Properties of metric spaces which are not coarsely embeddable into a Hilbert space

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Comment on ‘asymptotic’

- The topic of my talk is “asymptotic” in a somewhat different sense than topics of most of the other talks of this workshop. Instead of considering ‘dimension $\to \infty$’ we consider properties of metric spaces which reflect their behavior only as ‘distances $\to \infty$’.

Definitions, Examples

- More precisely, the main definition of this talk is the following **Definition** (Gromov). A mapping $F : (X, d_X) \to (Y, d_Y)$ between two metric spaces is called a *coarse embedding* if there exist two non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ such that ρ_2 has finite values (it is implicit in the condition above), $\lim_{t\to\infty} \rho_1(t) = \infty$, and

$$\forall u, v \in X \;\; \rho_1(d_X(u, v)) \le d_Y(F(u), F(v)) \le \rho_2(d_X(u, v)).$$

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$$\forall u,v\in X\ \ \rho_1(d_X(u,v))\le d_Y(F(u),F(v))\le\rho_2(d_X(u,v)).$$

- **Example 1.** The mapping $F:\mathbb{R}\to\mathbb{N}$ given by $F(x)=\lfloor x\rfloor$ is a coarse embedding.
- **Example 2.** The vertex set V of an infinite dyadic tree T rooted at vertex O with its graph metric can be coarsely embedded into ℓ_2 in the following way: we consider a bijection between the set of all edges of T and vectors of an orthonormal basis $\{e_i\}$ in ℓ_2, and map each vertex from V onto the sum of those vectors from $\{e_i\}$ which correspond to a path from a root O of T to the vertex, O is mapped to 0.

History

- The restrictions in the definition of a coarse embedding are weak and it is far from being obvious that there exist separable metric spaces which are not coarsely embeddable into a Hilbert space. Gromov (1993) wrote: "There is no known geometric obstruction for coarse embeddings into infinite dimensional Banach spaces."

History

- The restrictions in the definition of a coarse embedding are weak and it is far from being obvious that there exist separable metric spaces which are not coarsely embeddable into a Hilbert space. Gromov (1993) wrote: "There is no known geometric obstruction for coarse embeddings into infinite dimensional Banach spaces."
- Writing this Gromov was unaware of Enflo's work (1969) in which it was shown that there are no uniform (that is, uniformly continuous with the uniformly continuous inverse) embeddings of the Banach space c_0 into a Hilbert space. Dranishnikov, Gong, V. Lafforgue, and Yu showed (2002) that the construction due to Enflo (1969) can be used to construct locally finite metric spaces which are not coarsely embeddable into ℓ_2. (A metric space is called *locally finite* is all balls in it have finitely many elements.)

Applications

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- The idea of Gromov was to approach some well-known problems in Topology using coarse embeddings of certain finitely generated groups with their word metrics into “good” Banach spaces.
- This idea turned out to be very fruitful, see the survey of Yu [in: International Congress of Mathematicians. Vol. II, 1623–1639, Eur. Math. Soc., Zürich, 2006].

Coarse embeddings into Hilbert space

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  - The first example of a metric space with bounded geometry which is not coarsely embeddable into a Hilbert space (and, more generally, into ℓ_p with $1 \leq p < \infty$) was found by Gromov (2000). All inequalities needed for the example were known before: I mean the papers of Linial, London, and Rabinovich (1995, Hilbert space case), and Matoušek (1997, ℓ_p-case).

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- We define the embedding in the following way: we map each function from $L_1(\mathbb{R})$ to the indicator function of the set between the graph of the function and the x-axis. This indicator function is considered as an element of $L_2(\mathbb{R}^2)$. One can check that this mapping has the desired properties.

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- These results show that to prove coarse embeddability/non-embeddability results for a Hilbert space it suffices to prove similar results for L_1.

Expanders

- Definition
  For a graph G with vertex set V and a subset $F \subset V$ by ∂F we denote the set of edges connecting F and $V \backslash F$. The *expanding constant* (also known as the *Cheeger constant*) of G is

$$h(G) = \inf \left\{ \frac{|\partial F|}{\min\{|F|, |V \backslash F|\}} : \; F \subset V, \; 0 < |F| < +\infty \right\}.$$

  A sequence $\{G_n\}$ of graphs is called a *family of expanders* if all of G_n are finite, connected, k-regular for some $k \in \mathbf{N}$, their expanding constants $h(G_n)$ are bounded away from 0 (that is, there exists $\varepsilon > 0$ such that $h(G_n) \geq \varepsilon$ for all n), and their sizes (numbers of vertices) $\to \infty$ as $n \to \infty$.

