

Properties of isoperimetric, spectral-gap and log-Sobolev inequalities via concentration

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Workshop on Asymptotic Geometric Analysis and Convexity
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Outline

- Define isoperimetric inequalities and recall role in Asymptotic Geometric Analysis.
- Recall connection to Sobolev and concentration inequalities in general and in presence of convexity.
- Survey tools for deducing, transferring and analyzing stability of these inequalities in semi-convex setting.

Isoperimetric Inequalities

(Ω, d, μ) - measure metric space; d - metric, μ - Borel probability measure.

Assume: $\Omega \subset (M^n, g)$ Riemannian manifold, d induced geodesic distance on M , $\mu = h \text{vol}_M|_{\Omega}$.

Isoperimetric Inqs compare between $\mu(A)$ and $\mu^+(A)$ (Minkowski's exterior boundary measure):

$$\mu^+(A) := \liminf_{\varepsilon \rightarrow 0} \frac{\mu(A_\varepsilon^d) - \mu(A)}{\varepsilon},$$

$$A_\varepsilon^d := \{x \in \Omega; d(x, A) < \varepsilon\}.$$

Isoperimetric profile: $\mathcal{I} : [0, 1] \ni v \mapsto \inf \{\mu^+(A); \mu(A) = v\}$.

Typically $\mu^+(A) = \mu^+(\Omega \setminus A)$, and so $\mathcal{I}(1 - v) = \mathcal{I}(v)$.

Hence, we restrict to $\mu(A) \leq 1/2$, and $\mathcal{I} : [0, 1/2] \rightarrow \mathbb{R}_+$.

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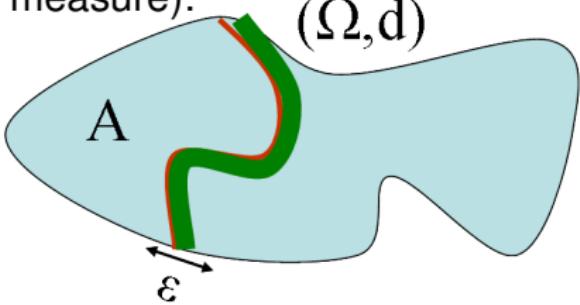
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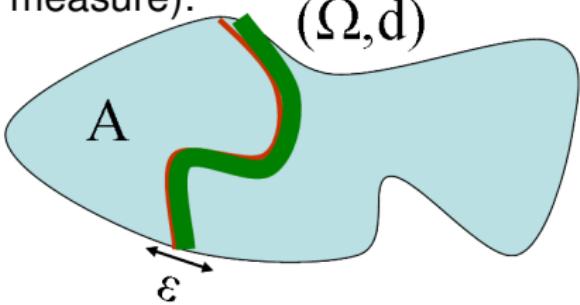
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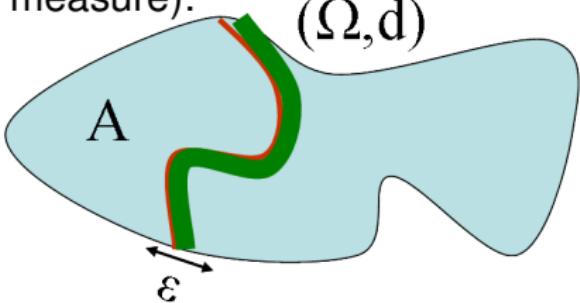
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Motivation from Asymptotic Geometric Analysis

Given $K \subset \mathbb{R}^n$ convex bdd domain, consider $(K, |\cdot|, \mu_K)$.

$$D_{Che}(K) := \min_{v \in [0, 1/2]} \frac{\mathcal{I}_{(K, |\cdot|, \mu_K)}(v)}{v} = \inf_{\mu_K(A) \leq 1/2} \frac{\mu_K^+(A)}{\mu_K(A)}.$$

Conjecture (Kannan–Lovász–Simonovits 90's): up to **dim independent universal** constant, enough to only consider half-spaces $A = \{\langle x, \theta \rangle \leq a\}$.

