The geometry of p-convex intersection bodies

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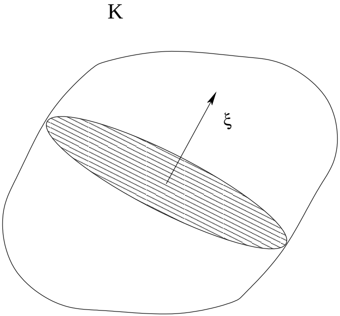
Authors

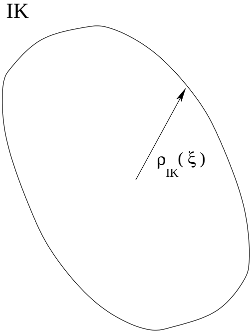
This is joint work with Jaegil Kim and Artëm Zvavitch.

Background

Lutwak (1988) introduced the notion of the intersection body IK of a star body K. IK is defined by its radial function

$$\rho_{IK}(\xi) = |K \cap \xi^{\perp}|, \qquad \text{for } \xi \in S^{n-1}.$$





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What if K is not convex?

Let $p \in (0,1]$. A body K is said to be *p-convex* if, for all $x, y \in \mathbb{R}^n$,

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Note also that a p_1-convex body is p_2-convex for all $0 < p_2 \leqslant p_1$.

Results

Theorem 1

Let K be an origin-symmetric p-convex body in $\mathbb{R}^n$, $p \in (0,1]$, and E a $(k-1)$-dimensional subspace of $\mathbb{R}^n$ for $1 \leqslant k \leqslant n$. Then the map

$$u \longmapsto \frac{|u|}{\left|K \cap \mathrm{span}(u, E)\right|_k}, \quad u \in E^{\perp}$$

defines the Minkowski functional of a q-convex body in $E^{\perp}$ with $q = [(1/p - 1)\, k + 1]^{-1}$.

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Proof is similar to the one in [Milman, Pajor].

Results

As a corollary of the previous theorem we get the following

Theorem 2

Let K be an origin-symmetric p-convex body in $\mathbb{R}^n$ for $p \in (0,1]$. Then the intersection body IK of K is q-convex for $q = [(1/p-1)(n-1)+1]^{-1}$.

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Note that this theorem does not hold without the symmetry assumption.

What about sharpness of Theorem 2?

Results

Theorem 3

There exists a p-convex body $K \subset \mathbb{R}^n$ such that IK is q-convex with

$$q \leqslant [(1/p - 1)(n-1) + 1 + g_n(p)]^{-1},$$

where $g_n(p)$ is a function that satisfies
1) $g_n(p) \geqslant -\log_2(n-1)$,
2) $\lim_{p \to 1^-} g_n(p) = 0$.

Proof.

Let

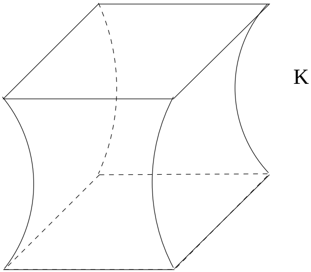
$$C_1 = \{|x_1| \leqslant 1, ..., |x_{n-1}| \leqslant 1, x_n = 1\}$$

and

$$C_{-1} = \{|x_1| \leqslant 1, ..., |x_{n-1}| \leqslant 1, x_n = -1\}.$$

For a fixed $0 < p < 1$ define $K \subset \mathbb{R}^n$ as follows:

$$K = \{z \in \mathbb{R}^n : z = t^{1/p}x + (1-t)^{1/p}y, x \in C_1, y \in C_{-1}, 0 \leqslant t \leqslant 1\}.$$



Proof

Let $L = IK$ be the intersection body of K.
In order to estimate q, we will use the inequality

$$\left\|\sqrt{2}e_n\right\|_L^q \leqslant \left\|\frac{e_n + e_1}{\sqrt{2}}\right\|_L^q + \left\|\frac{e_n - e_1}{\sqrt{2}}\right\|_L^q = 2\left\|\frac{e_n + e_1}{\sqrt{2}}\right\|_L^q,$$

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Computing $\rho_L(e_n)$ and $\rho_L((e_1 + e_n)/\sqrt{2})$ we get

$$q \leqslant \left[\left(\frac{1}{p} - 1\right)(n-1) + 1 + \log_2 \frac{(n-2)\left(\frac{2}{2^{1/p}}\right)^{n-1} + 1}{n-1}\right]^{-1}.$$

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Recall that the Banach-Mazur distance between two origin-symmetric star bodies is defined by

$$d_{BM}(K, L) = \inf\{b/a : \exists T \in GL(n) : aK \subset TL \subset bK\}.$$

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Question. *Does IK have to be closer to the ball than K?*

Our next theorem shows that the answer is "No". There are p-convex bodies for which the intersection body is farther from the Euclidean ball.