- Theorem (Gromov, Linial-London-Rabinovich, Matoušek)

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- Lemma (Sobolev-type inequality)

  Let $G = (V, E)$ be a graph with the expanding constant h, and $f : V \to \mathbb{R}$ be a function with the median m. Then

$$\sum_{uv \in E} |f(u) - f(v)| \ge h \sum_{v \in V} |f(v) - m|. \tag{1}$$

Description of the proof

- The Theorem is proved as follows: Assume that there exists a coarse embedding $F : A \to L_1$. We may assume that the images of elements of A are continuous functions, so $F(x,t)$ is well-defined for $x \in A$, $t \in [0,1]$.

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- Writing the Sobolev-type inequalities for each graph G_n, for each $t \in [0,1]$, and integrating we get, for some $M_n \in L_1$:

$$\sum_{uv \in E(G_n)} ||F(u)-F(v)||_{L_1} \geq h(G_n) \sum_{v \in V(G_n)} ||F(v)-M_n||_{L_1} \quad (2)$$

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- Inequality (2) implies that there exists a constant D independent of n such at least half of the images of $F(V(G_n))$ is in a ball of radius D centered at M_n. On the other hand, in a k-regular graph the number of vertices in a ball of radius R can be estimated in terms of k only. From here one can derive a contradiction with the definition of a coarse embedding.

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- Theorem (MO (2009), Tessera (2009))

Let M be a locally finite metric space which is not coarsely embeddable into L_1. Then there exists a constant D, depending on M only, such that for each $n \in \mathbb{N}$ there exists a finite set $B_n \subset M \times M$ and a probability measure μ on B_n such that

  - $d_M(u, v) \geq n$ *for each* $(u, v) \in B_n$.
  - *For each Lipschitz function* $f : M \to L_1$ *we have*

$$\int_{B_n} ||f(u) - f(v)||_{L_1} d\mu(u, v) \leq D\mathrm{Lip}(f). \qquad (3)$$

- Lemma
Let M be a locally finite metric space which is not coarsely embeddable into L_1. There exists a constant C depending on M only such that for each Lipschitz function $f : M \to L_1$ there exists a subset $B_f \subset M \times M$ such that $\sup_{(x,y)\in B_f} d_M(x,y) = \infty$, but $\sup_{(x,y)\in B_f} ||f(x) - f(y)||_{L_1} \le C\mathrm{Lip}(f)$.

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  - PROOF. Assume the contrary. Then, for each $n \in \mathbb{N}$, the number n^3 cannot serve as C. This means, that for each $n \in \mathbb{N}$ there exists a Lipschitz mapping $f_n : M \to L_1$ such that for each subset $U \subset M \times M$ with

$$\sup_{(x,y)\in U} d_M(x,y) = \infty,$$

    we have

$$\sup_{(x,y)\in U} ||f_n(x) - f_n(y)|| > n^3\mathrm{Lip}(f_n).$$

- We choose a point in M and denote it by O. Without loss of generality we may assume that $f_n(O) = 0$.

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$$f : M \to \left(\sum_{n=1}^{\infty} \oplus L_1\right)_1 \subset L_1$$

given by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{Kn^2} \cdot \frac{f_n(x)}{\mathrm{Lip}(f_n)},$$

where $K = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

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- It is clear that the series converges and $\mathrm{Lip}(f) \leq 1$. It is not difficult to verify that f is a coarse embedding. We get a contradiction.

- Lemma

  Let C be the constant whose existence is proved in the previous Lemma and let $\varepsilon > 0$ be arbitrary. For each $n \in \mathbb{N}$ we can find a finite subset $M_n \subset M$ such that for each Lipschitz mapping $f : M \to L_1$ there is a pair $(u_{f,n}, v_{f,n}) \in M_n \times M_n$ such that

    - $d_M(u_{f,n}, v_{f,n}) \geq n$.
    - $||f(u_{f,n}) - f(v_{f,n})|| \leq (C + \varepsilon)\mathrm{Lip}(f)$.

