On the volume of random convex sets

Peter Pivovarov (joint work with Grigoris Paouris)

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Outline

I. Expected volume of random sets

(1) Random convex hulls

(2) Random zonotopes

(3) L_p-centroid bodies

II. A generalization

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Main theorem

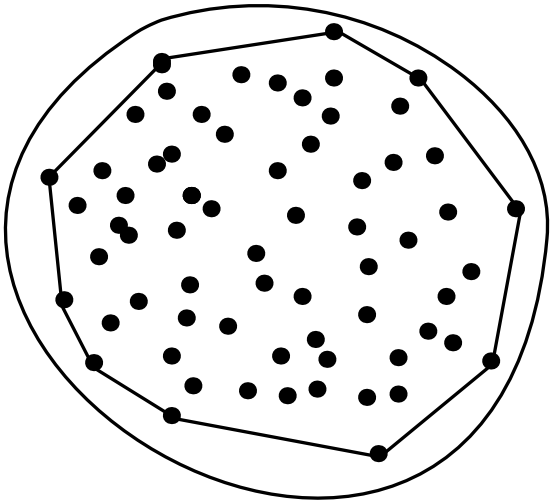
(1) Random convex hulls

Assume

- $K \subset \mathbb{R}^n$ - convex body, $\mathrm{vol}\,(K) = 1$.
- X - random vector distributed uniformly in K, i.e.,

$$\mathbb{P}\,(X \in A) = \mathrm{vol}\,(A \cap K)$$

  for measurable $A \subset \mathbb{R}^n$.
- $X_1, \dots, X_N$ independent copies of X,
- convex hull $= \mathrm{conv}\,\{X_1, \dots, X_N\}$.



Theorem (Groemer, 1974)

Assume

- $K \subset \mathbb{R}^n$ - *convex body,* $\mathrm{vol}\,(K) = 1$.
- $\overline{B_2^n}$ - *Euclidean ball in* $\mathbb{R}^n$ *with* $\mathrm{vol}\left(\overline{B_2^n}\right) = 1$.

Set

$$\mathbb{E}(K, N) := \int_K \ldots \int_K \mathrm{vol}\left(\mathrm{conv}\left\{x_1, \ldots, x_N\right\}\right) dx_1 \ldots dx_N.$$

Then

$$\mathbb{E}(K, N) \geq \mathbb{E}(\overline{B_2^n}, N).$$

Equality holds only for ellipsoids.

- Analogous result holds for $\mathrm{conv}\left\{\pm x_1, \ldots, \pm x_N\right\}$.

Generalizations: [Pfiefer '90], [Giannopoulos-Tsolomitis '03], ...

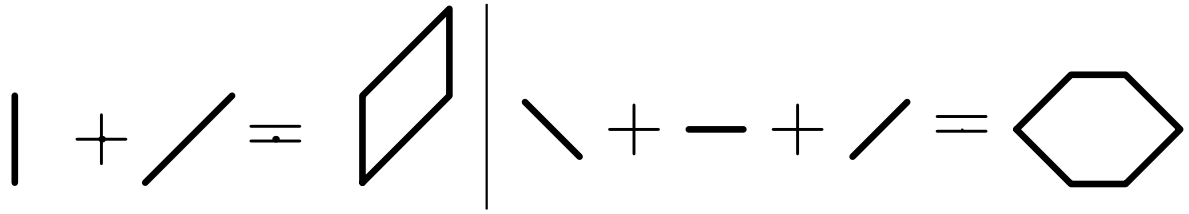
(2) Zonotopes

A *zonotope* is a Minkowski sum of line segments:

$$\sum_{i=1}^{N} [-x_i, x_i],$$

where the $x_i \in \mathbb{R}^n$ and $[-x_i, x_i] := \{\lambda x_i : -1 \leq \lambda \leq 1\}$.

