Shadow boundaries and the Fourier transform.

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Uniqueness results

Convex centrally symmetric bodies are uniquely determined by:

- Volumes of central sections (Minkowski’s theorem)
- Volumes of projections (Aleksandrov’s theorem)

Uniqueness results

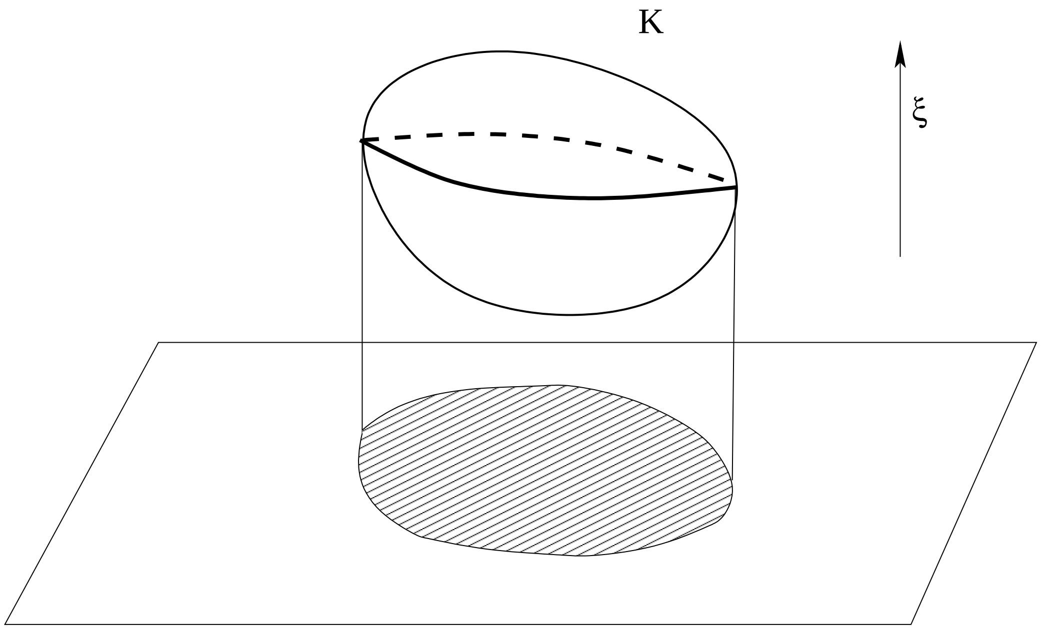
Convex bodies (not necessarily centrally symmetric):

- Falconer, Gardner: volumes of hyperplane sections passing through any two fixed points in the interior of the body
- Böröczky, Schneider: volumes and centroids of sections through 0
- Schneider: mean widths and Steiner points of projections

and many others...

Shadow boundaries

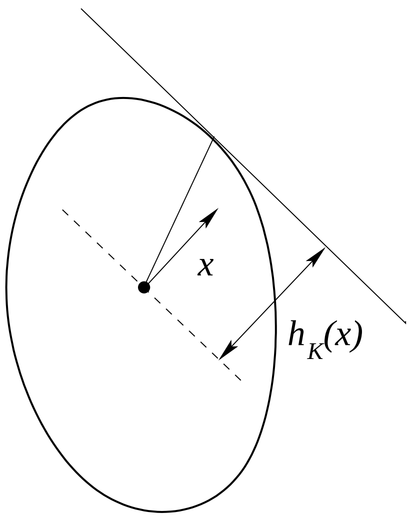
Let K be a convex body in $\mathbb{R}^n$. The shadow boundary of K under illumination parallel to $\xi \in S^{n-1}$ is defined as the set of all boundary points of K at which there are support lines of K parallel to ξ.