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  - PROOF. More-or-less standard ultraproduct argument.

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- According to the second Lemma, there is a finite subset $M_n \subset M$ such that for each 1-Lipschitz function f on M there is a pair (u, v) in M_n such that $d_M(u, v) \geq n$ and $||f(u) - f(v)|| \leq B$.

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- So we can consider this as a kind of a two-person game: one person picks a 1-Lipschitz function and the other a pair of points in M_n at distance $\geq n$. The second person wins if $||f(u) - f(v)|| \leq B$.

- Proof of the Theorem. Let D be a number satisfying $D > C$, and let B be a number satisfying $C < B < D$.
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- So we can consider this as a kind of a two-person game: one person picks a 1-Lipschitz function and the other a pair of points in M_n at distance $\geq n$. The second person wins if $||f(u) - f(v)|| \leq B$.
- The set of 1-Lipschitz functions is not finite. Nevertheless everything can be worked out in a suitable way. Applying the von Neumann minimax theorem we get the desired measure.

This is still far from the desired result

- Using the result of Johnson-Randrianarivony (2006) (or its strengthening due to Mendel-Naor (2008)) one can construct a locally finite metric space which is a subset of ℓ_p, p is some number satisfying $p > 2$, which is not coarsely embeddable into ℓ_2, and thus contains structures described above. Recall, on the other hand, that ℓ_p $(1 \leq p < \infty)$ do not contain expanders.

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- The following result was proved with the purpose to get from the previous result something more suitable for search of expander-like constructions.

Expansion properties of sets M_n.

- Let s be a positive integer. We consider graphs $G(n, s) = (M_n, E(M_n, s))$, where the edge set $E(M_n, s)$ is obtained by joining those pairs of vertices of M_n which are at distance $\leq s$. The graphs $G(n, s)$ have uniformly bounded degrees if the metric space M has bounded geometry.

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- (*) For some $s \in \mathbb{N}$ there is a number $h_s > 0$ and subgraphs H_n of $G(n, s)$ of indefinitely growing sizes (as $n \to \infty$) such that the expansion constants of $\{H_n\}$ are uniformly bounded from below by h_s.

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- (*) For some $s \in \mathbb{N}$ there is a number $h_s > 0$ and subgraphs H_n of $G(n, s)$ of indefinitely growing sizes (as $n \to \infty$) such that the expansion constants of $\{H_n\}$ are uniformly bounded from below by h_s.
- If we would prove that in the bounded geometry case the condition (*) is satisfied, it would solve the problem mentioned at the beginning of the talk: whether each metric space with bounded geometry which does not embed coarsely into a Hilbert space contains weak expanders? For spaces with bounded geometry weak expanders are defined as Lipschitz images $f_m(X_m)$ of (vertex sets) of a family of expanders with uniformly bounded Lipschitz constants of $\{f_m\}_{m=1}^{\infty}$ and without dominating pre-images in the sense that $\lim\limits_{m\to\infty} \max\limits_{z\in f_m(X_m)} (|f_m^{-1}(z)|/|X_m|) = 0$.

- At this point we are able to prove only the following weaker expansion property of the graphs $G(n, s)$. We introduce the measure ν_n on M_n by $\nu_n(A) = \mu_n(A \times M_n)$. Let F be an induced subgraph of $G(n, s)$. We denote the vertex boundary of a set A of vertices in F by $\delta_F A$.

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- Theorem (MO (2009))

  Let s and n be such that $2n > s > 8D$. Let $\varphi(D,s) = \frac{s}{4D} - 2$. Then $G(n,s)$ contains an induced subgraph F with d_M-diameter $\geq n - \frac{s}{2}$, such that each subset $A \subset F$ of d_M-diameter $< n - \frac{s}{2}$ satisfies the condition: $\nu_n(\delta_F A) > \varphi(D,s)\nu_n(A)$.

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  - The proof uses the exhaustion process similar to the one used by Linial-Saks (1993) and "random" signing of functions similar to the way it was used by Rao (1999) in his work on Lipschitz embeddings of planar graphs into ℓ_2.

Final comment

- The problem on relation between the expansion condition from the last theorem and the desired expansion resembles the well-known open problem: whether each sequence $\{G_n\}$ of k-regular ($k \geq 3$) graphs with indefinitely growing girth contains weak expanders?