E.g.:



Theorem (Bourgain-Meyer-Milman-Pajor, 1988)

Let $K \subset \mathbb{R}^n$ be a convex body $\mathrm{vol}\,(K) = 1$, $p > 0$. Set

$$\mathcal{I}_p(K, N) := \int_K \cdots \int_K \mathrm{vol}\left(\sum_{i=1}^{N} [-x_i, x_i]\right)^p dx_1 \cdots dx_N.$$

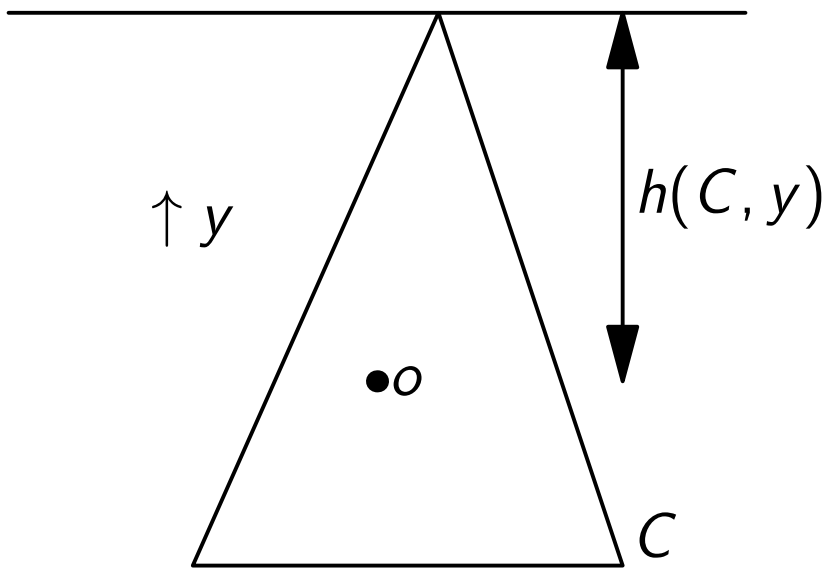
Then

$$\mathcal{I}_p(K, N) \geq \mathcal{I}_p(\overline{B_2^n}, N).$$

Notation:

For a convex body $C \subset \mathbb{R}^n$, let $h(C, \cdot)$ be its support function, i.e.,

$$h(C, y) := \sup_{x \in C} \langle x, y \rangle \quad (y \in S^{n-1}).$$



(3) L_p-centroid bodies

Definition

Let $K \subset \mathbb{R}^n$ be compact with $\mathrm{vol}\,(K) = 1$. Assume K is star-shaped, i.e.,

- $\alpha K \subset K$ for each $\alpha \in [0, 1]$.
- radial function $\rho_K(x) := \sup\{\lambda \geq 0 : \lambda x \in K\}$ is positive and continuous.