The support function of a convex body K is defined by

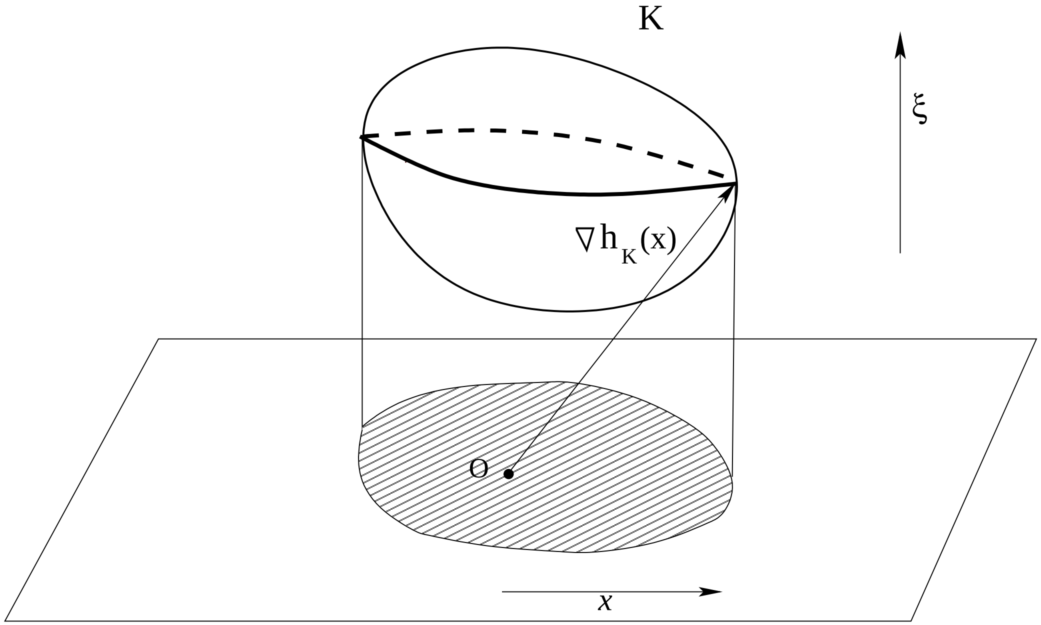
$$h_K(x) = \max_{\xi \in K}(x, \xi), \qquad x \in \mathbb{R}^n.$$

The geometric meaning of $h_K(x)$ if $x \in S^{n-1}$:



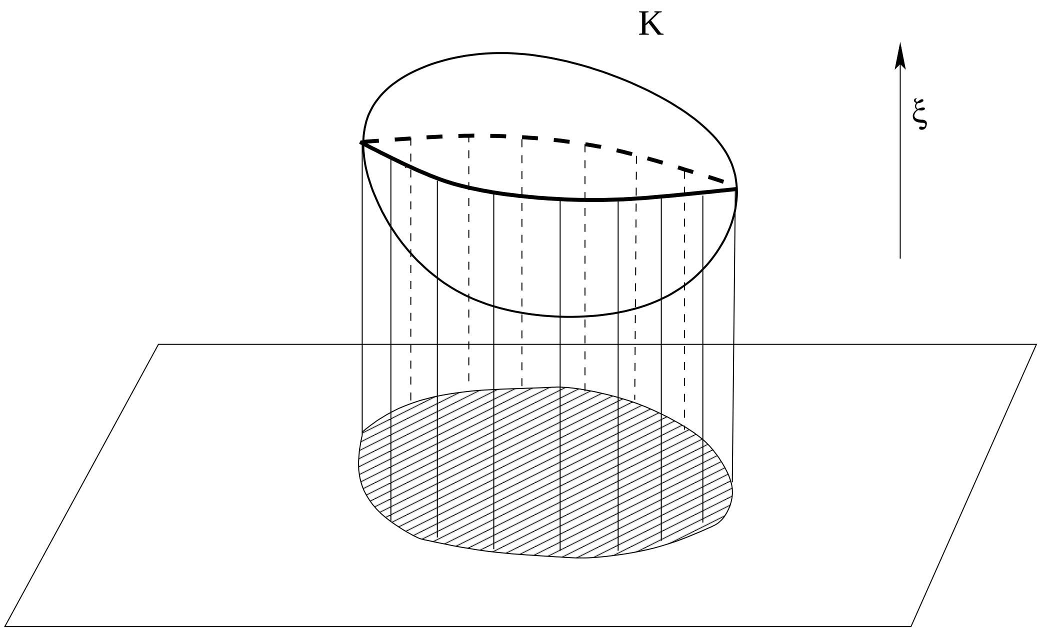
Let K be a strictly convex convex.

Fact: $\nabla h_K(x)$ is the point of contact with K of the support plane with outward normal x.



