On the reconstruction of inscribable sets in discrete tomography

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(joint work with P. Dulio)

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The term **discrete tomography** was introduced by L.A. Shepp at a mini-symposium he organized at DIMACS in 1994.

Discrete tomography deals with the reconstruction of discrete objects, e.g. lattice sets, binary matrices, digital images etc, from a **small number of "projections"**.

This reconstruction task can be interpreted as a discrete inverse problem and classified as an ill-posed problem, by specifying suitable variants of the Hadamard criteria

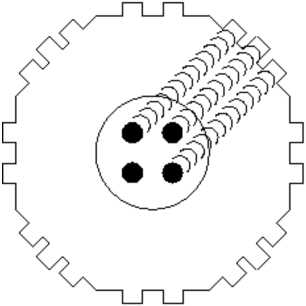
- For general data there need not exist a solution.
- If the data is consistent, the solution need not be uniquely determined.
- Even in the case of uniqueness, the solution may change dramatically with small change of the data.
- The problem of determining how the measurements should be corrected in order to provide consistency of the data is $\mathbb{NP}$-complete for $m \geq 3$ directions, but easy for $m \leq 2$.
- The problem of finding a set which best fits the data is $\mathbb{NP}$-hard for $m \geq 3$, but can be solved in polynomial time (in the size of the data) for $m \leq 2$.

Let $D = \{u_1, ..., u_m\}$ be a set of distinct lattice directions.

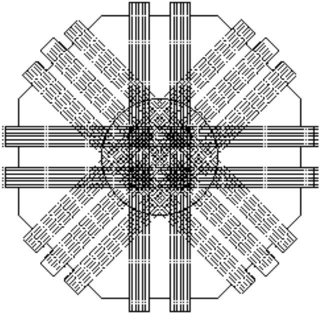
P.C. Fishburn - L.A. Shepp, 1999.

A finite set $F \subset \mathbb{Z}^2$ is ***D*-additive** if for each u_j there is a function f_j constant on each line parallel to u_j such that

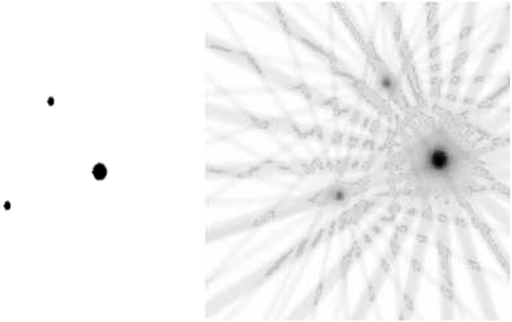
$$x \in F \quad \Leftrightarrow \quad \sum_{j=1}^{m} f_j(x) > 0$$

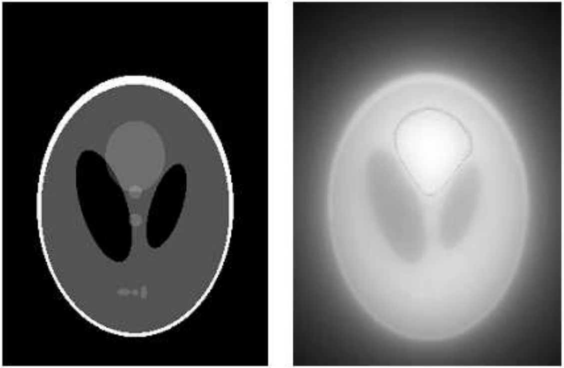


Projection



Backprojection





P.C. Fishburn - L.A. Shepp, 1999.

- D-additive lattice sets are determined by their X-rays taken in the directions in D.
- If $|D| = 2$, then lattice sets are determined by their X-rays taken in the directions in D if and only if they are D-additive.
- If $|D| \geq 3$, then there exist lattice sets which are determined by their X-rays taken in the directions in D but they are not D-additive.
- D-additive lattice sets are reconstructible in polynomial time (in the size of the data) by use of linear programming.

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A **convex lattice set** is a finite subset F of $\mathbb{Z}^2$ such that

$$F = (convF) \cap \mathbb{Z}^2.$$

R.J. Gardner - P. Gritzmann, 1997.

