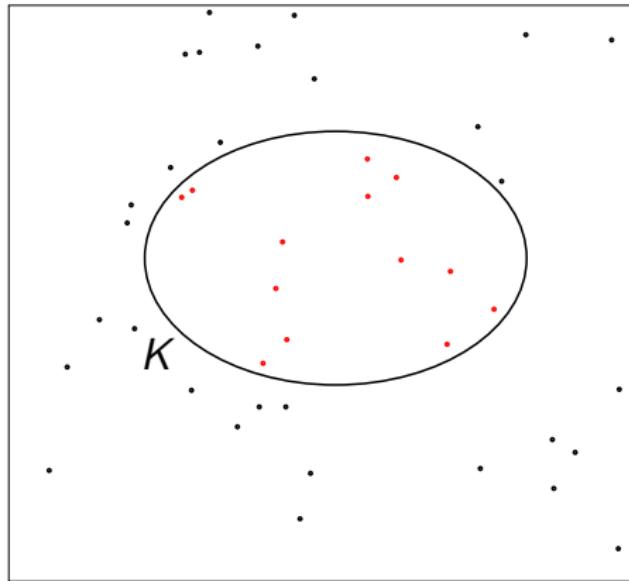


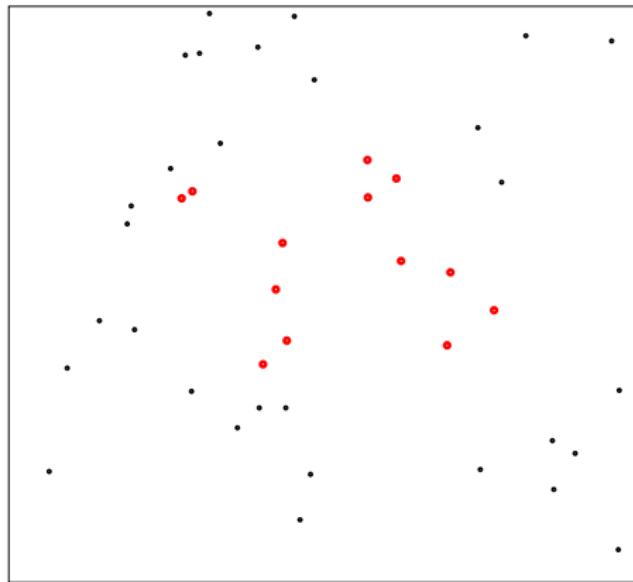
Poisson-Voronoi approximation

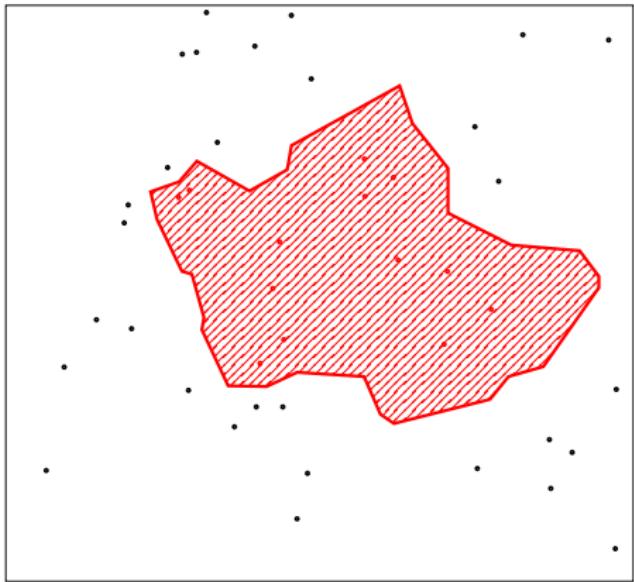
joint work with Matthias Heveling and Matthias Schulte

