Testing Quantum States for Purity

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1. Quantum State Estimation
   - Introduction
   - Basics of State Estimation
   - Likelihood Analysis

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- These probabilities are parameterized by a parameter which is called a state.

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- In the macroscopic world, we usually take measurement for granted.
- Day-to-day, if we want to know how long an object is, we simply use a ruler, measuring tape, etc.
- In experiments, we recognize that our measuring devices and techniques are not perfect, so we append estimated uncertainties to our measurements.

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- In the situation where we can repeatedly produce and test states created with the same experimental settings, we can circumvent this restriction.
- Combining the results of multiple measurements lets us produce an estimate of the full state

Statistical Theory

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- Repeated measurement of identical quantum states will, in general, result in different outcomes.
- Clearly, we cannot simply measure a property *once*, pack-up and go home!!!!!

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- Denote the set of measurement operators by $\{X_1 \ldots X_m\}$, where each is a Hermitian matrix.
- Born's rule tells us that the probability of observing a particular outcome when measuring a system is given by

$$p_i(\theta) = \mathrm{Tr}\left(X_i \rho(\theta)\right)$$

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$$p(y_1 \dots y_m|\theta) = \prod_{i=1}^{m} \frac{\left(\lambda \mathrm{Tr}\left(X_i \rho\left(\theta\right)\right)\right)^{y_i}}{y_i!} e^{-\lambda \mathrm{Tr}(X_i \rho(\theta))}.$$

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- The above is the probability of observing the counts $\{y_1 \ldots y_m\}$ given the parameter $\theta \in \Theta$.

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- Letting $y_1 + \cdots + y_m = n$, by conditioning on n
- multinomial distribution

$$P\left(Y_i = y_1, \ldots, Y_m = y_m | n; \theta\right) = \frac{n}{y_1! \cdots y_s!}\, p_1(\theta)^{y_1} \cdots p_s(\theta)^{y_s}$$

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- the log-likelihood

$$\sum_{k=1}^{s-1} y_k \log p_k(\theta) + (n - \cdots - y_{s-1}) \log (1 - \cdots - p_{s-1}(\theta)) .$$

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- a generalized linear model with a linear link function!

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- the interior of Θ are the positive definite matrices
- the boundaries consist of disjoint union of rank 1 to rank $d - 1$ matrices.
- Again the pure states are rank 1 matrices.

Parametrization

$\hat{\rho}$ and $\tilde{\rho}$ MLE estimates under all states and pure states respectively

$$D(\hat{\rho}, \tilde{\rho}) = 2\left\{\ell(\hat{\rho}; y_1, \ldots, y_s) - \ell(\tilde{\rho}; y_1, \ldots, y_s)\right\}.$$

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$\theta = (\theta^{(1)}, \theta^{(2)})$ for which the pure states is given by $\theta^{(2)} = 0$.
The score statistic for testing for purity is

$$S(\tilde{\theta}) = \left(\left.\frac{\partial \ell}{\partial \theta^{(2)}}\right|_{\theta=\tilde{\theta}} \right) \mathcal{I}_{22\cdot 1}^{-1}(\tilde{\theta}) \left(\left.\frac{\partial \ell}{\partial \theta^{(2)}}\right|_{\theta=\tilde{\theta}} \right)',$$

$$\mathcal{I}_{22\cdot 1}(\theta) = \mathcal{I}_{22}(\theta) - \mathcal{I}_{21}(\theta)\mathcal{I}_{11}^{-1}(\theta)\mathcal{I}_{12}(\theta)$$

$$\mathcal{I}(\theta) = \begin{pmatrix} \mathcal{I}_{11}(\theta) & \mathcal{I}_{12}(\theta) \\ \mathcal{I}_{21}(\theta) & \mathcal{I}_{22}(\theta) \end{pmatrix}.$$

Large sample theory

The score statistic for testing for purity has the result

$$S(\tilde{\theta}) \sim \chi^2_{(d-1)^2},$$

as $n \to \infty$ provided we "enlarge" our parameter space to

$$\mathcal{S} = \{\rho = \rho(\theta) : \rho^* = \rho \ \mathrm{Tr}\,\rho = 1 \ \mathrm{Tr}\, X_j \rho > 0 \ j = 1, \ldots, m\}$$

1 Qubit Case

In order to provide some clarity, let us consider the simplest case. If our parameters are $\theta = (a_{12}, b_{12}, a_{22})$, let

$$\rho(\theta) = \begin{pmatrix} 1 - a_{2,2} & a_{1,2} + ib_{1,2} \\ a_{1,2} - ib_{1,2} & a_{2,2} \end{pmatrix}.$$

The following new parameters are defined:

$$\begin{aligned} \alpha_{1,2} &= a_{1,2} \\ \beta_{1,2} &= b_{1,2} \\ \alpha_{2,2} &= a_{2,2}(1 - a_{2,2}) - (a_{1,2}^2 + b_{1,2}^2). \end{aligned}$$

1 Qubit Case ctd.

We have:

$$
\begin{aligned}
a_{1,2} &= \alpha_{1,2} \\
b_{1,2} &= \beta_{1,2} \\
a_{2,2} &= \frac{1 \pm \sqrt{1 - 4(\alpha_{2,2} + \alpha_{1,2}^2 + \beta_{1,2}^2)}}{2}.
\end{aligned}
$$

Thus

$$
\rho(\alpha_{1,2}, \beta_{1,2}, \alpha_{2,2}) = \begin{pmatrix} \frac{1 \mp \sqrt{1-4(\alpha_{1,2}^2+\beta_{1,2}^2+\alpha_{2,2})}}{2} & \alpha_{1,2} + i\beta_{1,2} \\ \alpha_{1,2} - i\beta_{1,2} & \frac{1 \pm \sqrt{1-4(\alpha_{1,2}^2+\beta_{1,2}^2+\alpha_{2,2})}}{2} \end{pmatrix}.
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Control Over Rank and Uniqueness

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These properties extend beyond the single-qubit case.

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Optical Experiments

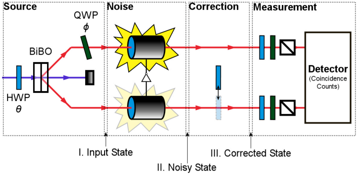

Figure: Schematic diagram of two-qubit experiment.

Single Qubit Experiments

Table: Deviances, score statistics, and purities of qubits.

	Data set				
	I	**II**	**III**	**IV**	**V**
Deviance, D	37.23	0.16	754.02	1.04	34.60
p value	.00	.69	.00	.31	.00
Score, S	41.14	0.16	838.37	1.04	34.75
p value	.00	.69	.00	.31	.00
Purity, $\hat{\gamma}$	.995	.999	.996	1.000	.996

Two Qubit Experiments

Table: Deviances, score statistics, and purities of qubit pairs.

	Data set								
	I	**II**	**III**	**IV**	**V**	**VI**	**VII**	**VIII**	**IX**
Deviance, D	25,146	892	3,958	148	9,835	981	199,658	4,232	205,642
Score, S	1,494	1,675	2,197	178	1,216	1,159	1.85×10^{13}	1.93×10^{10}	2.34×10^{11}
Purity, $\hat{\gamma}$	1.527	.992	1.355	.978	1.257	.935	.668	.937	.658