- Convex lattice sets in $\mathbb{Z}^2$ are determined by their X-rays in certain sets of four directions.
- Convex lattice sets in $\mathbb{Z}^2$ are determined by their X-rays in any set of seven lattice directions.

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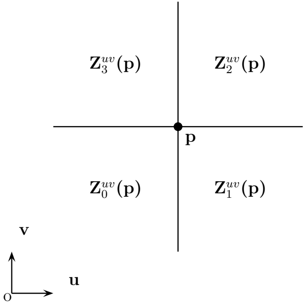
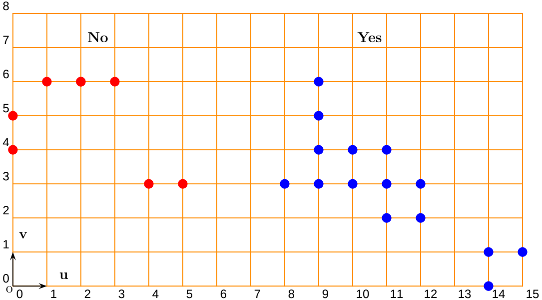
Let $p \in \mathbb{Z}^2$ and let u, v be two distinct lattice directions. Consider the following closed quadrants:



A lattice set F is **Q-convex** (quadrant-convex) w.r.t. $D = \{u, v\}$ if $Z_k^{uv}(p) \cap F \neq \emptyset$, for $k = 0, 1, 2, 3$, implies $p \in F$.

A lattice set F is **Q-convex** w.r.t. a finite set of lattice directions D if it is Q-convex w.r.t. every pair of directions included in D.



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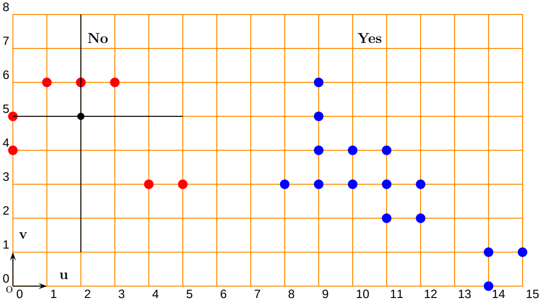


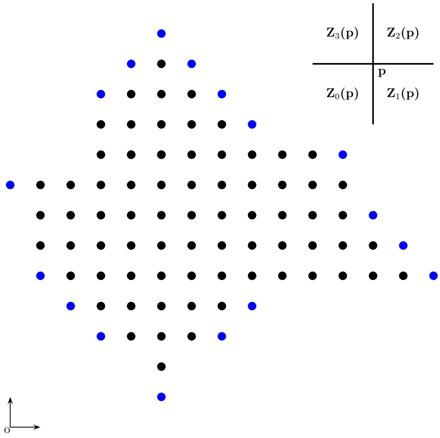
S. Brunetti - A. Daurat, 2005.

- Q-convex lattice sets w.r.t a set D of lattice directions are determined by X-rays taken in the directions in D if D is a set of uniqueness for convex lattice sets.
- Q-convex lattice sets w.r.t a set of directions of uniqueness are reconstructible in polynomial time.

S: set of the coordinate directions

Let F be a Q-convex with respect to S.

A box B is **inscribed** in F if $B \subset F$ and its vertices are salient points of F. (The box may be degenerate if it contains the intersection with F of each line through its vertices parallel to a coordinate direction)



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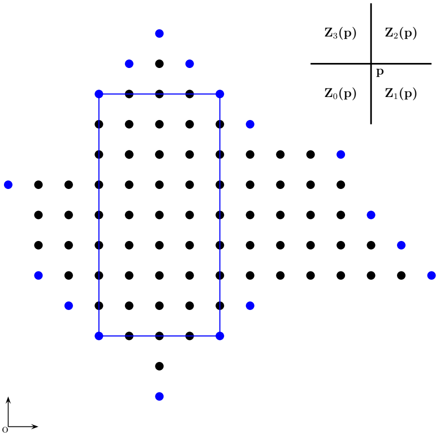

