$$
- $(Y_t)_{t \geq 0}$ is still regular affine

Generalized Riccati equations

$$
\begin{cases}
\partial_t A(t,u,v,w) = F(B(t,u,v,w),v,w) \\
\partial_t B_{\mathcal{I}}(t,u,v,w) = G(B(t,u,v,w),v,w) \\
B_{\mathcal{J}}(t,u,v,w) = \exp(\beta^T_{\mathcal{J}\mathcal{J}} t) u_{\mathcal{J}} \\
A(0,u,v,w) = 0, \qquad B_{\mathcal{I}}(0,u,v,w) = u_{\mathcal{I}},
\end{cases}
$$

Generalized Riccati equations

$$
\begin{aligned}
F(u,v,w) &= \langle au,u\rangle + \langle b,u\rangle + dv + c(w-1) \\
&+ \int_{D\setminus\{0\}} \left(e^{\langle u,\xi\rangle} - 1 - \langle u_{\mathcal{J}}, \chi_{\mathcal{J}}(\xi)\rangle\right) \nu(d\xi) \\
G_i(u,v,w) &= \langle \alpha_i u,u\rangle + \sum_{k=1}^{d} \beta_{ki} u_k + \delta_i v + \gamma_i(w-1) \\
&+ \int_{D\setminus\{0\}} \left(e^{\langle u,\xi\rangle} - 1 - \left\langle u_{\mathcal{J}\cup\{i\}}, \chi_{\mathcal{J}\cup\{i\}}(\xi)\right\rangle\right) \mu_i(d\xi)
\end{aligned}
$$

for $i \in \mathcal{I}$.

Candidate for $h_{t,x}$

- Define

$$l_{t,x}(z) = \exp(ze + A(t, z\varepsilon, z-1, z) + \langle B(t, z\varepsilon, z-1, z), x\rangle)$$

for $z \in V_t$, where

$$V_t := \{z \in \mathbb{C} : B_i(t, z\varepsilon, z-1, z) \text{ is finite for all } i \in \mathcal{I}\}.$$

- Since Y is affine, $h_{t,x}(iy) = l_{t,x}(iy)$ for all $y \in \mathbb{R}$
- $I_t :=$ largest interval around 0 contained in $V_t \cap \mathbb{R}$
- $V_t^0 :=$ connected component of V_t containing 0

Main result

Theorem

For all $(t,x) \in \mathbb{R}_+ \times D$, $U_{t,x}$ is an open subset of $\mathbb{C}$ containing $\{z \in \mathbb{C} : \mathrm{Re}(z) \in I_t\}$ and $h_{t,x}(z) = l_{t,x}(z)$ for each $z \in U_{t,x} \cap V_t^0$.

Idea of the proof:

1. $h_{t,x}$ is a characteristic function
2. $l_{t,x}$ is analytic on V_t^0.

Martingale property of the discounted stock price

Corollary

The condition

$$F(\varepsilon, 0, 1) = 0,\ G(\varepsilon, 0, 1) = 0 \text{ and } \beta_{\mathcal{J}\mathcal{J}} = 0 \quad \text{(M)}$$

is sufficient for the discounted stock price $\exp(s_t + \Lambda_t)1_{\{t<\tau\}}$ to be a martingale with respect to all $\mathbb{P}_x$, $x \in D$.

If all components of $\varepsilon_{\mathcal{J}}$ are different from 0, then (M) is also necessary.

Extension of Carr–Madan's inverse Fourier transform pricing formula

Call option with log strike k:

$$c_{t,x}(k) = \mathbb{E}_x\left[e^{-R_t}\left(S_t - e^k\right)^+\right].$$

Proposition

If $1+p \in U_{t,x}$ for some $p > 0$. Then

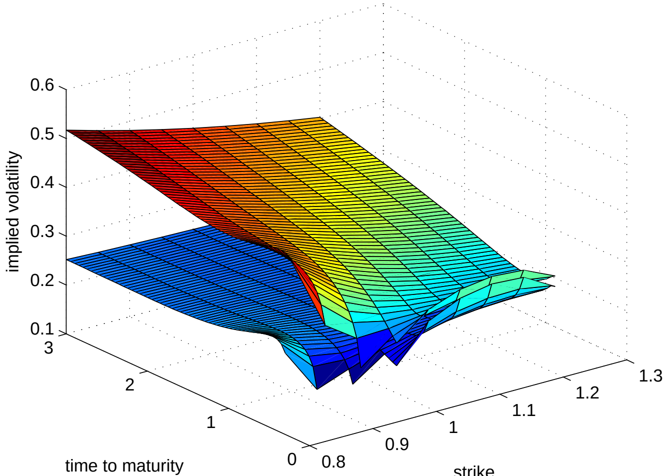
$$c_{t,x}(k) = \frac{e^{-pk}}{2\pi}\int_{\mathbb{R}} e^{-iyk} g_c(y)dy = \frac{e^{-pk}}{\pi}\int_0^\infty \mathrm{Re}\left(e^{-iyk} g_c(y)\right)dy,$$