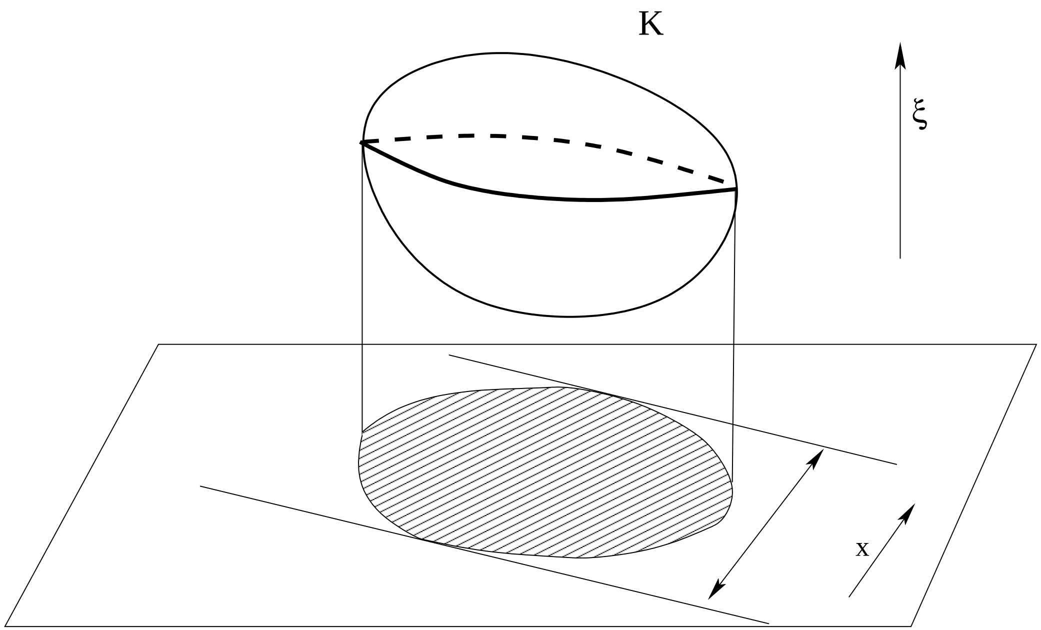
The average height of the shadow boundary of K in the direction of ξ:

$$H_K(\xi) = \int_{S^{n-1}\cap\xi^\perp} (\nabla h_K(x), \xi)\, dx.$$



The average width of the shadow boundary of K in the direction of ξ:

$$W_K(\xi) = \int_{S^{n-1}\cap\xi^\perp} (\nabla h_K(x), x)\, dx = \int_{S^{n-1}\cap\xi^\perp} h_K(x)\, dx.$$



Theorem [P. Goodey, V.Yaskin, MY].
Let K be a strictly convex body. K is uniquely determined by the average height and average width of all its shadow boundaries.

Let f be an infinitely smooth function on the sphere S^{n-1}. Denote

$$f_p(x) = f(x/|x|)|x|^{-n+p}$$

its homogeneous extension to $\mathbb{R}^n$ of degree $-n+p$.

We will be interested in its Fourier transform.

For $f \in C^{\infty}\left(S^{n-1}\right)$ and $\xi \in S^{n-1}$ define

$$F_{\xi}(t) = (1 - t^2)^{(n-3)/2} \int_{S^{n-1} \cap \xi^{\perp}} f(t\,\xi + \sqrt{1 - t^2}\,\zeta)\,d\zeta.$$

Theorem. [GKS] Let f be an even function. The Fourier transform of f_p is given by the following formulas.

i) If $0 < p < 2k+1$, $p \neq n$, p is not an odd integer, then

$$(f_p)^\wedge(\xi) = \cos\frac{p\pi}{2}\Gamma(p)\Big(\int_{-1}^{1} |t|^{-p} \ (F_\xi(t) - F_\xi(0) -$$

$$- \cdots - F_\xi^{(2k-2)}(0)\frac{t^{2k-2}}{(2k-2)!})dt +$$

$$+ \sum_{m=1}^{2k-2} F_\xi^{(m)}(0)\frac{2}{m!(1+m-p)}\Big)$$

ii) if $p = 2k-1 \neq n$, then

$$(f_{2k-1})^{\wedge}(\xi) = \pi(-1)^{k+1} F_{\xi}^{(2k-2)}(0).$$

Theorem. Let f be an odd function.