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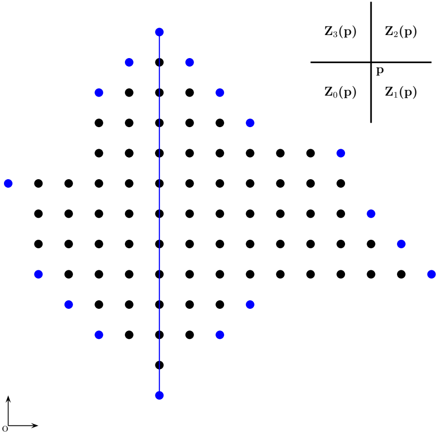
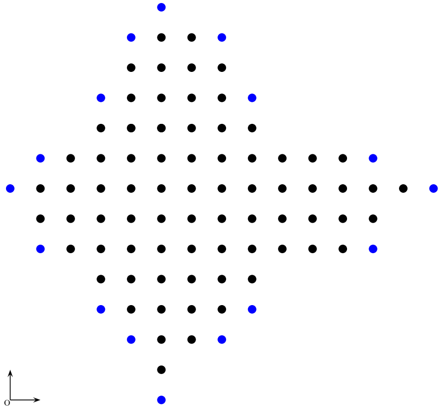


A Q-convex F with respect to S, is **S-inscribable** if for each salient point $p \in F$ there is a box inscribed in F with vertex p.



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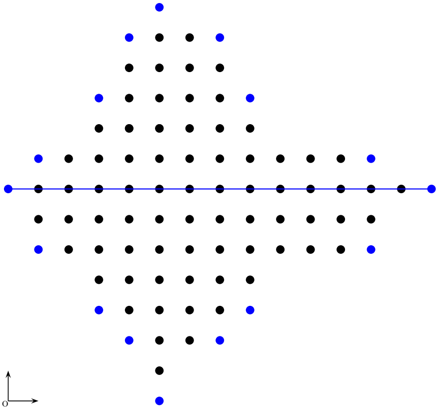

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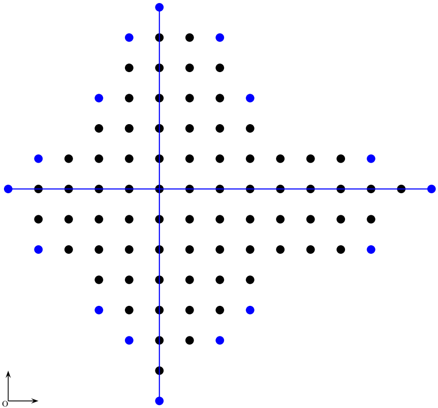

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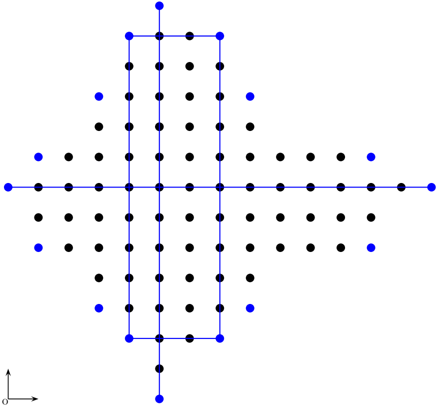

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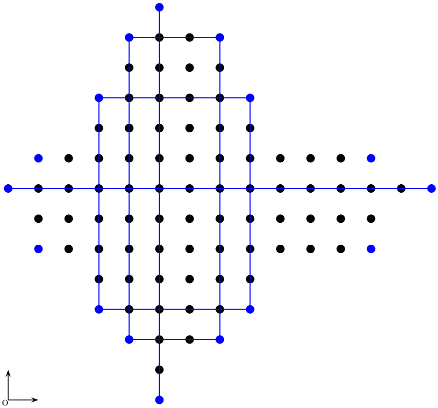