Results

Theorem 4

Let $p \in (0,1)$ and let c be any constant satisfying $1 < c < 2^{1/p-1}$. Then for all large enough n, there exists a p-convex body $K \subset \mathbb{R}^n$ such that

$$c^n d_{BM}(K, B_2^n) < d_{BM}(IK, B_2^n).$$

Proof

Consider K from the previous theorem. One can see that

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$$K \supset 2^{\frac{p-1}{p}} B_\infty^n \supset 2^{\frac{p-1}{p}} B_2^n,$$

and thus

$$d_{BM}(K, B_2^n) \leqslant 2^{\frac{1-p}{p}} \sqrt{n}.$$

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Therefore, $d \geqslant r$, where $r = \min\{t : conv(IK) \subset tIK\}$.
Thus,

$$d_{BM}(IK, B_2^n) \geqslant r = \max\left\{\frac{\rho_{conv(IK)}(\theta)}{\rho_{IK}(\theta)}, \theta \in S^{n-1}\right\} \geqslant \frac{\rho_{conv(IK)}(e_n)}{\rho_{IK}(e_n)}.$$

Proof

The convexity of $conv(IK)$ gives

$$\rho_{conv(IK)}(e_n)$$
$$\geqslant \left\| \frac{1}{2}\left(\rho_{IK}\left(\frac{e_n+e_1}{\sqrt{2}}\right)\frac{e_n+e_1}{\sqrt{2}} + \rho_{IK}\left(\frac{e_n-e_1}{\sqrt{2}}\right)\frac{e_n-e_1}{\sqrt{2}}\right)\right\|_2$$
$$= \frac{1}{\sqrt{2}}\rho_{IK}\left(\frac{e_n+e_1}{\sqrt{2}}\right).$$

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Using estimates from the previous theorem, we get

$$d_{BM}(IK, B_2^n) \geqslant \frac{\rho_{IK}\left(\frac{e_n+e_1}{\sqrt{2}}\right)}{\sqrt{2}\rho_{IK}(e_n)} \geqslant \left(\frac{2^{1/p}}{2}\right)^{n-1}\frac{1}{n-1}.$$

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Now compare this with

$$d_{BM}(K, B_2^n) \leqslant 2^{\frac{1-p}{p}}\sqrt{n}.$$

Generalization to log-concave measures

A measure μ on $\mathbb{R}^n$ is called log-concave if for any measurable $A, B \subset \mathbb{R}^n$ and $0 < \lambda < 1$, we have

$$\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^{\lambda}\mu(B)^{(1-\lambda)}$$

whenever $\lambda A + (1-\lambda)B$ is measurable.

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Theorem (Ball)

Let $f : \mathbb{R}^n \to [0, \infty)$ be an even log-concave function satisfying $0 < \int_{\mathbb{R}^n} f < \infty$ and let $k \geqslant 1$. Then the map

$$x \longmapsto \left[\int_0^{\infty} f(rx) r^{k-1} dr\right]^{-\frac{1}{k}}$$

defines a norm on $\mathbb{R}^n$.

Generalization to log-concave measures

Let μ be a measure on $\mathbb{R}^n$, absolutely continuous with respect to the Lebesgue measure m, and let its density function f be locally integrable on $n-1$-dimensional subspaces of $\mathbb{R}^n$.

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Define the intersection body $I_\mu K$ of a star body K with respect to μ by

$$\rho_{I_\mu K}(u) = \mu_{n-1}(K \cap u^\perp), \quad u \in S^{n-1}.$$

Generalization to log-concave measures

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If we apply Ball's Theorem to the log-concave function $1_K f$, we get a symmetric convex body L whose Minkowski functional is given by

$$\|x\|_L = \left[(n-1)\int_0^\infty (1_K f)(rx) r^{n-2} dr\right]^{-\frac{1}{n-1}}.$$

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Then for every $u \in S^{n-1}$,

$$\mu_{n-1}(K \cap u^\perp) = \int_{S^{n-1}\cap u^\perp} \int_0^\infty (1_K f)(r\theta) r^{n-2} dr d\theta$$

$$= \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \|\theta\|_L^{-n+1} d\theta = |L \cap u^\perp|.$$

Using Busemann's Theorem for the convex body L, one obtains the following version of Busemann's theorem for log-concave measures.

