# Moments of unconditional logarithmically concave vectors

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# Definitions

Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ with full dimensional support. We say that the distribution of $X$ is

- *logaritmically concave*, if $X$ has density of the form $e^{-h(x)}$ with $h\colon \mathbb{R}^n \to (-\infty, \infty]$ convex;
- *unconditional*, if $(\eta_1 X_1, \ldots, \eta_n X_n)$ has the same distribution as $X$ for any choice of signs $\eta_1, \ldots, \eta_n$;
- *isotropic*, if $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i X_j = \delta_{i,j}$.

If $\mathbb{E}|X|^2 < \infty$ then there exists an affine transformation $T$ such that $TX$ is isotropic.

If $X$ is unconditional and $\mathbb{E}|X|^2 < \infty$ then there exists a diagonal transformation $D$ such that $DX$ is unconditional and isotropic.

Logarithmically concave vectors have finite all moments.

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# Examples

During this talk we will assume that $X = (X_1, \ldots, X_n)$ is logconcave, isotropic and unconditional.

Basic examples:

- standard normal vector $X = (g_1, \ldots, g_n)$, where $g_i$ are i.i.d. $\mathcal{N}(0, 1)$;
- $X_i$ independent symmetric with (one dimensional) logarithmically concave distribution normalized in such a way that $\mathbb{E}X_i^2 = 1$;
- Uniform distributions on unconditional convex bodies normalized to satisfy $\mathbb{E}X_i^2 = 1$ (for example uniform distributions on $\alpha_{r,n}B_r^n$, $\alpha_{r,n} \sim n^{1/r}$).

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# Moments and tails

For a random variable $S$ (or more general a random vector with values in a normed space) and $p > 0$, we put $\|S\|_p := (\mathbb{E}|S|^p)^{1/p}$ (resp. $\|S\|_p := (\mathbb{E}\|S\|^p)^{1/p}$).

Moments and tails are strictly related. Chebyshev's inequality gives

$$\mathbb{P}(\|S\| \geq e\|S\|_p) \leq e^{-p}.$$

Moreover if $\|S\|_{2p} \leq \alpha\|S\|_p$ then by the Paley-Zygmund inequality

$$\mathbb{P}\Big(\|S\| \geq \frac{1}{C(\alpha)}\|S\|_p\Big) \geq e^{-\max\{C(\alpha),p\}}.$$

For scalar or vector valued combinations of coordinates of logconcave vectors and $p \geq 2$ we have $\|S\|_{2p} \leq C\|S\|_p$ ($C = 2$ in the scalar case).

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# Notation

- $(g_i)$ – a sequence of independent normal standard random variables $\mathcal{N}(0,1)$;
- $(\varepsilon_i)$ – a Bernoulli sequence (i.e. a sequence of i.i.d. symmetric $\pm 1$ r.v.'s) independent of other random variables;
- $(\mathcal{E}_i)$ – a sequence of i.i.d. symmetric exponential r.v.'s with variance 1 (i.e. the density $\frac{1}{\sqrt{2}}\exp(-\sqrt{2}|x|)$).
- $C$ - universal constants (that may take different values at each occurence).
- For two functions $f$ and $g$ we write $f \sim g$ if $\frac{1}{C} f \leq g \leq Cf$.

Our goal is to find a "reasonable" function $f = f_{X,p}$ such that for any scalars $a_i$, $\|\sum_i a_i X_i\|_p \sim f(a_1, \ldots, a_n)$ or more general for any vectors $v_i$, $\|\sum_i v_i X_i\|_p \sim f(v_1, \ldots, v_n)$.

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# Gaussian Case

For any scalars $a_i$ and $p \geq 1$,

$$\Big\| \sum_{i=1}^{n} a_i g_i \Big\|_p = \gamma_p \|a\|_2 \sim \sqrt{p} \|a\|_2,$$

where

$$\gamma_p = \|g_i\|_p = \sqrt{2} \Big[ \frac{1}{\sqrt{\pi}} \Gamma\big(\frac{p+1}{2}\big) \Big]^{1/p}.$$

Gaussian concentration implies that for any vectors $v_i$ in $(F, \|\ \|)$,

$$\Big\| \sum_{i=1}^{n} v_i g_i \Big\|_p \sim \Big\| \sum_{i=1}^{n} v_i g_i \Big\|_1 + \sup_{\|\varphi\|_* \leq 1} \sqrt{p} \|(\varphi(v_i))\|_2.$$

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# Lower estimate of moments - scalar case