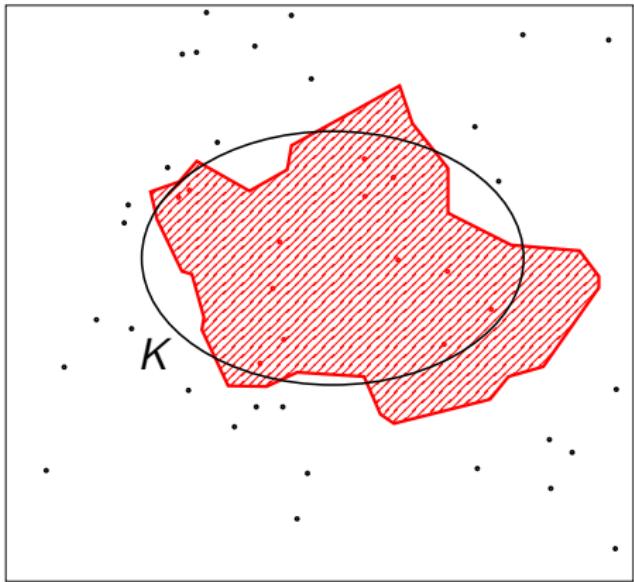
Matthias Reitzner











Poisson-Voronoi approximation

X Poisson point process in \mathbb{R}^d with intensity λ

$v_X(z)$ Voronoi cells, $z \in X$:

$$v_X(z) = \{t \in R^d \mid \|t - z\| = \min_{y \in X} \|t - y\|\}$$

Poisson-Voronoi approximation

X Poisson point process in \mathbb{R}^d with intensity λ

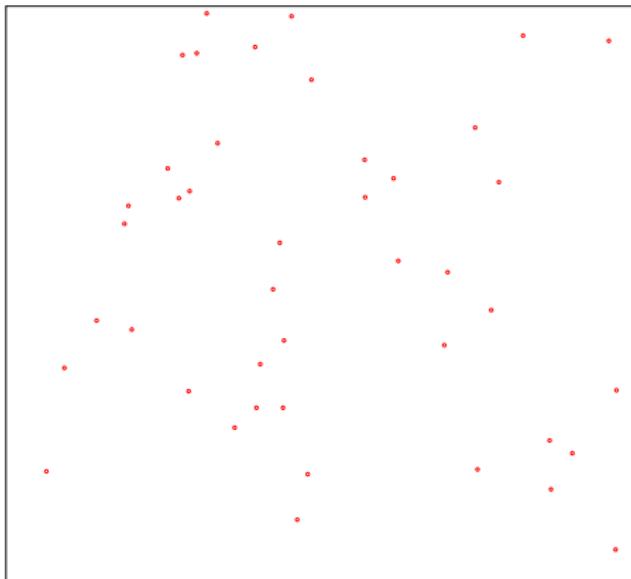
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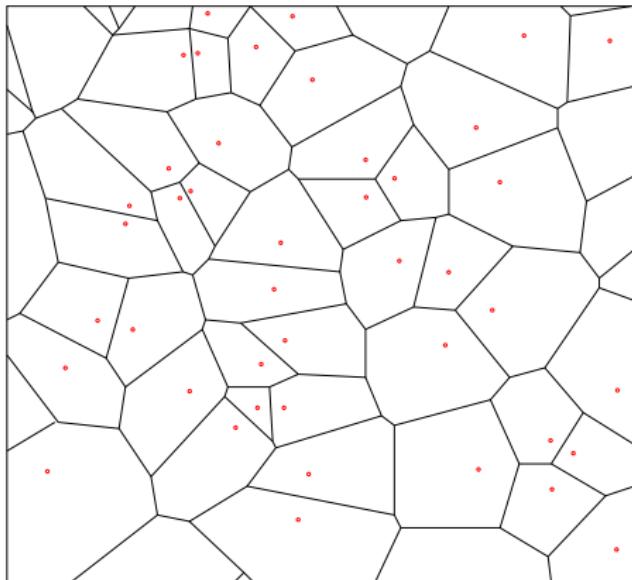
Poisson-Voronoi approximation of a set K :