Theorem

Let μ be a symmetric log-concave measure on $\mathbb{R}^n$ and K a symmetric convex body in $\mathbb{R}^n$. Then the intersection body $I_\mu K$ is convex.

Using Busemann's Theorem for the convex body L, one obtains the following version of Busemann's theorem for log-concave measures.

Theorem

Let μ be a symmetric log-concave measure on $\mathbb{R}^n$ and K a symmetric convex body in $\mathbb{R}^n$. Then the intersection body $I_\mu K$ is convex.

In order to generalize this Theorem to p-convex bodies, we will first prove a version of Ball's theorem for p-convex bodies.

Generalization to log-concave measures

Theorem 5

Let $f : \mathbb{R}^n \to [0, \infty)$ be an even log-concave function, $k \geqslant 1$, and K a p-convex body in $\mathbb{R}^n$ for $0 < p \leqslant 1$. Then the body L defined by the Minkowski functional

$$\|x\|_L = \left[\int_0^{\|x\|_K^{-1}} f(rx) r^{k-1} dr \right]^{-\frac{1}{k}}, \quad x \in \mathbb{R}^n,$$

is p-convex.

Proof

Fix two non-parallel vectors $x_1, x_2 \in \mathbb{R}^n$ and denote $x_3 = x_1 + x_2$. We claim that $\|x_3\|_L^p \leqslant \|x_1\|_L^p + \|x_2\|_L^p$.

Proof

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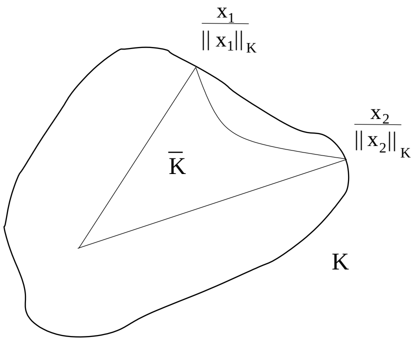
Consider the following 2-dimensional bodies in the plane $E = \mathrm{span}\{x_1, x_2\}$,

$$\bar{K} = \left\{ \frac{t_1 x_1}{\|x_1\|_K} + \frac{t_2 x_2}{\|x_2\|_K} : t_1, t_2 \geqslant 0, t_1^p + t_2^p \leqslant 1 \right\}$$

and

$$\bar{L} = \left\{ x \in \mathbb{R}^n : \|x\|_{\bar{L}} = \left[\int_0^{\|x\|_{\bar{K}}^{-1}} f(rx) r^{k-1} dr \right]^{-\frac{1}{k}} \leqslant 1 \right\}.$$

Proof



Clearly $\bar{K}$ is p-convex and $\bar{K} \subset K$. Also note that $\|x_i\|_{\bar{K}} = \|x_i\|_K$ for $i = 1, 2$, and $\|x_3\|_{\bar{K}} \geqslant \|x_3\|_K$.

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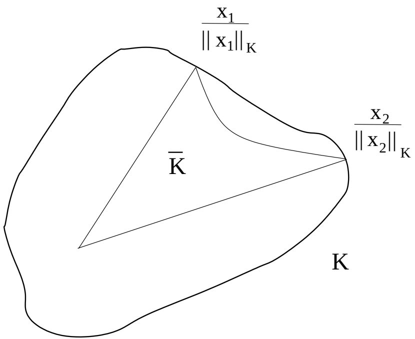

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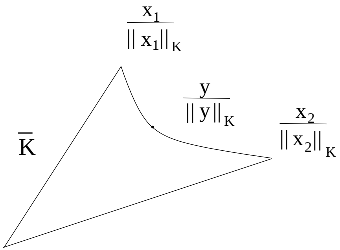
Therefore, $\|x_i\|_{\bar{L}} = \|x_i\|_L$ $(i = 1, 2)$, and $\|x_3\|_{\bar{L}} \geqslant \|x_3\|_L$.

Proof

Consider the point $y = \frac{\|x_1\|_{\bar{L}}}{\|x_1\|_{\bar{K}}} x_1 + \frac{\|x_2\|_{\bar{L}}}{\|x_2\|_{\bar{K}}} x_2$ in the plane E.
The point $\frac{y}{\|y\|_{\bar{K}}}$ lies on the p-arc connecting $\frac{x_1}{\|x_1\|_{\bar{K}}}$ and $\frac{x_2}{\|x_2\|_{\bar{K}}}$.