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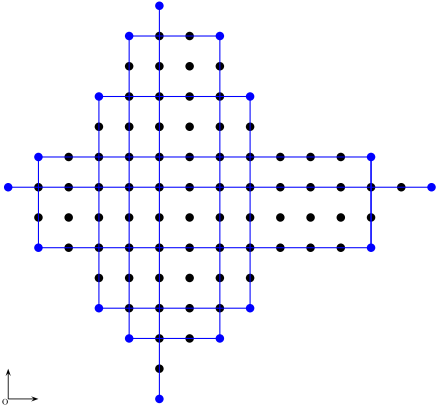

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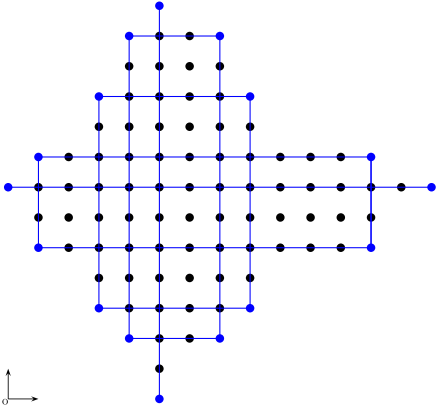

Geometric structure

(i) Each S-inscribable planar lattice set F is the union of inscribed boxes.

(ii) Each pair of non-degenerate inscribed boxes of F have interlacing boundaries, i.e. the edges of one alternatively meet and are disjoint from the other.

(iii) There exists at most one inscribed degenerate box for each direction in S. Such a segment intersects all the inscribed boxes of F.