$$PV_\lambda := \bigcup_{z \in X \cap K} v_X(z).$$

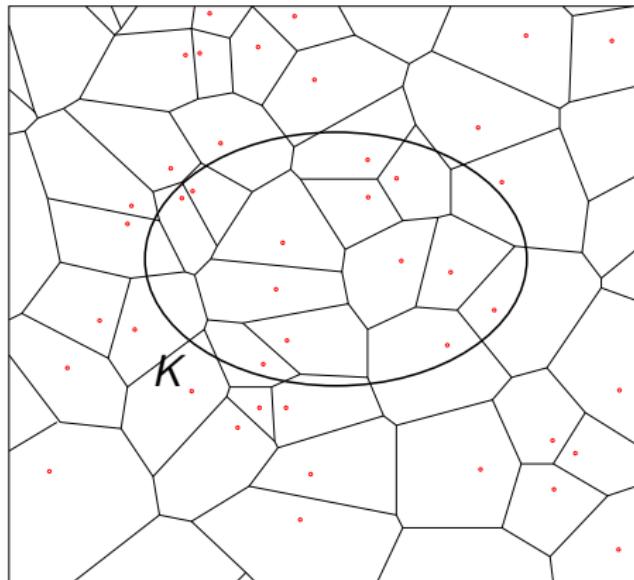
Poisson-Voronoi approximation



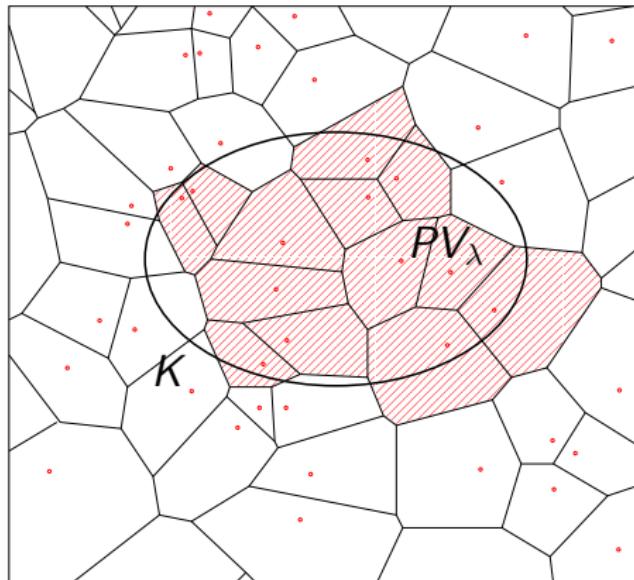
Poisson-Voronoi approximation



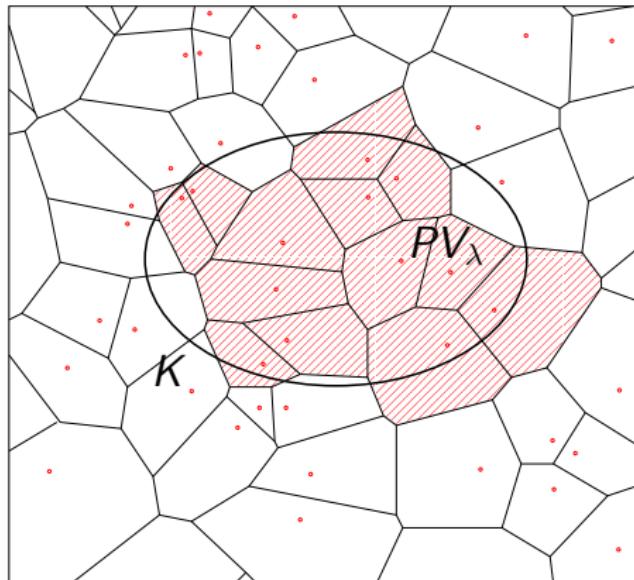
Poisson-Voronoi approximation



Poisson-Voronoi approximation