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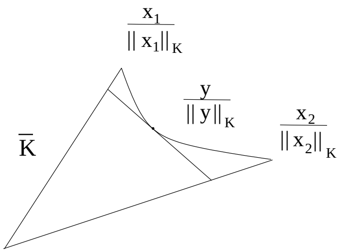

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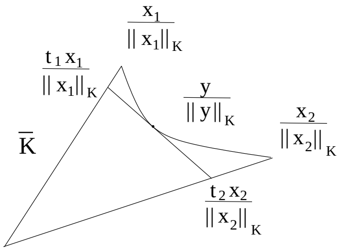

Consider the tangent line to this arc at the point $\frac{y}{\|y\|_{\bar{K}}}$. This line intersects the segments $[0, x_i/\|x_i\|_{\bar{K}}]$, $i = 1, 2$, at some points $\frac{t_i x_i}{\|x_i\|_{\bar{K}}}$ with $t_i \in (0, 1)$.

Proof

Since $\frac{t_1 x_1}{\|x_1\|_{\bar{K}}}$, $\frac{t_2 x_2}{\|x_2\|_{\bar{K}}}$ and $\frac{y}{\|y\|_{\bar{K}}}$ are on the same line, it follows that the coefficients of $\frac{t_1 x_1}{\|x_1\|_{\bar{K}}}$ and $\frac{t_2 x_2}{\|x_2\|_{\bar{K}}}$ in the equality

$$\frac{y}{\|y\|_{\bar{K}}} = \frac{1}{\|y\|_{\bar{K}}} \left(\frac{\|x_1\|_{\bar{L}}}{t_1} \cdot \frac{t_1 x_1}{\|x_1\|_{\bar{K}}} + \frac{\|x_2\|_{\bar{L}}}{t_2} \cdot \frac{t_2 x_2}{\|x_2\|_{\bar{K}}} \right)$$

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have to add up to 1.
Therefore,

$$\|y\|_{\bar{K}} = \frac{\|x_1\|_{\bar{L}}}{t_1} + \frac{\|x_2\|_{\bar{L}}}{t_2}.$$

Proof

Note that the line between $\frac{t_1 x_1}{\|x_1\|_{\bar{K}}}$ and $\frac{t_2 x_2}{\|x_2\|_{\bar{K}}}$ separates $\frac{x_3}{\|x_3\|_{\bar{K}}}$ from the origin, i.e. the three points $\frac{t_1 x_1}{\|x_1\|_{\bar{K}}}$, $\frac{t_2 x_2}{\|x_2\|_{\bar{K}}}$ and $\frac{x_3}{\|x_3\|_{\bar{K}}}$ are in the "convex position".

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Applying Ball's theorem to these three points,

$$\left[\int_0^{\frac{1}{\|x_3\|_{\bar{K}}}} f(rx_3) r^{k-1} dr\right]^{-\frac{1}{k}}$$

$$\leqslant \left[\int_0^{\frac{t_1}{\|x_1\|_{\bar{K}}}} f(rx_1) r^{k-1} dr\right]^{-\frac{1}{k}} + \left[\int_0^{\frac{t_2}{\|x_2\|_{\bar{K}}}} f(rx_2) r^{k-1} dr\right]^{-\frac{1}{k}}.$$

Proof

Let $s_i = \|x_i\|_{\bar{L}} \left[\int_0^{\frac{t_i}{\|x_i\|_{\bar{K}}}} f(rx_i) r^{k-1} dr \right]^{\frac{1}{k}}$ for $i = 1, 2$.

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By a change of variables, we get

$$s_i = t_i \|x_i\|_{\bar{L}} \left[\int_0^{\frac{1}{\|x_i\|_{\bar{K}}}} f(t_i r x_i) r^{k-1} dr \right]^{\frac{1}{k}}$$

$$\geqslant t_i \|x_i\|_{\bar{L}} \left[\int_0^{\frac{1}{\|x_i\|_{\bar{K}}}} f(r x_i) r^{k-1} dr \right]^{\frac{1}{k}} = t_i$$

for each $i = 1, 2$.

