Some geometric properties of Intersection Body Operator.

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Kent State University

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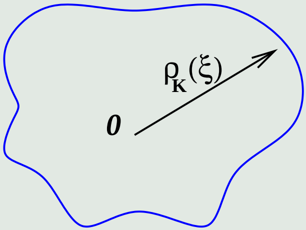
Standard definition.

Radial function: $\rho_K(\xi) = \sup\{a : a\xi \in K\}$, for $\xi \in S^{n-1}$.



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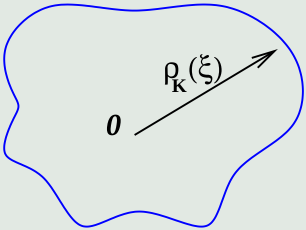

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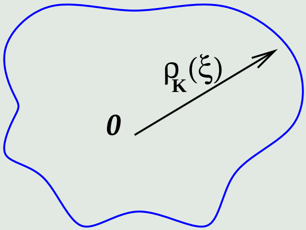

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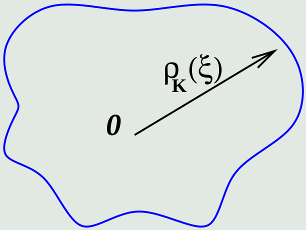

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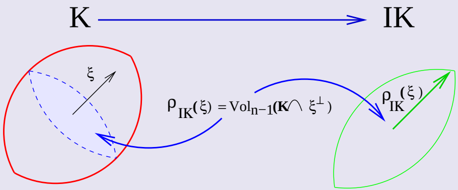
- $\xi^\perp = \{x \in \mathbb{R}^n : x \cdot \xi = 0\}$.

E. Lutwak: Intersection body, of a body K



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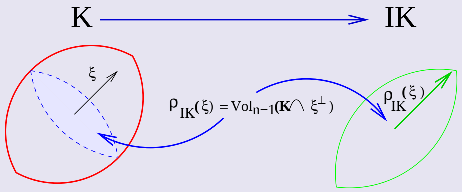

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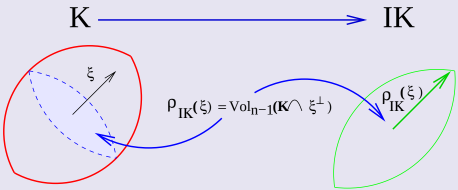

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Why do we need them?

Solution of Busemann-Petty problem. Definition of L_{-1}. Very nice questions in Harmonic Analysis & just for fun.

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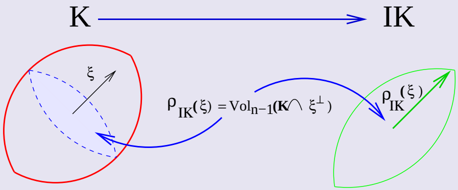

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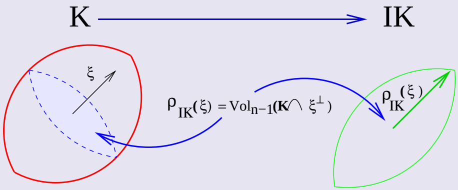

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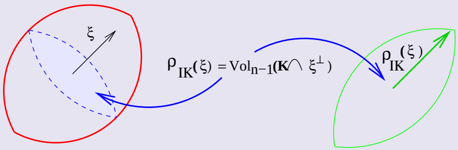

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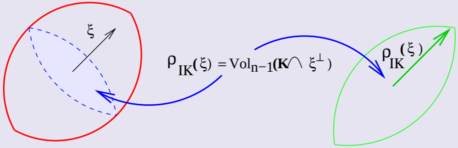

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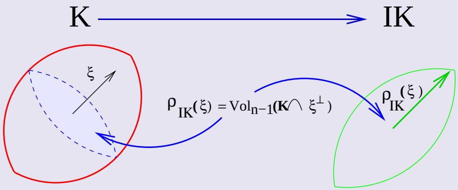

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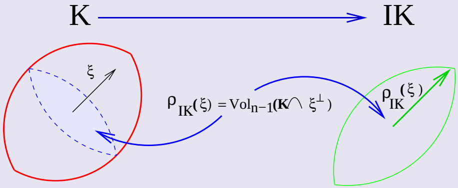

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- A. Koldobsky: B_p^n - intersection body for $p \in (0,2]$; NOT intersection body for $p > 2$, $n \geq 5$.