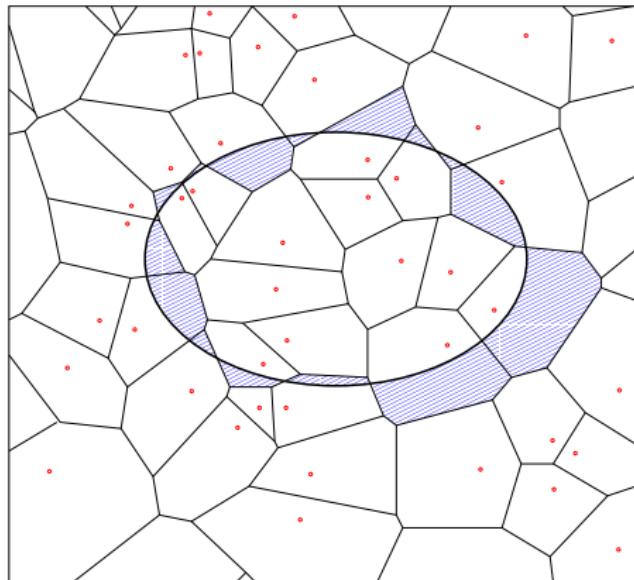


Poisson-Voronoi approximation



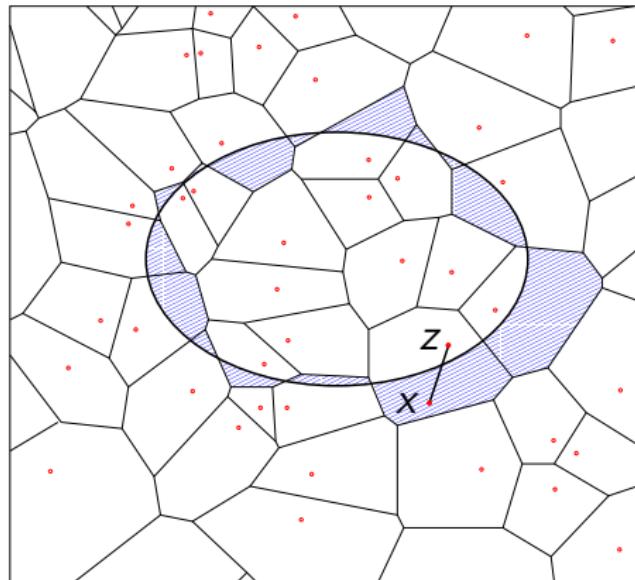
$$\mathbb{E} V(PV_\lambda) = V(K)$$

Poisson-Voronoi approximation: mean values



$$PV_{\lambda} \Delta K$$

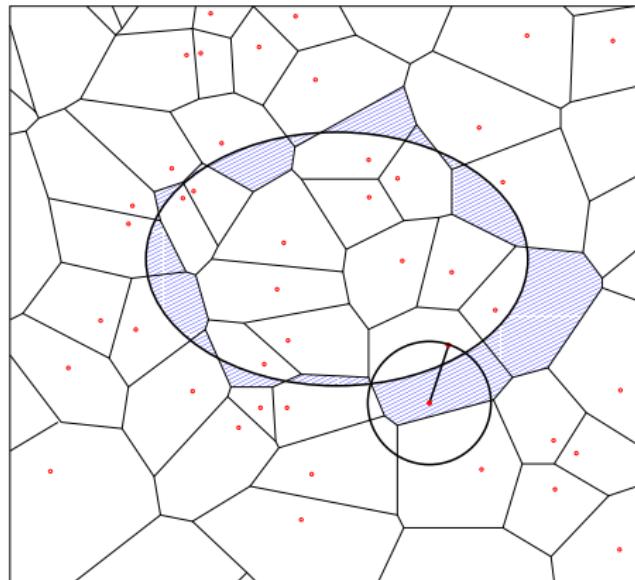
Poisson-Voronoi approximation: mean values