iii) If $0 < p < 2k+2$, $p \neq n$, p is not an even integer, then

$$(f_p)^\wedge(\xi) = i \sin\frac{p\pi}{2}\Gamma(p)\Big(\int_{-1}^{1} |t|^{-p}\, \text{sgn}\, t\ (F_\xi(t) - F_\xi'(0)t -$$

$$- \cdots - F_\xi^{(2k-1)}(0)\frac{t^{2k-1}}{(2k-1)!})dt +$$

$$+ \sum_{m=1}^{2k-1} F_\xi^{(m)}(0)\frac{2}{m!(1+m-p)}\Big)$$

iv) if $p = 2k \neq n$, then

$$(f_{2k})^{\wedge}(\xi) = i\pi(-1)^{k+1} F_{\xi}^{(2k-1)}(0).$$

In particular,

$$\hat{f}_{-1}(\xi) = -\frac{\pi}{2}\int_{S^{n-1}} |(u,\xi)| f(u)\,du$$

$$-i\int_{S^{n-1}} \big(1 + \Gamma'(1) - \ln|(u,\xi)|\big)\,(u,\xi) f(u)\,du,$$

$$\hat{f}_0(\xi) = \int_{S^{n-1}} \big(\Gamma'(1) - \ln|(u,\xi)|\big)\, f(u)\,du$$

$$+i\pi\left(\frac{1}{2}\int_{S^{n-1}} f(u)\,du - \int_{S^{n-1}\cap\xi^+} f(u)\,du\right),$$

$$\hat{f}_1(\xi) = \pi \int_{S^{n-1}\cap\xi^\perp} f(u)\,du - i\int_{-1}^{1} t^{-1}\left(F_\xi(t) - F_\xi(0)\right)\,dt,$$

$$\hat{f}_2(\xi) = -\int_{-1}^{1} t^{-2}\left(F_\xi(t) - F_\xi(0) - tF'_\xi(0)\right)\,dt + 2F_\xi(0) - i\pi F'_\xi(0),$$

where

$$F_\xi(t) = (1-t^2)^{(n-3)/2}\int_{S^{n-1}\cap\xi^\perp} f(t\,\xi + \sqrt{1-t^2}\,\zeta)\,d\zeta.$$

Theorem.

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Proof.

W_K: even information $\qquad$ H_K: odd information

Let h_K^+ and h_K^- be the even and odd parts of h_K correspondingly.

Even part:

$$(h_K^+(x/|x|_2)|x|_2^{-n+1})^\wedge(\xi) = \pi \int_{S^{n-1}\cap\xi^\perp} h_K^+(u)\, du = \pi W_K(\xi)$$

Even part:

$$(h_K^+(x/|x|_2)|x|_2^{-n+1})^\wedge(\xi) = \pi \int_{S^{n-1}\cap\xi^\perp} h_K^+(u)\,du = \pi W_K(\xi)$$

Odd part:

$$(h_K^-(x/|x|_2)|x|_2^{-n+2})^\wedge(\xi)$$

$$= -i\pi \frac{d}{dt}\left(\int_{S^{n-1}\cap\xi^\perp} h_K^-(t\,\xi + \sqrt{1-t^2}\,u)\,du\right)_{t=0}$$

$$= -i\pi \frac{d}{dt}\left(\int_{S^{n-1}\cap\xi^\perp} h_K(t\,\xi + \sqrt{1-t^2}\,u)\,du\right)_{t=0}$$

$$= -i\pi \int_{S^{n-1}\cap\xi^\perp} (\nabla h_K(u), \xi)du = -i\pi H_K(\xi)$$

Stability results

The Hausdorff distance is defined as

$$\delta_\infty(K, L) = \min\{\lambda \geq 0 | K \subset L + \lambda B^n, L \subset K + \lambda B^n\}.$$

Alternatively,

$$\delta_\infty(K, L) = \max_{\theta \in S^{n-1}} |h_K(\theta) - h_L(\theta)|.$$

Theorem.

Let K and L be convex bodies in $\mathbb{R}^n$ $(n \geq 5)$ which are contained in a ball of radius R. If, for some $\epsilon > 0$,

$$\|W_K - W_L\|_2 + \|H_K - H_L\|_2 < \epsilon$$

then there is a constant $c(n, R)$, dependent only on the dimension n and the radius R, such that

$$\delta_\infty(K, L) \leq c(n, R)\epsilon^{4/(n(n+1))}.$$

Let $0 < p < n$. Let $I_p : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ be the operator defined by

$$I_p(f) = \widehat{f_p},$$

where

$$f_p(x) = f\left(\frac{x}{|x|}\right)|x|^{-n+p}.$$

Theorem.
Let K and L be convex bodies in $\mathbb{R}^n$, contained in a ball of radius R, with infinitely smooth support functions. Let $0 < p < n$. If for some $\epsilon \geq 0$

$$\|I_p(h_K) - I_p(h_L)\|_2 \leq \epsilon,$$

then

$$\delta_\infty(K, L) \leq \begin{cases} C(n,p,R)\epsilon^{\frac{4}{(n-2p+2)(n+1)}}, & \text{if } n > 2p, \\ C(n,p,R)\epsilon^{\frac{2}{(n+1)}}, & \text{if } n \leq 2p. \end{cases}$$

Here $C(n,p,R)$ is a constant that depends only on n, p, R.