Connection to Spherical Radon Transform

Spherical coordinates in $\xi^\perp$

$$\rho_{IK}(\xi) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{n-1} \int\limits_{S^{n-1}\cap\xi^\perp} \rho_K^{n-1}(\theta)d\theta = \frac{1}{n-1} R\rho_K^{n-1}(\xi).$$

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More general definition of Intersection Body (C^∞-case).

A symmetric star body L is an intersection body if $R^{-1}\rho_L \geq 0$.

Example: Local/Equatorial characterization.

Intersection Bodies: Fix $\varepsilon \in (0, 1/10)$

Consider body K such that for every $u \in S_{n-1}$ there exits an intersection body K_u, which coincide with K on a ε-neighborhood of u. Is it true that K must be an intersection body itself?

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Original Dual problem for Zonoids: The same answer: Local - W. Weil; Local equatorial: G. Panina; W. Weil and P. Goodey – even dimensions; F. Nazarov, D. Ryabogin, A.Z. – odd dimensions.

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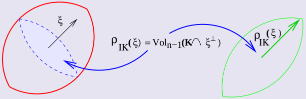

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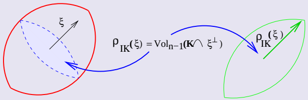

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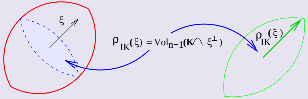

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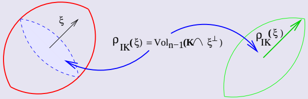

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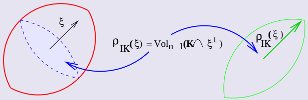

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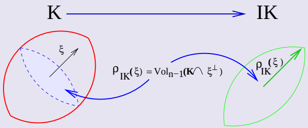

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- $d_{BM}(\mathrm{I}TK, \mathrm{I}TL) = d_{BM}(\mathrm{I}K, \mathrm{I}L)$.
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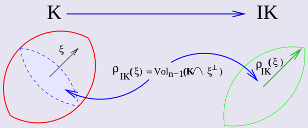

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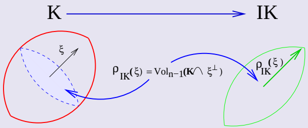

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- $d_{BM}(E, \mathrm{I}E) = 1$.
- $d_{BM}(K, \mathrm{I}K) = 1$, $K \subset \mathbb{R}^2$, K-symmetric.

Questions:

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A. Fish, F. Nazarov, D. Ryabogin, A.Z.:

Consider a star body $K \subset \mathbb{R}^n$, $n \geq 3$, is it true that

$$d_{BM}(I^m K, B_2^n) \to 1, \text{ as } m \to \infty,$$

i.e. iterations of intersection body operator of a star body K will converge to B_2^n in d_{BM}?

Support function: $h_L(\theta) = \sup\{x \cdot \theta,\ x \in L\}$.

Dual story – Projection body (convex, sets only!)

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ΠL – projection body of L:

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- $\Pi B_2^n = c_n B_2^n$.
- $\Pi B_\infty^n = c_n B_\infty^n$, where $B_\infty^n = [-1,1]^n$.

Fixed point is NOT unique! W. Weil (71) described polytopes that satisfy this property. General case is still open.

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A. Fish, F, Nazarov, D. Ryabogin, A.Z., (2009)

$\exists \varepsilon_n > 0$ such that $\forall K \subset \mathbb{R}^n$ such that K-start body, $d_{BM}(K, B_2^n) < 1 + \varepsilon_n$, we get

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Remarks:

- We do not assume convexity of K. Such an assumption will much simplify the proofs.
- Busemann theorem: If K-convex symmetric, then $\mathrm{I}K$ is convex symmetric.
- Even if K is convex symmetric, then $d_{BM}(K, B_2^n) \le \sqrt{n}$, which is very far from ε_n.

