Another observation about operator compressions

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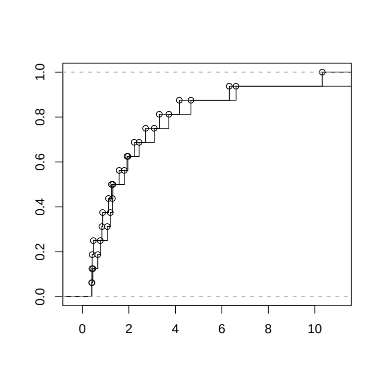

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For M given, let A be chosen uniformly at random from all $k \times k$ principal submatrices. Let F_A denote the empirical distribution function of A; that is,

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$$\mathbb{E}\|F_A - F\|_\infty \leq \frac{13 + \sqrt{8}\log(k)}{\sqrt{k}}.$$

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2. The transposition random walk on S_n has uniform measure as stationary distribution.
3. Good bounds (due to Diaconis and Shahshahani) on the spectral gap of the transposition random walk are available.
4. It's not too hard to get from concentration of $F_A(x)$ near $F(x)$ to concentration of $\|F_A - F\|_\infty$.

Away from matrices: the coordinate-free viewpoint

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Let T be a self-adjoint operator on an n-dimensional Hilbert space $\mathcal{H}$. Let E be a subspace of $\mathcal{H}$, and let $\pi_E : \mathcal{H} \to E$ denote orthogonal projection.

The *compression* of T to E is the operator

$$T_E := \pi_E T|_E = \pi_E T \pi_E^*.$$

The spectral distribution of T_E is defined to be the measure

$$\mu_E := \frac{1}{k} \sum_{j=1}^{k} \delta_{\lambda_j(T_A)},$$

where $\lambda_1(T_A) \geq \cdots \geq \lambda_k(T_A)$ are the eigenvalues of T_A.

Our observation

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For T a given self-adjoint operator on an n-dimensional Hilbert space $\mathcal{H}$ and $1 \leq k \leq n$, most compressions of T to k-dimensional subspaces have spectral distributions which are about the same.

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If E is a k-dimensional random subspace of $\mathcal{H}$ distributed according to the rotation-invariant probability measure on the Grassmannian, μ_E is the spectral measure of T_E and $\mu := \mathbb{E}\mu_E$, then

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$$\mathbb{P}\left[d_1(\mu_E, \mu) \geq A\frac{\sigma_k(T)^{4/7}\rho(T)^{3/7}}{(kn)^{2/7}} + t\right] \leq B\exp\left[-C\frac{knt^2}{\sigma_k^2(T)}\right].$$

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Here,

$$\rho(T) := \frac{1}{2}\left[\lambda_1(T) - \lambda_n(T)\right] \qquad \sigma_k(T) := \inf_\lambda \sqrt{\sum_{i=1}^{k} s_i^2(T - \lambda I)}.$$

A note on distance

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We use the Kantorovich-Rubenstein distance

$$\begin{aligned} d_1(\mu,\nu) &= \inf_{\pi} \int_{\mathbb{R}\times\mathbb{R}} |x-y| d\pi(x,y) \\ &= \sup_{f} \left| \int f d\mu - \int f d\nu \right| \\ &= \|F_\mu - F_\nu\|_{L_1(\mathbb{R})}, \end{aligned}$$

where π varies over probability measures on $\mathbb{R} \times \mathbb{R}$ with margins μ and ν, and f varies over functions on $\mathbb{R}$ with $\|f'\|_\infty \leq 1$.

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where π varies over probability measures on $\mathbb{R} \times \mathbb{R}$ with margins μ and ν, and f varies over functions on $\mathbb{R}$ with $\|f'\|_\infty \leq 1$.

This distance is not directly comparable to the Kolmogorov distance $\|F_\mu - F_\nu\|_\infty$ in general, although some comparison can be made here due to the finite support of the measures in question.

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Recall that one can define a metric $d(E, F)$ on $\mathfrak{G}_k(\mathcal{H})$ by

$$d(E, F) := \inf \sqrt{\sum_{i=1}^{k} \|e_i - f_i\|^2},$$

where the infimum is over orthonormal bases $\{e_i\}_{i=1}^k$ and $\{f_i\}_{i=1}^k$ of E and F, respectively.

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One has concentration about a fixed value for functions on $\mathfrak{G}_k(\mathcal{H})$ which are Lipschitz with respect to the distance $d(\cdot, \cdot)$.

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Theorem (Gromov-Milman)

Let $f : \mathfrak{G}_k(\mathcal{H}) \to \mathbb{R}$ be 1-Lipschitz with respect to $d(\cdot,\cdot)$, and let E be distributed according to the rotation-invariant probability measure on $\mathfrak{G}_k(\mathcal{H})$. Then there is are absolute constants C, c such that

$$\mathbb{P}\left[\left|f(E) - \mathbb{E}f(E)\right| \geq t\right] \leq Ce^{-cnt^2}.$$

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We want to apply this theorem to the function $f(E) := d_1(\mu_E, \mu)$, where d_1 is the Kantorovich-Rubenstein distance, μ_E is the spectral distribution of the compression of T to E, and $\mu = \mathbb{E}\mu_E$.

The Lipschitz constant of $f(E) = d_1(\mu_E, \mu)$ can be bounded using the coupling $\pi = \frac{1}{k}\sum_{i=1}^{k} \delta_{(\lambda_i(T_E), \lambda_i(T_F))}$.

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Recall that $d_1(\mu_E,\mu) = \sup_f \left|\int f d\mu_E - \int f d\mu\right|$; we need to bound the expected maximum of a stochastic process indexed by $\{f : \|f'\|_\infty \le 1\}$.

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Theorem (Dudley)

Let $\{X_t\}_{t\in T}$ be a stochastic process indexed by a metric space T with distance d. Suppose that there is a constant c such that X_t satisfies the increment condition

$$\forall u, \quad \mathbb{P}\left[|X_t - X_s| \geq u\right] \leq c \exp\left(-\frac{u^2}{2d(s,t)^2}\right).$$

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Then there is a constant C such that

$$\mathbb{E} \sup_{t \in T} X_t \leq C \int_0^\infty \sqrt{\log N(T, d, \epsilon)}\, d\epsilon,$$

where $N(T, d, \epsilon)$ is the ϵ-covering number of T with respect to the distance d.

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- Concentration of measure on $\mathfrak{G}_k(\mathcal{H})$ thus gives that

$$\mathbb{P}[|X_f - X_g| > u] \leq \mathbb{P}[|X_{f-g}| \geq u] \leq C \exp\left[-c\frac{nku^2}{\sigma_k^2 |f-g|_L}\right],$$

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Bad News: The covering number of $\{f : \|f'\|_\infty \leq 1\}$ with respect to $\|\cdot\|'$ is infinite. In fact, it suffices to consider $\{f : \|f\|_{C^1} \leq 1 + 2\rho\}$, but that still has infinite covering number.

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Smoothing functions in $\{f : \|f\|_{C^1} \leq 1 + 2\rho\}$ and optimizing over the various parameters yields

$$\mathbb{E}d_1(\mu_E, \mu) \lesssim \frac{\sigma_k^{4/7}\rho^{3/7}}{(kn)^{2/7}}.$$

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- The C-L result is only interesting for $k \gg 1$, whereas our result has content for any k as long as $n \gg 1$.
- Our result is sensitive, via the appearance of $\sigma_k(T)$ and $\rho(T)$, to the proximity of T to the space of scalar matrices

Thank you.