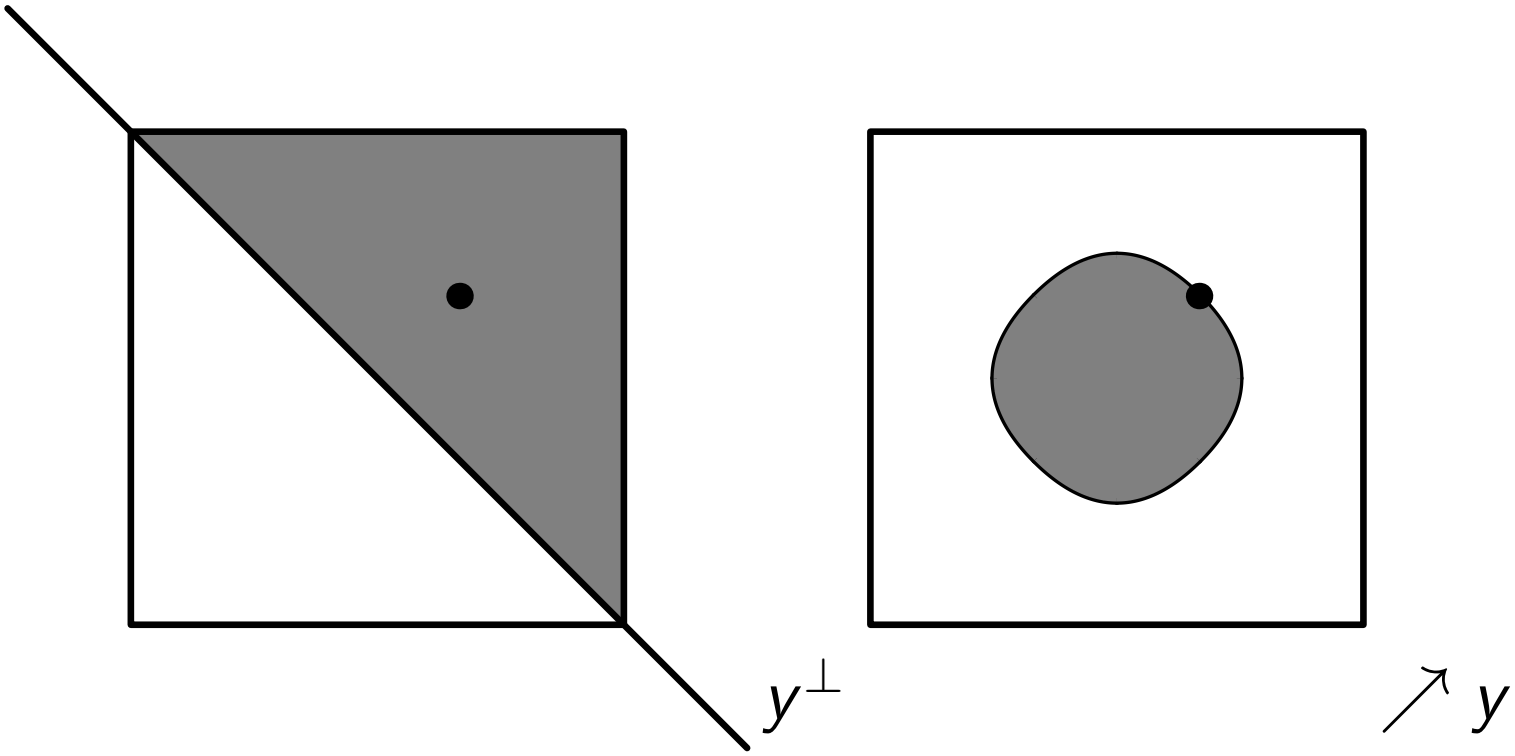
For $p \geq 1$, the L_p-centroid body $Z_p(K)$ is defined by its support function

$$h(Z_p(K), y) := \left(\int_K |\langle x, y \rangle|^p \, dx \right)^{1/p} \quad (y \in S^{n-1}).$$

Introduced by [Lutwak, Yang, & Zhang, 2000] (with a different normalization).

Example: $K = [-\frac{1}{2}, \frac{1}{2}]^2$, $p = 1$.

$$h(Z_1(K), y) := \int_K |\langle x, y\rangle| \, dx \quad (y \in S^{n-1}).$$



Support function of $Z_p(K)$:

$$h(Z_p(K), y) := \left(\int_K |\langle x, y \rangle|^p \, dx \right)^{1/p} \quad (y \in S^{n-1}).$$

Theorem (Lutwak-Yang-Zhang, 2000)

Let $K \subset \mathbb{R}^n$ be compact, star-shaped with $\mathrm{vol}\,(K) = 1$. Then

$$\mathrm{vol}\,(Z_p(K)) \geq \mathrm{vol}\,\left(Z_p(\overline{B_2^n})\right).$$

Equality holds only for ellipsoids.