KLS conjecture implies Thin-Shell and Slicing conjectures (Ball 05, Eldan-Klartag 10).

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$$\forall r > 0 \quad \forall A \subset \Omega \quad \mu(A) \geq 1/2 \quad \Rightarrow \quad \mu(\Omega \setminus A_r^d) \leq \mathcal{K}(r);$$

$$[\quad \forall r > 0 \quad \forall \text{1-Lip } f \quad \mu\{x; f(x) - \text{med}_\mu f \geq r\} \leq \mathcal{K}(r).]$$

Examples: (Cheeger, Maz'ya; Gromov–V. Milman; Ledoux, Beckner; Herbst)

$$\mathcal{I}(v) \geq Dv \quad \Rightarrow \quad \|\nabla f\|_2 \geq \frac{D}{2} \sqrt{\text{Var}(f)} \quad \Rightarrow \quad \mathcal{K}(r) \leq \exp(-cDr)$$

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$$\text{Var}_\mu(f) := \mu(f^2) - \mu(f)^2 \quad \text{Ent}_\mu(g) := \mu(g \log g) - \mu(g)\mu(\log g).$$

Remark: **reverse** implications are in general **false**
due to **bottlenecks** in space (geometry of (Ω, d) or measure μ).

In presence of **convexity**, all levels are **equivalent**: $\sqrt{\lambda_1^N} \simeq D_{\text{Che}}$.
KLS conj $\Leftrightarrow \lambda_1^N(K)$ essentially attained on linear functionals.

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Hierarchy Reversal in Presence of Convexity

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Setting: $(M^n, g), d, \Omega \subset M$ convex, $\mu = \exp(-\psi) vol_M|_\Omega$.

Bakry–Émery Curvature-Dimension Condition:

$$Ric_g + Hess_g \psi \geq -\kappa g, \quad \kappa \geq 0.$$

- $\kappa = 0$ - “convex case” (e.g. $(\mathbb{R}^n, |\cdot|, \mu)$ log-concave)
- $\kappa > 0$ - “semi-convex case” (e.g. double-well potentials).

Dimension independent hierarchy reversal from:

- Sobolev inqs (Buser, Ledoux, Bakry–Ledoux, M.).
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"**weakest** concentration implies **linear** isop. (hence **exponential** conc.)"

Thm (M. 08,09,10): If $\exists \lambda_0 \in (0, 1/2) \exists r_0 > 0$ so that:

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Then:

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"**stronger than exp** concentration implies stronger than linear isop."

Thm (M. 09):

$$\mathcal{K}(r) \leq \exp(-\alpha(r)) \Rightarrow \mathcal{I}(v) \geq \min(cv \frac{\log 1/v}{\alpha^{-1}(\log 1/v)}, c_\alpha) \quad \forall v \in [0, 1/2] .$$

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Remark: growth condition is necessary even in 1-D case
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$$\begin{cases} \text{Under } Ric_g + Hess_g \psi \geq -\kappa g, \kappa \geq 0 \\ \text{additional growth condition if } \kappa > 0 \end{cases}$$

Recover all previous dim-dependent results:

- Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov: $\int_{\Omega} \exp(\beta(d(x, x_0))) d\mu(x) < \infty$
 $\Rightarrow \mathcal{K}(r) \leq \text{Markov}$
- Bérard, Besson, Gallot, Li, Yau, . . . : $\text{diameter}(\Omega) < D$
 $\Rightarrow \mathcal{K}(r) = 0 \quad \forall r > D.$
- Generalize everything to Riemannian setting.
- Some generalizations require $CAT(\lambda)$ property.

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Isoperimetric Inqs \Rightarrow Sobolev Inqs \Rightarrow Concentration Inqs.



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Let $\mu_i = \exp(-V_i)dx$, V_i convex, on $(\mathbb{R}^n, |\cdot|)$, $i = 1, 2$.