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- (J. Kim, V. Yaskin, A.Z.) Wrong without assumption of convexity! there is a star body (p-convex) K such that $d_{BM}(\mathrm{I}K, B_2^n) >> d_{BM}(K, B_2^n)$.

Main Idea: Spherical Radon Transform

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Denote by $\mathcal{R} = \frac{1}{\mathrm{Vol}_{n-2}(S^{n-2})} R$, i.e. $\mathcal{R}1 = 1$.

Question: $(n \geq 3)$

Consider symmetric function $f : S^{n-1} \to \mathbb{R}^+$, such that $f = \mathcal{R}f^{n-1}$, is it true that then $f = 1$?

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Formulas Exists: Clebsch–Gordan coefficients — but they are hard, not clear (to me!) how to use for this problem.

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Thus we need to KILL H_2^{ϕ}. HOW ? Main idea – in the end of the day, H_2^{ϕ} is just quadratic polynomial make it constant on S^{n-1}, using linear transformation. YES, "like" isotropic position, BUT in Fourier coordinates.

Linear Transform $T \in GL(n)$ applied to function f on S^{n-1}

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- If $f, g \in \mathcal{U}_\alpha$, then $fg \in \mathcal{U}_\alpha$ and $\|fg\|_{\mathcal{U}_\alpha} \leq C\|f\|_{\mathcal{U}_\alpha}\|g\|_{\mathcal{U}_\alpha}$.
- Let $T \in GL(n)$ with $\|T\|, \|T^{-1}\| \leq 2$. Then if, $f \in \mathcal{U}_\alpha$, we have $Tf \in \mathcal{U}_{\alpha-1/2}$ and $\|Tf\|_{\mathcal{U}_{\alpha-1/2}} \leq C_{1/2}\|f\|_{\mathcal{U}_\alpha}$.
- If $f \in \mathcal{U}_\alpha$, then $\mathcal{R}f \in \mathcal{U}_{\alpha+n-2}$ and $\|\mathcal{R}f\|_{\mathcal{U}_{\alpha+n-2}} \leq C\|f\|_{\mathcal{U}_\alpha}$.

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3. Let $\beta > \alpha$. Then for every $\delta > 0$, there exists $C = C_{\alpha,\beta,\delta}$, such that $\|f\|_{\mathcal{U}_\alpha} \le C\|f\|_{L_\infty} + \delta\|f\|_{\mathcal{U}_\beta}$.

Fix $\beta > \alpha > 0$. Let $f = 1 + \varphi$, $\|\varphi\|_{L^\infty} < \varepsilon < 1/2$.

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Using (1) and (2): $f_k \in \mathcal{U}_\beta$ for sufficiently large k and $\|f_k\|_{\mathcal{U}_\beta} \le C(k)$. Note

$$(1-\varepsilon)^{(n-1)^k} \le f_k \le (1+\varepsilon)^{(n-1)^k}.$$

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Let $\mu = \int f_k$. If $\varepsilon > 0$ is sufficiently small, then $|\mu - 1|$ is small and $\mu^{-1} f_k = 1 + \psi$ where $\int \psi = 0$ and $\|\psi\|_{L^\infty}$ is small. Note that

$$\|\psi\|_{\mathcal{U}_\beta} \leq 1 + \mu^{-1}\|f_k\|_{\mathcal{U}_\beta} \leq C'(k),$$

by (3), $\|\psi\|_{\mathcal{U}_\alpha}$ is also small ($\|\psi\|_{\mathcal{U}_\beta} < C(k)$ and $\|\psi\|_{L^\infty} \to 0$ as $\varepsilon \to 0$).

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Applying this to the function ρ_K, we conclude that if K is sufficiently close to B_n, then, after proper normalization, $\rho_{\mathrm{I}^k K}$ can be written as $1 + \varphi$ with $\|\varphi\|_{\mathcal{U}_\alpha}$ as small as we want,