Proof

Putting all together, we have

$$\|x_3\|_L \leqslant \|x_3\|_{\bar{L}} \leqslant \frac{\|x_1\|_{\bar{L}}}{s_1} + \frac{\|x_2\|_{\bar{L}}}{s_2} \leqslant \frac{\|x_1\|_{\bar{L}}}{t_1} + \frac{\|x_2\|_{\bar{L}}}{t_2} = \|y\|_{\bar{K}}.$$

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Using the p-convexity of $\bar{K}$, we have

$$\|y\|_{\bar{K}}^p \leqslant \left\| \frac{\|x_1\|_{\bar{L}}}{\|x_1\|_{\bar{K}}} x_1 \right\|_{\bar{K}}^p + \left\| \frac{\|x_2\|_{\bar{L}}}{\|x_2\|_{\bar{K}}} x_2 \right\|_{\bar{K}}^p = \|x_1\|_{\bar{L}}^p + \|x_2\|_{\bar{L}}^p = \|x_1\|_L^p + \|x_2\|_L^p,$$

and therefore $\|x_3\|_L^p \leqslant \|x_1\|_L^p + \|x_2\|_L^p$. □

Corollary

Let μ be a symmetric log-concave measure and K a symmetric p-convex body in $\mathbb{R}^n$ for $p \in (0, 1]$. Then the intersection body $I_\mu K$ of K is q-convex with $q = \Big[(1/p - 1)(n-1) + 1\Big]^{-1}$.

Proof

Let f be the density function of μ. By Theorem 5, the body L with the Minkowski functional

$$\|x\|_L = \left[(n-1)\int_0^{\|x\|_K^{-1}} f(rx)r^{n-2}dr\right]^{\frac{-1}{n-1}}, \quad x \in \mathbb{R}^n,$$

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On the other hand,

$$\begin{aligned}
\rho_{I_\mu K}(u) &= \mu_{n-1}(K \cap u^\perp) \\
&= \int_{\mathbb{R}^n} 1_{K\cap u^\perp}(x) f(x)dx = \int_{S^{n-1}\cap u^\perp} \int_0^{\|u\|_K^{-1}} f(rv)r^{n-2}drdv \\
&= \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \|v\|_L^{-n+1}dv = |L \cap u^\perp|_{n-1} \\
&= \rho_{IL}(u),
\end{aligned}$$

which means $I_\mu K = IL$.

Proof

IL is q-convex with $q = \left[(1/p-1)(n-1)+1\right]^{-1}$, and therefore so is $I_\mu K$.

□

Generalization to log-concave measures

Remark

We have an example that shows that the condition on f to be even in Theorem 5 cannot be dropped.

Non-symmetric cases and s-concave measures

However, it is possible to give a version of Theorem 5 for non-symmetric s-concave measures.

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Borell introduced the classes $\mathfrak{M}_s(\Omega)$, $(-\infty \leqslant s \leqslant \infty$, $\Omega \subset \mathbb{R}^n$ open convex) of s-concave measures, satisfying

$$\mu(\lambda A + (1-\lambda)B) \geqslant \left[\lambda\mu(A)^s + (1-\lambda)\mu(B)^s\right]^{\frac{1}{s}}$$

holds for all nonempty compact $A, B \subset \Omega$ and all $\lambda \in (0,1)$.

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holds for all nonempty compact $A, B \subset \Omega$ and all $\lambda \in (0,1)$. In particular, $s = 0$ gives the class of log-concave measures.

Non-symmetric cases and s-concave measures

Let us consider the case $0 < s < 1/n$. According to Borell, μ is s-concave if and only if the support of μ is n-dimensional and $d\mu = fdm$ for some $f \in L^1_{loc}(\Omega)$ such that $f^{\frac{s}{1-ns}}$ is a concave function on Ω.

Non-symmetric cases and s-concave measures

Theorem 6

Let μ be an s-concave measure on $\Omega \subset \mathbb{R}^n$ with density f, for $0 < s < 1/n$, and K a p-convex body in Ω, for $p \in (0,1]$. If $k \geqslant 1$, then the body L whose Minkowski functional is given by

$$\|x\|_L = \Big[\int_0^\infty 1_K(rx) f(rx) r^{k-1} dr \Big]^{-\frac{1}{k}}, \quad x \in \mathbb{R}^n$$

is q-convex with $q = \Big[\big(\frac{1}{p} - 1\big)\big(\frac{1}{s} - n\big)\frac{1}{k} + \frac{1}{p} \Big]^{-1}$.

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Remark

We have an example showing that the above theorem is sharp.

THANK YOU!!!