$$\mathbb{E} \mathbb{I}(x \in v_X(K)) =$$

$$\mathbb{E} \sum_{z \in X \cap K} \mathbb{I}(x \in v_X(z))$$

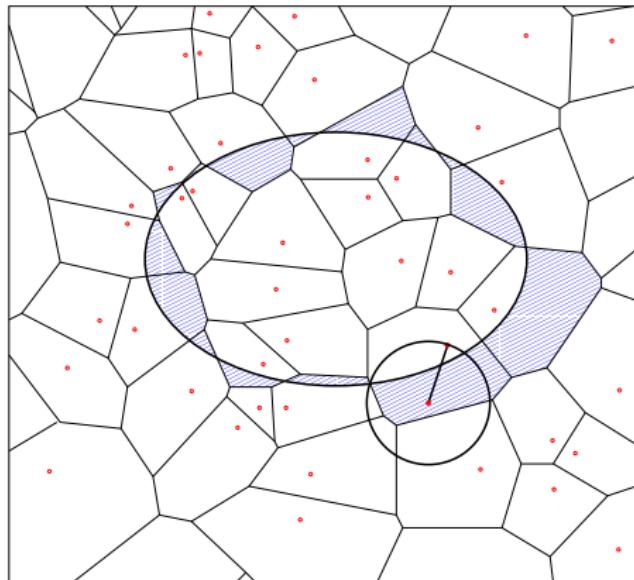
Poisson-Voronoi approximation: mean values



$$\mathbb{E} \mathbb{I}(x \in v_X(K)) =$$

$$\lambda \int_K \mathbb{P}(x \in v_X(z)) dz$$

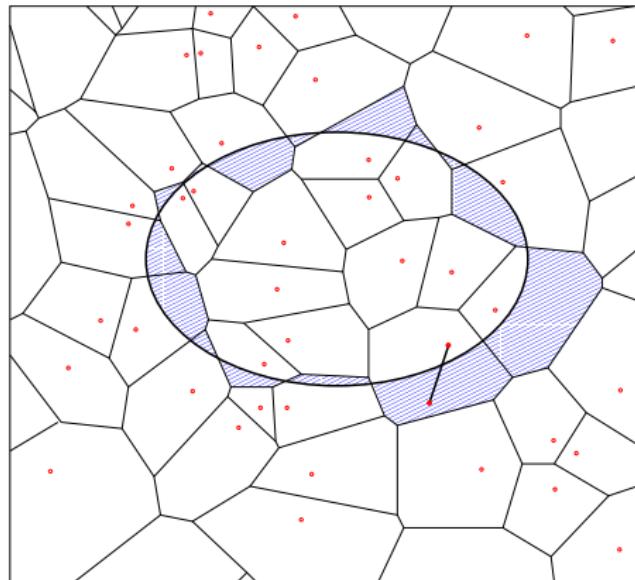
Poisson-Voronoi approximation: mean values



$$\mathbb{E} \mathbb{I}(x \in v_X(K)) =$$

$$\lambda \int_K e^{-\lambda \kappa_d \|z-x\|^d} dz$$

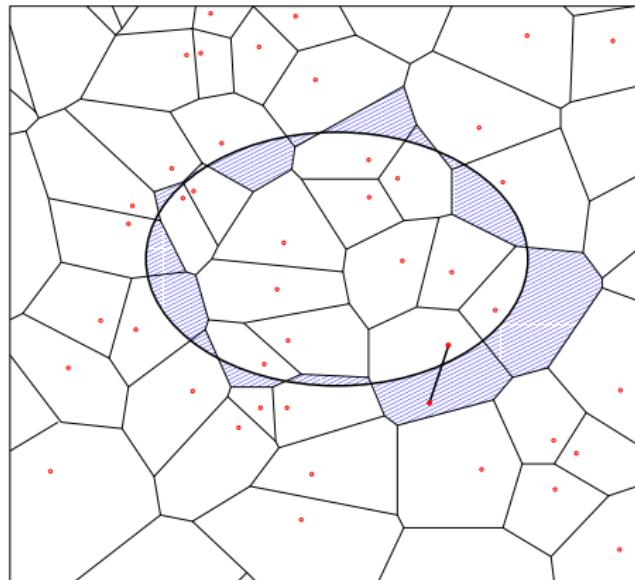
Poisson-Voronoi approximation: mean values



$$\mathbb{E} V(x \in v_X(K)) =$$

$$\lambda \int_{\mathbb{R}^d \setminus K} \int_K e^{-\lambda \kappa_d \|z-x\|^d} dz dx$$