where

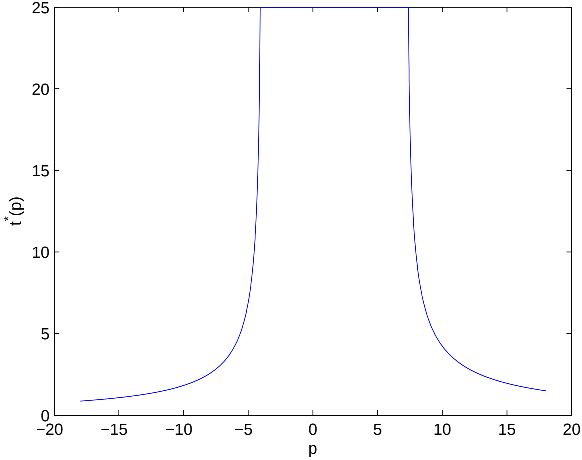
$$g_c(y) = \frac{h_{t,x}(p+1+iy)}{p^2+p-y^2+iy(2p+1)}.$$

Pricing in a Heston model with stochastic interest rates and jump to default

Implied volatility surface: with and without default



Moment explosions



Pricing of European options with arbitrary payoff φ

Integrability condition

$$L_{t,x} = \{\varphi : \mathbb{R}_+ \to \mathbb{R} : \mathbb{E}_x\left[e^{-R_t}|\varphi(S_t)|\right] < \infty\}.$$

Procedure:

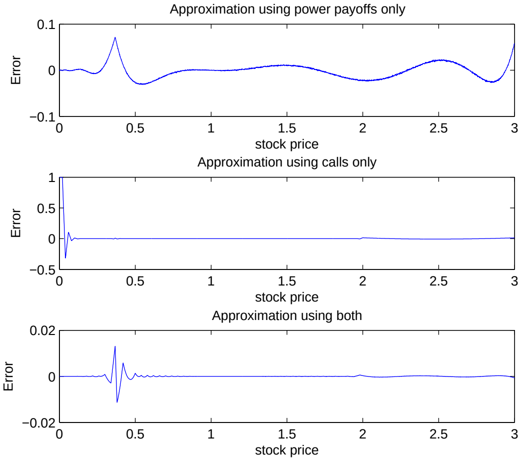
1. Let $\varphi \in L_{t,x}$.
2. Take a set $\mathcal{K}$ of strikes of European calls.
3. Take a set $\mathcal{P}$ of powers of power payoffs in $L_{t,x}$.
4. Use regression weighted by the heuristic density of S_t in order to find the best approximation. For better numerical performance use Gram-Schmidt in order to orthogonalize the power payoffs.

Application: truncated log payoff

- Payoff function: $\varphi(x) = \log(x) \vee k$.
- Example: $S_0 = 1$, $k = -1$.
- Approximating assets:
  1. call options with strikes $\mathcal{K} = \{0.02, 0.04, \dots, 2\}$ ($\mathcal{P} = \emptyset$)
  2. power payoffs of powers $\mathcal{P} = \{0, 0.05, \dots, 4.95\}$ ($\mathcal{K} = \emptyset$)
  3. using $\mathcal{K} = \{0.02, 0.06, \dots, 1.98\}$ and $\mathcal{P} = \{0, 0.1, \dots, 4.9\}$.
- Heuristic density for S_t:

$$\rho(x) = \begin{cases} \exp(-10x) & x < 0.5 \\ \exp(-10|x-1|) & 0.5 \le x \le 1.5 \\ \exp(-5) & x > 1.5. \end{cases}$$

Comparison of different approximation methods



Application: variance swap pricing

- Variance swaps on futures on the stock with maturity t
- Future price $dF_u = \sigma_u F_u dW_u$ (no jumps)
- $S_t = F_t$
- Cap $C > 0$ (usually $C = 2.5K$)
- Payoff at maturity:

$$\Sigma_t = \max\left(\frac{1}{t}\sum_{i=1}^{I}\left(\log\frac{F_{t_i}}{F_{t_{i-1}}}\right)^2 - K, C\right)$$

Application: variance swap pricing

Without default:

- Neglect the cap (it is very unlikely to be hit)
- Approximation by Dupire (1993) and Neuberger (1994):

$$\begin{aligned} & \sum_{i=1}^{I} \left(\log \frac{F_{t_i}}{F_{t_{i-1}}} \right)^2 \approx \int_0^t \sigma_u^2 du \\ = \quad & 2 \left(\int_0^t \frac{1}{F_u} dF_u - \log S_t + \log F_0 \right), \end{aligned}$$