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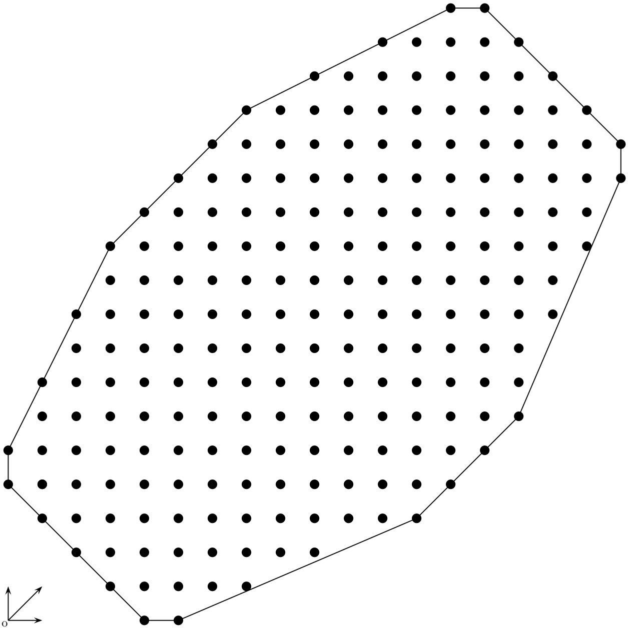
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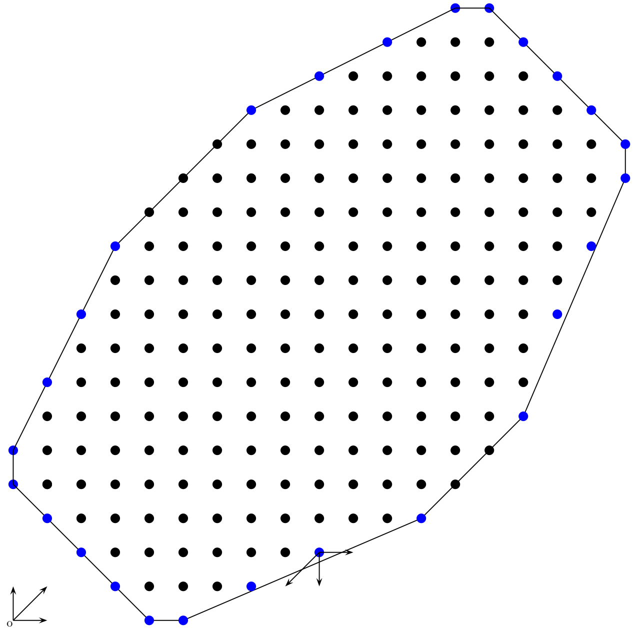
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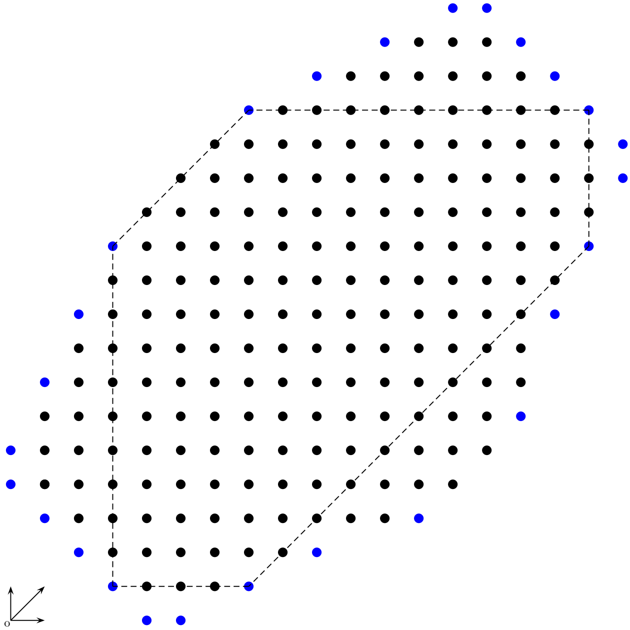
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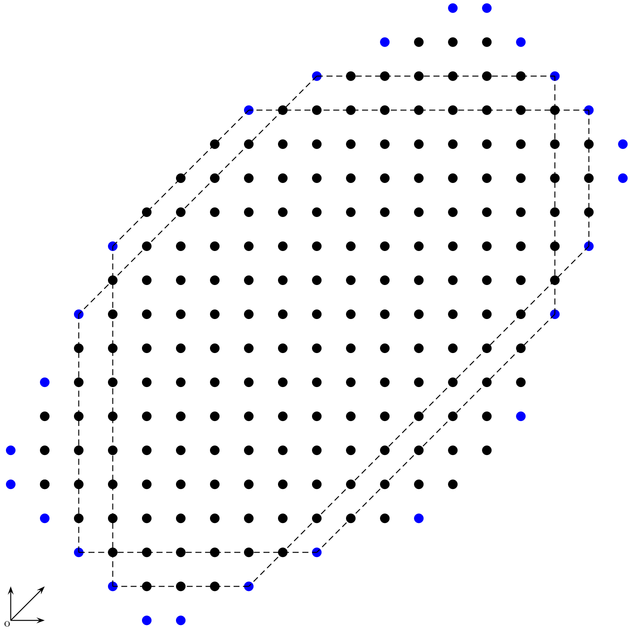
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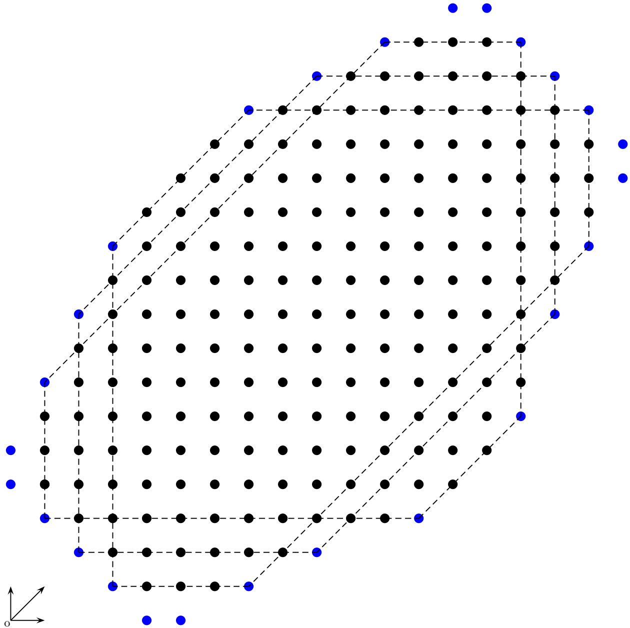