For any scalars $a_i$ and $p \geq 1$,

$$\Big\|\sum_{i=1}^n a_i X_i\Big\|_p = \Big\|\sum_{i=1}^n a_i \varepsilon_i |X_i|\Big\|_p \geq \Big\|\sum_{i=1}^n a_i \varepsilon_i \mathbb{E}|X_i|\Big\|_p \geq \frac{1}{C}\Big\|\sum_{i=1}^n a_i \varepsilon_i\Big\|_p.$$

Montgomery-Smith'90 and Hitczenko'93

$$\Big\|\sum_{i=1}^n a_i \varepsilon_i\Big\|_p \sim \sum_{i\leq p} a_i^* + \sqrt{p}\Big(\sum_{i>p}(a_i^*)^2\Big)^{1/2},$$

where $(a_i^*)$ denotes the noincreasing rearrangement of $(a_i)$.

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# Upper estimate of moments - scalar case

Less trivial upper bound follows by Bobkov-Nazarov'03 result

$$\Big\|\sum_{i=1}^{n} a_i X_i\Big\|_p \leq C\Big\|\sum_{i=1}^{n} a_i \mathcal{E}_i\Big\|_p \quad p \geq 2.$$

Gluskin and Kwapień'95 showed that

$$\Big\|\sum_{i=1}^{n} a_i \mathcal{E}_i\Big\|_p \sim p\|a\|_\infty + \sqrt{p}\|a\|_2.$$

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# Two-sided estimate - independent scalar case

**Theorem (Gluskin-Kwapień'95)**

*Let $Y_i$ be independent symmetric r.v's with logconcave tails such that $\mathbb{E}Y_i^2 = 1$. We put $N_i(t) = -\ln \mathbb{P}(|Y_i| \geq t)$ for $t > 1$ and $N_i(t) = t^2$ for $t \in [0,1]$. Then for any $p \geq 2$,*

$$\Big\| \sum_{i=1}^n a_i Y_i \Big\|_p \sim \sup \Big\{ \sum_{i=1}^n a_i b_i : \ \sum_i N_i(|b_i|) \leq p \Big\}.$$

It is not hard to notice that

$$\sup \Big\{ \sum_{i=1}^n a_i b_i : \ \sum_i N_i(|b_i|) \leq p \Big\}$$
$$\sim \sqrt{p} \Big( \sum_{i \notin I_p} a_i^2 \Big)^{1/2} + \sup \Big\{ \sum_{i \in I_p} a_i b_i : \ \sum_{i \in I_p} N_i(|b_i|) \leq p \Big\},$$

where $(|a_i|)_{i \in I_p}$ are $\min\{p,n\}$ largest values of $|a_i|$.

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# More precise estimate - scalar independent case

In fact one can get

**Theorem**

*Let $Y_i$, $N_i$ and $I_p$ be as before. Then for any $p \geq 2$,*

$$\max\Big\{\gamma_p\Big(\sum_{i\notin I_{p/2}} a_i^2\Big)^{1/2}, \frac{1}{C}\sup\Big\{\sum_{i\in I_{p/2}} a_i b_i : \sum_{i\in I_{p/2}} N_i(|b_i|) \leq p\Big\}\Big\}$$

$$\leq \Big\|\sum_{i=1}^n a_i Y_i\Big\|_p$$

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# Two-sided estimate - general scalar case

**Theorem**

*There exists a constant $C$ such that for any $p \geq 2$,*

$$\Big\| \sum_{i=1}^{n} a_i X_i \Big\|_p \sim \inf_{\#I=\min\{\lfloor p \rfloor, n\}} \sup\Big\{ \sum_{i\in I} a_i x_i + \sqrt{p}\big(\sum_{i\notin I} a_i^2\big)^{1/2} \colon \; g_I(x) \geq e^{-Cp} \Big\}$$
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*where $g_I$ is a density of $(X_i)_{i\in I}$ and $(|a_i|)_{i\in I_p}$ are $\min\{p,n\}$ largest values of $|a_i|$.*

# Uniform distribution on $B_r^n$

If $X$ has a uniform distribution on $\alpha_{r,n}B_r^n$ then for $p \geq 2$

$$\Big\|\sum_{i=1}^n a_i X_i\Big\|_p \sim \min\{p,n\}^{1/r}\Big(\sum_{i\leq p}|a_i^*|^{r'}\Big)^{1/r'} + \sqrt{p}\Big(\sum_{i>p}|a_i^*|^2\Big)^{1/2},$$

where $\frac{1}{r}+\frac{1}{r'}=1$ (Barthe,Guedon,Mendelson,Naor'05).

