On the Homothety Conjecture

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Convex Floating Body, Schütt and Werner (1990)

Let K be a convex body in $\mathbb{R}^n$. The convex floating body K_δ is the intersection of all halfspaces H^+ whose defining hyperplanes H cut off a set of volume at most δ from K,

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$$K_\delta = \bigcap_{|H^- \cap K| \leq \delta} H^+.$$

- $K_0 = K$, and $K_s \subset K_t$ if $s \geq t \geq 0$.
- K_δ is convex.
- $(B_2^n)_\delta = c(\delta, n) B_2^n$ for all $\delta > 0$.

Properties of Convex Floating Body

- For all (invertible) affine maps $T : \mathbb{R}^n \to \mathbb{R}^n$ and for all $\delta > 0$,

$$(TK)_\delta = T\big(K_{\frac{\delta}{|\det(T)|}}\big).$$

Here $|\det(T)|$ is the absolute value of the determinant of T.

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- An ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ equals $T(B_2^n)$ for some (invertible) affine map T on $\mathbb{R}^n$. Then

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- Schütt and Werner (1994): K_δ is strictly convex for all $\delta > 0$. (If in addition K is origin-symmetric, Meyer and Reisner (1991)). Hence K_δ cannot be a polytope for all $\delta > 0$.

Properties of Convex Floating Body

- Connection with the classical affine surface area (Schütt and Werner (1990)):

$$c_n \lim_{\delta \to 0} \frac{|K| - |K_\delta|}{\delta^{\frac{2}{n+1}}} = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} \, d\mu_K(x) = as(K),$$

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- Connection with $L_{-n(n+2)}$-affine surface area (Meyer and Werner (2000), Werner and Ye (2008)):

$$\begin{aligned} c_n \lim_{\delta\to 0} \frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} &= \int_{S^{n-1}} f_{K^\circ}(u)^{\frac{n+2}{n+1}} h_{K^\circ}(u)^2\, d\sigma(u) \\ &= as_{\frac{-n}{n+2}}(K^\circ) = as_{-n(n+2)}(K), \end{aligned}$$

for K smooth enough.

Note that if K is an origin-symmetric ellipsoid, then, for all $\delta > 0$, K_δ is an ellipsoid that dilates to K.

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- Hereafter, an ellipsoid means an origin-symmetric ellipsoid.

What is known?

Schütt and Werner (1994)

If there is a sequence $\delta_i \to 0$, such that, K_{δ_i} is homothetic to K for all $i \in \mathbb{N}$, then K is an ellipsoid.

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Stancu (2009):

Let K be a convex body with boundary of class C^2_+. There exists a positive number $\delta(K)$, such that, K_δ is homothetic to K for some $\delta < \delta(K)$, then and only then K is an ellipsoid.

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- Stancu (2006) proved similar results for $K \in C^{\geq 4}$.

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Werner and Ye, 2010

Let $B_p^n, 1 \leq p \leq \infty$ be the unit ball of l_p^n. Let $0 < \delta < \frac{|B_p^n|}{2}$. Then $(B_p^n)_\delta = cB_p^n$ for some $0 < c < 1$ if and only if $p = 2$.

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- Recall $(TK)_\delta = T\left(K_{\delta/|\det(T)|}\right)$. Let $K = T\left(B_p^n\right)$, $1 \leq p \leq \infty$. Let $0 < \delta < \frac{|K|}{2}$ be a constant. $K_\delta = cK$ for some constant $0 < c < 1$ if and only if K is an ellipsoid.

Proof by a "One direction technique"

For $p = 1, \infty$: $(B_p^n)_\delta$ is strictly convex, then $(B_p^n)_\delta$ cannot be homothetic to B_p^n.

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The curvature $\kappa_{B_p^n}$ at $x \in \partial B_p^n$ has the following form (see Schütt and Werner (2004))

$$\kappa_{B_p^n}(x) = \frac{(p-1)^{n-1} \prod_{i=1}^n x_i^{p-2}}{\left(\sum_{i=1}^n |x_i|^{2(p-1)}\right)^{\frac{1}{2}}}.$$

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Let K be a convex body in $\mathbb{R}^n$. There exists a positive number $\delta(K)$, such that, the following are equivalent:

(i) K_δ is homothetic to K for some $0 < \delta < \delta(K)$;

(ii) K is an ellipsoid.

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(ii) K is an ellipsoid.

- We provide an estimate for $\delta(K)$ if $K \in C^3_+$.

Sketch of proof.

- Step 1: We show that there exists $\delta_0(K) > 0$, such that, if K_δ is homothetic to K for some $\delta \leq \delta_0(K)$, then K_δ (and hence K) is of class C^2_+.

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- Step 2: Let $\{x_\delta\} = \partial(K_\delta) \cap [0, x]$, $c_n = 2\left(\frac{|B_2^{n-1}|}{n+1}\right)^{\frac{2}{n+1}}$, and

$$f_\delta(x) = \frac{c_n}{n\ \delta^{\frac{2}{n+1}}}\left[1 - \left(\frac{\|x_\delta\|}{\|x\|}\right)^n\right].$$

Schütt and Werner (1990) proved that for all $x \in \partial K$,

$$\lim_{\delta \to 0} f_\delta(x) = \left(\kappa_K(x)\right)^{\frac{1}{n+1}} \langle x, N_K(x) \rangle^{-1} = f(x),$$

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- Step2: For all $x \in \partial K$,

$$\lim_{\delta\to 0} \frac{c_n}{n\ \delta^{\frac{2}{n+1}}} \left[1 - \left(\frac{\|x_\delta\|}{\|x\|}\right)^n\right] = \frac{\left(\kappa_K(x)\right)^{\frac{1}{n+1}}}{\langle x, N_K(x)\rangle} = f(x),$$

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- Step 4: It follows that for δ small enough, $f_\delta(x)$ is not constant, which is impossible if K is homothetic to K_δ. Hence K is an ellipsoid.

References

- E. Werner and D. Ye, *On the Homothety Conjecture.* Accepted by Indiana University Mathematics Journal, 2010. arXiv:0911.0642

Thank you.