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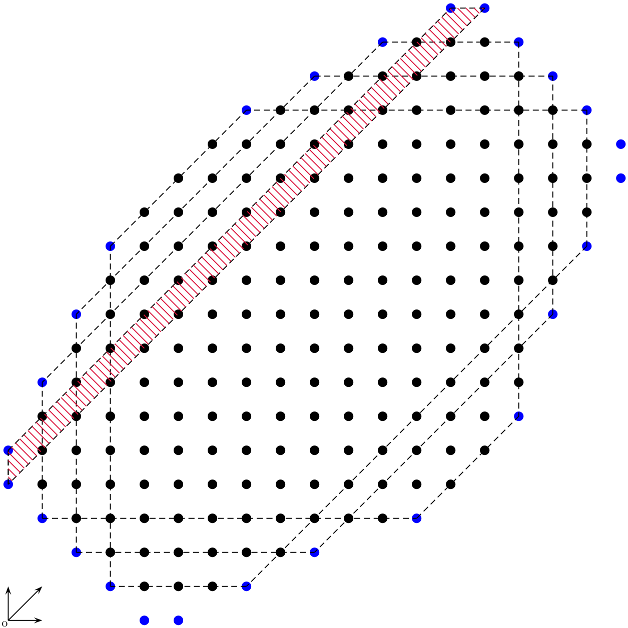
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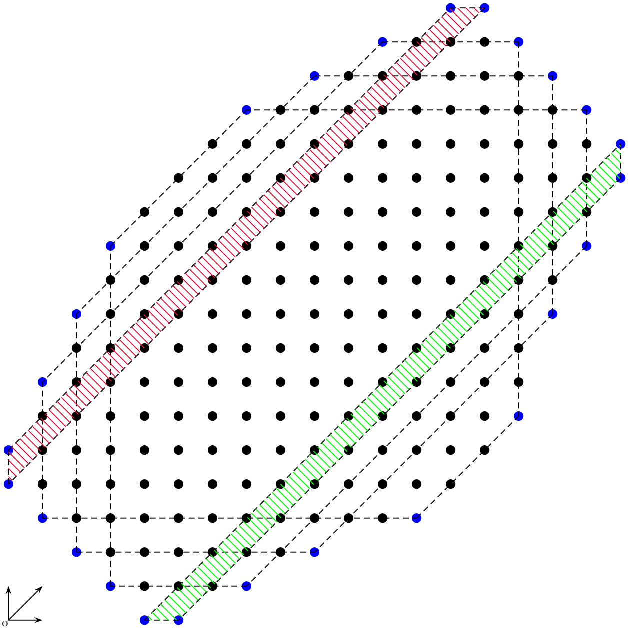
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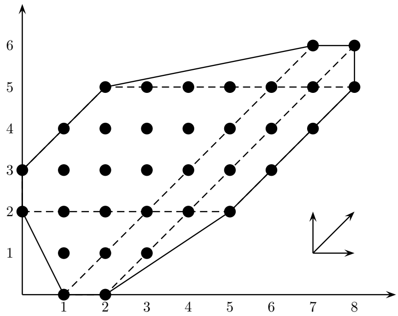
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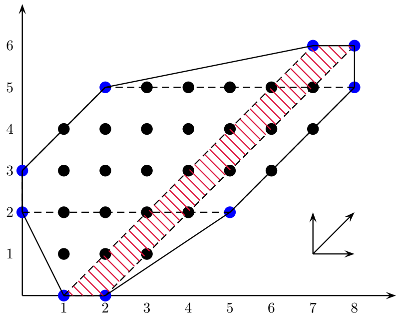