(1)

$$\mathbb{E}(K, N) \geq \mathbb{E}(\overline{B_2^n}, N).$$

(2)

$$\mathcal{I}_p(K, N) \geq \mathcal{I}_p(\overline{B_2^n}, N).$$

(3)

$$\mathrm{vol}\left(Z_p(K)\right) \geq \mathrm{vol}\left(Z_p(\overline{B_2^n})\right).$$

Linear operator viewpoint

Assume

- $e_1, \dots, e_N$ - standard unit vector basis in $\mathbb{R}^N$.
- $x_1, \dots, x_N \in \mathbb{R}^n$.
- $T_N : \mathbb{R}^N \to \mathbb{R}^n$,

$$e_i \mapsto x_i, \quad i = 1, \dots, N.$$

As a matrix,

$$T_N = [x_1 \cdots x_N].$$

Write

$$B_1^N = \operatorname{conv}\{\pm e_1, \dots, \pm e_N\}, \quad B_\infty^N = [-1, 1]^N.$$

Then

1. $T_N B_1^N = \operatorname{conv}\{\pm x_1, \dots, \pm x_N\}$.
2. $T_N B_\infty^N = \left\{ \sum_i b_i x_i : (b_i) \in B_\infty^N \right\} = \sum_{i=1}^N [-x_i, x_i]$.

Main result

Theorem (P., Paouris, 2010)

Assume

- $K \subset \mathbb{R}^n$ - *bounded, Borel measurable,* $\mathrm{vol}\,(K) = 1$.
- $T_N(x_1, \dots, x_N) : \mathbb{R}^N \to \mathbb{R}^n$, *defined by*

$$T_N e_i = x_i, \quad i = 1, \dots, N.$$

- $C \subset \mathbb{R}^N$ *compact, convex.*

Set

$$\mathbb{E}(K, N, C) := \int_K \dots \int_K \mathrm{vol}\,(T_N(x_1, \dots, x_N)C)\, dx_N \dots dx_1.$$

Then

$$\mathbb{E}(K, N, C) \geq \mathbb{E}(\overline{B_2^n}, N, C).$$

Equality $\Longleftrightarrow$ K *is equal to an ellipsoid a.e.*

Consider independent random vectors

$$X_1, X_2, \ldots \in K \text{ and } Y_1, Y_2, \ldots \in \overline{B_2^n}.$$

For $N = 1, 2, \ldots$, set

$$T_N := [X_1 \cdots X_N] \text{ and } S_N = [Y_1 \cdots Y_N].$$

Corollary

Let $C_N \subset \mathbb{R}^N$, be compact, convex sets for $N = 1, 2, \ldots$ Let $R > 0$. Assume

$$T_N C_N \subset R B_2^n \quad \text{and } S_N C_N \subset R B_2^n$$

and (in the Hausdorff metric)

$$\mathcal{C}(K) := \lim_{N\to\infty} T_N C_N \ (a.s.) \text{ and } \mathcal{C}(\overline{B_2^n}) := \lim_{N\to\infty} S_N C_N \ (a.s.).$$

Then

$$\mathbb{E}\,\mathrm{vol}\,(\mathcal{C}(K)) \geq \mathbb{E}\,\mathrm{vol}\,\left(\mathcal{C}(\overline{B_2^n})\right).$$

Checking convergence ...

Hausdorff metric: for convex bodies $C, D \subset \mathbb{R}^n$,

$$\delta^H(C, D) := \sup_{y \in S^{n-1}} |h(C, y) - h(D, y)| .$$

Recall $T_N = [X_1 \cdots X_N]$. If $C = -C \subset \mathbb{R}^N$ is a convex body, then

$$h(T_N C, y) \quad = \quad \|(\langle X_i, y\rangle)_{i=1}^N\|_{C^\circ} \quad (y \in S^{n-1}).$$

where

$$C^\circ := \{x \in \mathbb{R}^N : \langle x, y\rangle \leq 1 \quad \forall y \in C\}.$$

Thus if $1/p + 1/q = 1$,

$$h(T_N B_q^N, y)^p = \sum_{i=1}^N |\langle X_i, y\rangle|^p .$$

By SLLN,

$$Z_p(K) = \lim_{N \to \infty} N^{-1/p} T_N B_q^N \text{ (a.s.)}$$

Orlicz centroid bodies

Theorem (Lutwak-Yang-Zhang, 2010)

Assume

- $K \subset \mathbb{R}^n$ - *convex body with* $\mathrm{vol}\,(K) = 1$.
- $\psi : [0, \infty) \to [0, \infty)$ - *convex,* $\nearrow$, $\psi(0) = 0$.