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If: $d_{TV}(\mu_1, \mu_2) := \frac{1}{2} \int \left| \frac{d\mu_1}{dx} - \frac{d\mu_2}{dx} \right| dx \leq 1 - \varepsilon < 1$

Then: $LVar_{\mu_1}(f) \leq \int |\nabla f|^2 d\mu_1 \Rightarrow C_\varepsilon LVar_{\mu_2}(f) \leq \int |\nabla f|^2 d\mu_2 .$

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Proof: let $D_{p,q}(\mu)$ be the best constant in:

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Stability of λ_1^N w.r.t. Lipschitz Maps

Fact: "Lipschitz maps transfer isoperimetric inequalities".

If $T : (\Omega_1, d_1, \mu_1) \rightarrow (\Omega_2, d_2, \mu_2)$, $T_*(\mu_1) = \mu_2$ and $\|T\|_{Lip} \leq L$:

$$\|T\|_{Lip} := \sup_{x,y} \frac{d_2(T(x), T(y))}{d_1(x, y)}.$$

Then $\mathcal{I}(\Omega_2, d_2, \mu_2) \geq \frac{1}{L} \mathcal{I}(\Omega_1, d_1, \mu_1)$.

"On-average Lipschitz maps transfer D_{Che} when $\kappa = 0$ ".

Thm (M. 08): If (Ω_2, d_2, μ_2) is convex ($\kappa = 0$), then:

$$D_{Che}(\Omega_2, d_2, \mu_2) \geq \frac{c}{\int_{\Omega_1} \|dT\|_{op}(x) d\mu_1(x)} D_{Che}(\Omega_1, d_1, \mu_1).$$

$$\|dT\|_{op}(x) := \limsup_{y \rightarrow x} \frac{d_2(T(x), T(y))}{d_1(x, y)}.$$

Applications: Barthe–Wolf, Fleury, Huet.

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Thm (M. 08): If (Ω_2, d_2, μ_2) is convex ($\kappa = 0$), then:

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Applications: Barthe–Wolf, Fleury, Huet.

Stability of λ_1^N w.r.t. Lipschitz Maps

Fact: "Lipschitz maps transfer isoperimetric inequalities".

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Remark: results apply to general Sobolev and isoperimetric inqs.

Let $\mu_i = \exp(-V_i)dx$ probability measures on $(\mathbb{R}^n, |\cdot|)$, $i = 1, 2$.

Lemma (Holley–Stroock 87): Assume $V_1 + D_+ \geq V_2 \geq V_1 - D_-$. Then:

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Thm (M. 09,10): Assume $V_2 \geq V_1 - D_-$ and $\text{Hess}V_2 \geq -\kappa \text{Id}$. Then:

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Remark: false without assuming $\text{Hess}V_2 \geq -\kappa \text{Id}$.

Cor (M. 10): Assume $\mu_1 = \exp(-V)dx$, V convex. Let $A \subset \mathbb{R}^n$ denote a convex event, $\mu_1(A) = p$, and set $\mu_2 = \frac{\mu_1|_A}{\mu_1(A)}$. Then:

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The \widetilde{W}_{Ψ_1} metric

Define " ψ_1 Lipschitz metric" on exponentially integrable prob. measures:

$$\widetilde{W}_{\Psi_1}(\mu_1, \mu_2) := \sup \left\{ \frac{|\log \int \exp(f) d\mu_1 - \log \int \exp(f) d\mu_2|}{\|f\|_{Lip}} \right\}.$$

Using $f = \varepsilon g$ and letting $\varepsilon \rightarrow 0$:

$$W_1(\mu_1, \mu_2) = \sup \left\{ \frac{\int g d\mu_1 - \int g d\mu_2}{\|g\|_{Lip}} \right\} \leq \widetilde{W}_{\Psi_1}(\mu_1, \mu_2),$$

\widetilde{W}_{Ψ_1} has some unclear relation to:

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Questions:

- Is there some optimal transport interpretation ?
- $\widetilde{W}_{\Psi_1}(\mu_1, \mu_2) \simeq W_1(\mu_1, \mu_2)$ for log-concave measures μ_i ?
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Exercise: concentration inqs are stable w.r.t. \widetilde{W}_{Ψ_1} .

Idea of proof: formulate via Laplace transform:

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