Application: variance swap pricing

With default:

- Cap is important
- Approximate price for $k = \log(F_0) - t(C+K)/2$:

$$
\begin{aligned}
\Sigma_t &\approx 1_{\{\tau > t\}} \left\{ \frac{1}{t} \int_0^t \sigma_u^2 du - K \right\} + 1_{\{\tau \le t\}} C \\
&\approx \frac{2}{t} \left\{ \int_0^{t \wedge \tau} \frac{1}{F_{u-}} dF_u - (\log(S_t) \vee k) + \log F_0 \right\} \\
&\quad -K - 1_{\{\tau \le t\}} \frac{2}{t} \int_0^{\tau} \frac{1}{F_{u-}} dF_u
\end{aligned}
$$

Outline

Completing the market

- Number of assets must match the number of sources of risk and the sources of risk must be "hedgeable"
- Jumps must be discrete
- $\nu = \sum_{q=1}^{M} v_q \delta_{x_q}$ for some $v_q > 0$ $x_q \in D \backslash \{0\}$
- $\mu_i = \sum_{q=1}^{M_i} v_{iq} \delta_{x_{iq}}$ for all $i \in \mathcal{I}$ with $v_{iq} \in \mathbb{R} \backslash \{0\}$ and $x_{iq} \in D \backslash \{0\}$.
- Assume X has a realization as

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \sum_{i=0}^{m} \sum_{k=1}^{N_t^i} Z_k^i$$

Where W is an N-dimensional BM, N^i are Poisson processes and Z_1^i, Z_2^i... are i.i.d.

Completing the market

- $L = N + M + \sum_{i=1}^{m} M_i + 1$ instruments needed
- Porfolio of basic instruments (European options): $\Phi = \{(t_1, \varphi_1),\ \ldots,\ (t_L, \varphi_L)\}$

Sensitivities

- Classical Greeks: For $q = 1, \ldots, N$:

$$H^q_{t,x} = \frac{\partial}{\partial x_q} \mathbb{E}_x[\exp(-R_t)\varphi(S_t)]$$

- Sensitivity to jumps corresponding to ν:
  For $q = 1, \ldots, M$:

$$J^q_{t,x} = \mathbb{E}_{x+x_q}[\exp(-R_t)\varphi(S_t)] - \mathbb{E}_x[\exp(-R_t)\varphi(S_t)]$$

- Sensitivity to jumps corresponding to μ_i:
  For $q = 1, \ldots, M_i$:

$$J^q_{t,x} = \mathbb{E}_{x+x_{iq}}[\exp(-R_t)\varphi(S_t)] - \mathbb{E}_x[\exp(-R_t)]\varphi(S_t)]$$

- Sensitivity to default:

$$D = \mathbb{E}_x[\exp(-R_t)\varphi(0)] - \mathbb{E}_x[\exp(-R_t)\varphi(S_t)]$$

Replicating portfolio

$$
\begin{aligned}
H^1_{t,x} &= \sum_{l=1}^{L} \vartheta^l(t,x) H^{l,1}_{t,x} \quad \dots \quad H^N_{t,x} = \sum_{l=1}^{L} \vartheta^l(t,x) H^{l,N}_{t,x} \\
J^1_{t,x} &= \sum_{l=1}^{L} \vartheta^l(t,x) J^{l,1}_{t,x} \quad \dots \quad J^M_{t,x} = \sum_{l=1}^{L} \vartheta^l(t,x) J^{l,M}_{t,x} \\
&\vdots \\
&\vdots \\
D_{t,x} &= \sum_{l=1}^{L} \vartheta^l(t,x) D^l_{t,x}
\end{aligned}
$$

Completeness

- The market is complete iff the system of linear equations has a unique solution for all $\varphi \in L_{t,x}$.
- In the Heston model with stochastic interest rates and jump to default the market is complete if one can trade a stock, a government bond, a corporate bond and a vanilla option and the following holds for all $0 \leq s \leq t$:

$$\partial_{x_1} c_{t,x}(k) B_2^0(s,0) - \partial_{x_2} c_{s,x}(k) B_1^0(s,0) \neq$$
$$[B_1^0(s,0) B_2(s,0) - B_2^0(s,0) B_1(s,0)][\partial_{x_3} c_{s,x}(k) - c_{s,x}(k)]$$

Conclusion

- General affine model for equity derivatives that incorporates stochastic volatility, stochastic interest rates and jump to default
- Notion of discounted characteristic function for cases when $S_t = 0$ with positive probability
- Pricing in semi-closed form for most common European equity derivatives; otherwise: approximation
- Under additional assumptions: completeness/hedging