Define $Z_\psi(K)$ *by its support function*

$$h(Z_\psi(K), y) := \inf \left\{ \lambda > 0 : \int_K \psi\left(\frac{|\langle x, y \rangle|}{\lambda} \right) dx \leq 1 \right\}.$$

Then

$$\mathrm{vol}\,(Z_\psi(K)) \geq \mathrm{vol}\left(Z_\psi(\overline{B_2^n})\right).$$

Equality holds only for ellipsoids.

Remark: For $\psi(s) := s^p$, $Z_\psi(K) = Z_p(K)$.

Orlicz-centroid bodies in our framework

As before,

- $X_1, \dots, X_N$ independent random vectors in K.
- $T_N = [X_1 \cdots X_N]$.

In the Hausdorff metric

$$Z_\psi(K) = \lim_{N\to\infty} T_N B^\circ_{\psi/N} \quad \text{(a.s.)},$$

where

$$B_{\psi/N} := \{t \in \mathbb{R}^N : \|t\|_{\psi/N} \le 1\},$$

and

$$\|t\|_{\psi/N} := \inf\left\{\lambda > 0 : \frac{1}{N}\sum_{i=1}^N \psi\left(\frac{|t_i|}{\lambda}\right) \le 1\right\}.$$

Elements of the proof

Let $\mu_1, \dots, \mu_N$ be positive Borel measures on $\mathbb{R}^n$, absolutely continuous w.r.t. Lebesgue measure, say,

$$d\mu_i(x) = f_i(x)dx, \qquad i = 1, \dots, N.$$

Set

$$\mathcal{F}(f_1, \dots, f_N) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mathrm{vol}\,(T_N(x_1, \dots, x_N)C) \prod_{i=1}^{N} f_i(x_i)dx_1 \dots dx_N$$

Then

$$\mathcal{F}(f_1, \dots, f_N) \geq \mathcal{F}(f_1^*, \dots, f_N^*).$$

Here f_i^* is the symmetric decreasing rearrangement of f_i.

Rearrangements:

If $A \subset \mathbb{R}^n$ is measurable with $\mathrm{vol}\,(A) < \infty$, then A^* is the (open) ball with the same volume as A:

$$A^* = rB_2^n \text{ with } \mathrm{vol}\,(A) = \mathrm{vol}\,(rB_2^n)\,.$$

If $f : \mathbb{R}^n \to [0, \infty)$ is integrable its *symmetric decreasing rearrangement* is

$$f^*(x) = \int_0^\infty \chi_{\{f(x)>s\}^*}\, ds$$

Properties of f^*:

- $f^*(x) = f^*(y)$ if $|x| = |y|$.
- $f^*(x) \geq f^*(y)$ if $|x| \leq |y|$.

Lemma ((Draghici, 2006), (Pfiefer, 1990))

Let $L \subset \mathbb{R}^N$ be a centrally-symmetric convex set. Suppose $g_1, \dots, g_N : \mathbb{R} \to [0, \infty)$ are integrable. Then

$$\int_L g_1(t_1) \cdots g_N(t_N) dt \leq \int_L g_1^*(t_1) \cdots g_N^*(t_N) dt,$$

where $dt = dt_1 \dots dt_N$.

Applying the lemma:

For fixed $\theta \in S^{n-1}$ and $y_1, \dots, y_N \in \theta^\perp$,

- set $g_i(s) = f(y_i + s\theta)$
- consider $F_Y(t) : \mathbb{R}^N \to [0, \infty)$ defined by

$$F_Y(t) = \mathrm{vol}\left(T(y_1 + t_1\theta, \dots, y_N + t_N\theta)C\right).$$

- $L =$ levels sets of F_Y.

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