Poisson-Voronoi approximation: mean values



$$\mathbb{E} V(PV_\lambda \Delta K) =$$

$$2\lambda \int_{\mathbb{R}^d \setminus K} \int_K e^{-\lambda \kappa_d \|z-x\|^d} dz dx$$

Poisson-Voronoi approximation: covariogram

$$\mathbb{E}V(PV_\lambda \triangle K) = 2\lambda \int_{\mathbb{R}^d \setminus K} \int_K e^{-\lambda \kappa_d \|z-x\|^d} dz dx$$

Poisson-Voronoi approximation: covariogram

$$\mathbb{E} V(PV_\lambda \triangle K) = 2\lambda \int_{\mathbb{R}^d} e^{-\lambda \kappa_d \|t\|^d} \int_{\mathbb{R}^d} \mathbb{I}(x \in (K + t) \cap K^c) dx dt$$

Poisson-Voronoi approximation: covariogram

$$\mathbb{E}V(PV_\lambda \triangle K) = 2\lambda \int_{\mathbb{R}^d} e^{-\lambda \kappa_d \|t\|^d} V_d((K + t) \cap K^c) dt$$

Poisson-Voronoi approximation: covariogram

$$\mathbb{E}V(PV_\lambda \triangle K) = 2\lambda \int_{\mathbb{R}^d} e^{-\lambda \kappa_d \|t\|^d} \underbrace{V_d((K+t) \cap K^c)}_{V_d(K) - V_d((K+t) \cap K))} dt$$

Poisson-Voronoi approximation: covariogram

$$\mathbb{E} V(PV_\lambda \triangle K) = 2\lambda \int_{\mathbb{R}^d} e^{-\lambda \kappa_d \|t\|^d} (g_K(0) - g_K(t)) dt$$

Poisson-Voronoi approximation: covariogram

$$\mathbb{E} V(PV_\lambda \triangle K) = 2\lambda \int_{\mathbb{R}^d} e^{-\lambda \kappa_d \|t\|^d} (g_K(0) - g_K(t)) dt$$

$K \in \mathcal{K}^d$:

$$\mathbb{E} V(PV_\lambda \triangle K) = c_d \lambda^{-\frac{1}{d}} S(K) (1 + O(\lambda^{-\frac{1}{d}}))$$

Poisson-Voronoi approximation: mean values

$K \in \mathcal{K}^d$:

$$\mathbb{E}V(PV_\lambda) = V(K)$$

$$\mathbb{E}V(PV_\lambda \triangle K) = c_d \lambda^{-\frac{1}{d}} S(K) (1 + O(\lambda^{-\frac{1}{d}}))$$

..... higher moments, ... ?

Wiener-Ito chaos expansion

$f = f(X)$, X Poisson point process in \mathbb{R}^d with intensity λ

- $f(X) - \mathbb{E}f(X) = \sum_{n=1}^{\infty} I_n(f_n) \dots$ Wiener-Ito chaos expansion

Wiener-Ito chaos expansion

$f = f(X)$, X Poisson point process in \mathbb{R}^d with intensity λ

- $f(X) - \mathbb{E}f(X) = \sum_{n=1}^{\infty} I_n(f_n) \dots$ Wiener-Ito chaos expansion
- $\mathbb{E}(I_n(f_n)) = 0$
- $\mathbb{E}(I_n(f_n)I_j(f_j)) = 0 \text{ for } n \neq j$
- $\mathbb{E}f(X)^2 = (\mathbb{E}f(X))^2 + \sum_1^{\infty} n! \mathbb{E}I_n(f_n)^2$

\implies variance estimates, CLT, deviation inequalities, ... ?