Proof.
Schur's Lemma:

$$I_p(H_m) = \lambda_m(n,p)H_m,$$

where H_m is a spherical harmonic of degree m.

$$|\lambda_m(n,p)| = \frac{2^p \pi^{n/2} \Gamma((m+p)/2)}{\Gamma((m+n-p)/2)}$$

Put $f = h_K - h_L$ and denote its associated series by

$$\sum_{m=0}^{\infty} Q_m.$$

We will estimate the L_2-norm of f instead of the sup-norm.

Vitale's theorem:

$$c_1(n)\delta_2(K, L) \leq \delta_\infty(K, L) \leq c_2(n) D^{2(n-1)/(n+1)} \delta_2(K, L)^{2/(n+1)},$$

where $D = \mathrm{diam}(K \cup L)$ and c_1, c_2 are constants depending on n only.

i) Assume first that $n > 2p$.

$$\delta_2(K,L)^2 = \|f\|_2^2 = \sum_{m=0}^{\infty} \|Q_m\|_2^2$$

$$= \sum_{m=0}^{\infty} \left(|\lambda_m|^{\frac{4}{n-2p+2}} \|Q_m\|_2^{\frac{4}{n-2p+2}} \right) \cdot \left(|\lambda_m|^{-\frac{4}{n-2p+2}} \|Q_m\|_2^{\frac{2n-4p}{n-2p+2}} \right)$$

$$\leq \left(\sum_{m=0}^{\infty} |\lambda_m|^2 \|Q_m\|_2^2 \right)^{\frac{2}{n-2p+2}} \left(\sum_{m=0}^{\infty} |\lambda_m|^{-\frac{4}{n-2p}} \|Q_m\|_2^2 \right)^{\frac{n-2p}{n-2p+2}}$$

Note that $I_p f$ has spherical harmonic expansion

$$\sum_{m=0}^{\infty} \lambda_m Q_m.$$

Parseval's equality:

$$\sum_{m=0}^{\infty} |\lambda_m(n,p)|^2 \|Q_m\|_2^2 = \|I_p f\|_2^2$$

Stirling's formula (as m tends to infinity):

$$|\lambda_m(n,p)|^{-\frac{4}{n-2p}} \approx C(n,p) m^2.$$

Therefore

$$\|f\|_2^2 \leq C(n,p)\left(\|I_p f\|_2^2\right)^{\frac{2}{n-2p+2}} \times$$

$$\times \left(\|Q_0\|_2^2 + \sum_{m=1}^{\infty} m(m+n-2)\|Q_m\|_2^2\right)^{\frac{n-2p}{n-2p+2}}$$

$$\leq C(n,p)\epsilon^{\frac{4}{n-2p+2}}\left(\epsilon^2 + \|\nabla_o h_K - \nabla_o h_L\|_2^2\right)^{\frac{n-2p}{n-2p+2}}$$

$$\leq C(n,p)\epsilon^{\frac{4}{n-2p+2}}\left(\epsilon^2 + R^2\right)^{\frac{n-2p}{n-2p+2}}$$

ii) If $n \leq 2p$, then $|\lambda_m(n,p)|$ does not approach zero as m tends to infinity.

Therefore there exists $C(n,p)$ such that

$$C(n,p)|\lambda_m(n,p)|^2 \geq 1$$

for all m.

$$\|f\|_2^2 = \sum_{m=0}^{\infty} \|Q_m\|_2^2 \leq C(n,p) \sum_{m=0}^{\infty} |\lambda_m(n,p)|^2 \|Q_m\|_2^2 =$$

$$= C(n,p)\|I_p f\|_2^2 \leq C(n,p)\epsilon^2$$

Q.E.D.

Corollary. Any convex body K is uniquely determined by the average height and average width of almost all its shadow boundaries.

Thank you!!!