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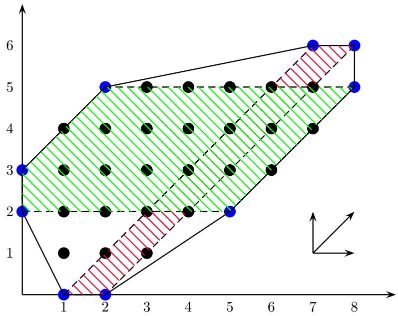

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P. Dulio - C. P.

- Each non degenerate D-inscribable set is D-additive.
- Non degenerate D-inscribable sets are determined by their X-rays in the directions in D.

Problem. Can a *small error* in the measurements of the X-rays lead to a reconstruction which is entirely different from the original object?

The **tomographic distance** between E and F, w.r.t. D, is defined by

$$\delta_D(E,F) = \sum_{u \in D} \|X_u E - X_u F\|_1.$$

A. Alpers - P. Gritzmann, 2006. If $|E| = |F|$, $|D| = m$, and

$$\delta_D(E,F) < 2(m-1)$$

then E and F are tomographically equivalent (i.e. $X_{u_i} E = X_{u_i} F$).

A. Alpers - P. Gritzmann - L. Thorens, 2001;
A. Alpers - P. Gritzmann, 2004, 2006.

Let D be a set of m distinct lattice directions, $m \geq 3$.
For $\alpha \in \mathbb{N}$, there exist two finite lattice sets E, F which are uniquely determined by their X-rays in the directions in D, such that

- $|E| = |F| \geq \alpha$
- $\delta_D(E, F) = 2(m-1)$
- $E \cap F = \emptyset$

A. Alpers - S. Brunetti, 2005

For any set $D = \{u, v\}$ of two lattice directions, and any two finite lattice sets E, F such that

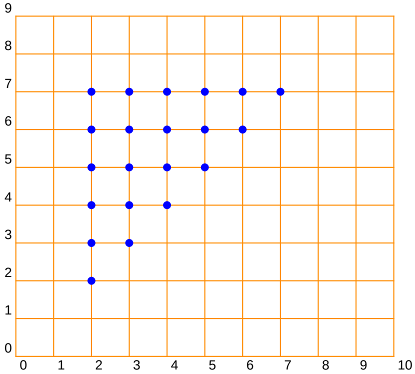
- the set E is determined by its X-rays taken in the directions u, v
- $|E| = |F|$
- $\delta_D(E, F) = 2$

then

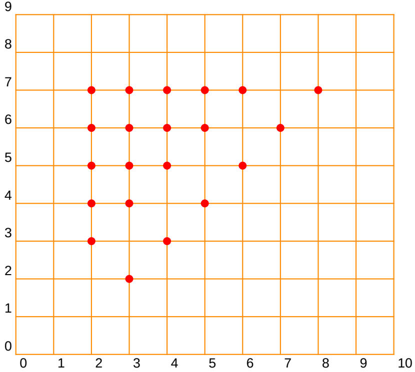
$$|E \triangle F| \leq \sqrt{8|E| + 1} - 1.$$

This bound is sharp

($E \triangle F$ is the symmetric difference)



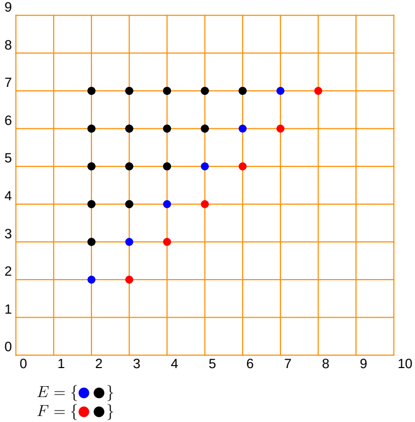
$E = \{\bullet\}$



$F = \{$●$\}$

$$|E| = |F| = 21, \quad |E \triangle F| = 12, \quad \sqrt{8|E|+1} - 1 = 12$$

$$\delta_D(E, F) = 2$$



For a given set F, let E be uniquely determined by its X-rays taken in the coordinate directions such that $X_{(1,0)}E = X_{(1,0)}F$. Define

$$\alpha = \frac{\delta_D(E, F)}{2} = \frac{1}{2}\|X_{(0,1)}E - X_{(0,1)}F\|_1$$

B. van Dalen, 2009

$$|E \triangle F| \leq \alpha(\sqrt{8|E| + 1} - 1).$$

If F_1 and F_2 have the same X-rays in the coordinate directions then

$$|F_1 \triangle F_2| \leq 2\alpha(\sqrt{8|E| + 1} - 1).$$

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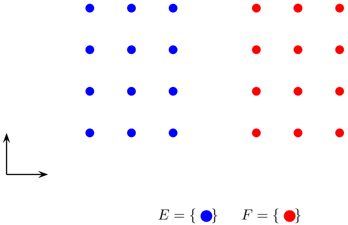
$$|F_1 \triangle F_2| \leq 2\alpha(\sqrt{8|E| + 1} - 1).$$