Wiener-Ito chaos expansion

The kernel f_n :

- $D_y f(X) = f(X \cup \{y\}) - f(X)$. . . difference operator D_y

$$f_1 = f_1(y_1) = \mathbb{E} D_{y_1} f(X)$$

Wiener-Ito chaos expansion

The kernel f_n :

- $D_y f(X) = f(X \cup \{y\}) - f(X)$. . . difference operator D_y

$$f_1 = f_1(y_1) = \mathbb{E} D_{y_1} f(X)$$

- $D_{y_1, \dots, y_n}^n f(X) = D_{y_n} D_{y_1, \dots, y_{n-1}}^{n-1} f(X)$

$$f_n = f_n(y_1, \dots, y_n) = \frac{1}{n!} \mathbb{E} D_{y_1, \dots, y_n}^n f(X)$$

Wiener-Ito chaos expansion

The (multiple) Wiener-Ito integral of $f_n(y_1, \dots, y_n)$:

$$I_1(f_1)(X) = \sum_{y_1 \in X} f_1(y_1) - \mathbb{E} \sum_{y_1 \in X} f_1(y_1)$$

$$I_n(f_n)(X) = "I_1 I_{n-1}(f_n(y_1, \dots, y_n))"$$

Wiener-Ito chaos expansion

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$$I_n(f_n)(X) = "I_1 I_{n-1}(f_n(y_1, \dots, y_n))"$$

isometry:

$$\mathbb{E}[I_n(f_n)(X)]^2 = \|f_n\|^2$$

Variance inequalities

$$\mathbb{E}f(X)^2 = (\mathbb{E}f(X))^2 + \sum_1^{\infty} n! \mathbb{E}I_n(f_n)^2$$

Variance inequalities

$$\mathbb{V}f(X) = \sum n! \mathbb{E} I_n(f_n)^2$$

Variance inequalities

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Variance inequalities

$$\mathbb{V}f(X) = \sum n! \mathbb{E} I_n(f_n)^2 = \sum_1^\infty n! \|f_n\|^2$$

Poincaré inequality

$$\mathbb{V}f(X) \leq \lambda \int_{\mathbb{R}^d} \mathbb{E}(D_y f(X))^2 dy$$

Nualart and Vives, Heveling and Reitzner, Last and Penrose, ...

Variance inequalities

$$\mathbb{V}f(X) = \sum n! \mathbb{E} I_n(f_n)^2 = \sum_1^\infty n! \|f_n\|^2$$

jackknife inequality

$$\mathbb{V}f(X) \leq \mathbb{E} \sum_{x \in X} (f(X) - f(X \setminus \{x\}))^2$$

Nualart and Vives, Heveling and Reitzner, Last and Penrose, ...

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Nualart and Vives, Heveling and Reitzner, Last and Penrose, ...

$$\mathbb{V}f(X) \geq \|f_1\|^2$$

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$$\mathbb{V}f(X) = \sum n! \mathbb{E} I_n(f_n)^2 = \sum_1^\infty n! \|f_n\|^2$$

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Nualart and Vives, Heveling and Reitzner, Last and Penrose, ...

exchange inequality

$$\mathbb{V}f(X) \geq \mathbb{E} \sum_{x \in X} (f(X) - f(X \setminus \{x\}))(f(Y) - f(Y \cup \{x\}))$$

Poisson-Voronoi approximation: variance

$K \in \mathcal{K}^d$:

$$\underline{c}_d \lambda^{-1-\frac{1}{d}} S(K) \leq \mathbb{V}V(PV_\lambda) \leq \bar{c}_d \lambda^{-1-\frac{1}{d}} S(K)$$

$$\underline{c}_d \lambda^{-1-\frac{1}{d}} S(K) \leq \mathbb{V}V(PV_\lambda \triangle K) \leq c_d \lambda^{-1-\frac{1}{d}} S(K)$$

Heveling, Reitzner and Schulte

... work in progress