Theorem. Let $E, F \subset \mathbb{Z}^2$ be two non-degenerate S-inscribable lattice sets. Then

$$|E \triangle F| \leq \delta_S(E, F).$$

This inequality is sharp.

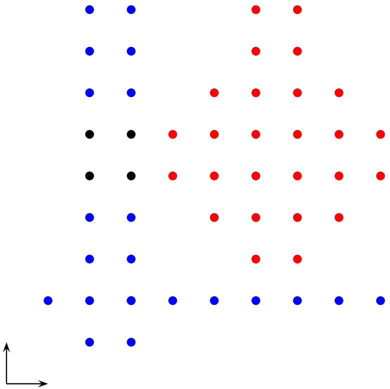


Lemma Let $E, F \subset \mathbb{Z}^2$ be two non-degenerate S-inscribable lattice sets. If $\|X_u E - X_u F\|_1 = 0$ for some $u \in S$, then

$$|E \triangle F| = \delta_S(E, F).$$

Idea of the proof.

We partition the set $E \triangle F$ into horizontal and vertical subsets



$E = \{$ ●● $\}$ $F = \{$ ●● $\}$

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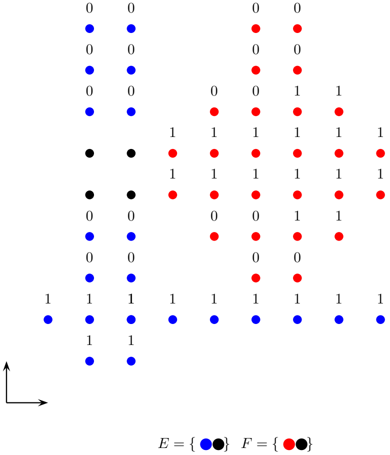

row-function values

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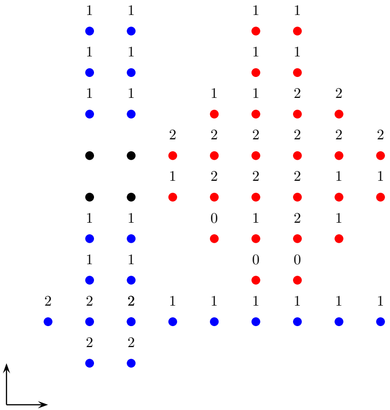


$E = \{$ ●● $\}$ $F = \{$ ●● $\}$

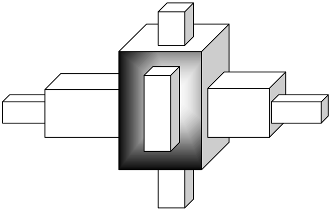
δ-function values

$$
\begin{aligned}
A_0 &= \{p \in E \triangle F : \ \delta(p) = 0\}, \\
A_1 &= \{p \in E \triangle F : \ \delta(p) = 1\}, \\
A_2 &= \{p \in E \triangle F : \ \delta(p) = 2\}.
\end{aligned}
$$

$$|E \triangle F| = |A_0| + |A_1| + |A_2|$$

$$\delta_S(E, F) = \sum_{p \in E \triangle F} \delta(p) = |A_1| + 2|A_2|.$$

$$|E \triangle F| \leq \delta_S(E, F) \quad \Leftrightarrow \quad |A_0| \leq |A_2|$$



S: set of the coordinate directions

Theorem. Let $E, F \subset \mathbb{Z}^n$ be two non-degenerate S-inscribable lattice sets. Then

$$|E \triangle F| \leq \delta_S(E, F).$$