In particular

$$\Big\|\sum_{i=1}^n a_i X_i\Big\|_p \sim \Big\|\sum_{i=1}^n a_i X_i^*\Big\|_p \quad \text{for } 2 \leq p \leq n, \tag{1}$$

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# Uniform distribution on Orlicz balls

The result of Pilipczuk and Wojtaszczyk'08 implies that

$$\Big\|\sum_{i=1}^{n} a_i X_i\Big\|_p \leq C\Big\|\sum_{i=1}^{n} a_i X_i^*\Big\|_p \text{ for } p \geq 2 \tag{2}$$

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# Moments - vector case

**Theorem**

*For any vectors $v_i$ in a normed space*

$$\frac{1}{C}\Big\|\sum_{i=1}^{n} v_i\varepsilon_i\Big\|_p \le \Big\|\sum_{i=1}^{n} v_iX_i\Big\|_p \le C\Big\|\sum_{i=1}^{n} v_i\mathcal{E}_i\Big\|_p.$$

Lower estimate may be shown as in the scalar case. Upper follows from Bobkov-Nazarov's result and Talagrand's estimation of suprema of linear combinations of exponential random variables (generic chaining technique).

**Corollary**

*For any $t > 0$*

$$\frac{1}{C}\mathbb{P}\Big(\Big\|\sum_{i=1}^{n} v_i\varepsilon_i\Big\| \ge Ct\Big) \le \mathbb{P}\Big(\Big\|\sum_{i=1}^{n} v_iX_i\Big\| \ge t\Big) \le C\mathbb{P}\Big(\Big\|\sum_{i=1}^{n} v_i\mathcal{E}_i\Big\| \ge \frac{t}{C}\Big)$$

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$$\frac{1}{C}\mathbb{P}\Big(\Big\|\sum_{i=1}^{n} v_i\varepsilon_i\Big\| \geq Ct\Big) \leq \mathbb{P}\Big(\Big\|\sum_{i=1}^{n} v_i X_i\Big\| \geq t\Big) \leq C\mathbb{P}\Big(\Big\|\sum_{i=1}^{n} v_i\mathcal{E}_i\Big\| \geq \frac{t}{C}\Big)$$

# Moments - vector case

**Theorem**

*For any vectors $v_i$ in a normed space*

$$\frac{1}{C}\Big\|\sum_{i=1}^{n} v_i\varepsilon_i\Big\|_p \le \Big\|\sum_{i=1}^{n} v_iX_i\Big\|_p \le C\Big\|\sum_{i=1}^{n} v_i\mathcal{E}_i\Big\|_p.$$

Lower estimate may be shown as in the scalar case. Upper follows from Bobkov-Nazarov's result and Talagrand's estimation of suprema of linear combinations of exponential random variables (generic chaining technique).

**Corollary**

*For any $t > 0$*

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# Weak and strong moments

Using Talagrand's two level concentration for the product exponential distribution one can prove that

**Theorem**

*If $X_i$ are independent, symmetric, logconcave then for $p \geq 1$,*

$$\Big\| \sum_{i=1}^{n} v_i X_i \Big\|_p \sim \Big\| \sum_{i=1}^{n} v_i X_i \Big\|_1 + \sup_{\|\varphi\|_* \leq 1} \Big\| \sum_{i=1}^{n} \varphi(v_i) X_i \Big\|_p . \qquad (3)$$

**Conjecture**

*Estimate (3) holds for any (unconditional) logconcave random vector $X$.*

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# Connection with concentration

Comparison of weak and strong moments is related to the following concentration problem for symmetric (unconditional) logconcave measures $\mu$:

Is it true that

$$1 - \mu(A + \mathcal{Z}_\mu(p)) \leq e^{-p/C} \quad \text{if } \mu(A) \geq 1/2,$$

where

$$\mathcal{M}_\mu(p) := \left\{ t \in \mathbb{R}^n \colon \int |\langle t, x\rangle|^p d\mu(x) \leq 1 \right\}$$

and

$$\begin{aligned} \mathcal{Z}_\mu(p) &:= (\mathcal{M}_\mu(p))^\circ \\ &= \{y \in \mathbb{R}^n \colon |\langle t, y\rangle|^p \leq \int |\langle t, x\rangle|^p d\mu(x) \text{ for all } t \in \mathbb{R}^n\}? \end{aligned}$$

# Weaker question

Is it true that

$$\Big\|\sum_{i=1}^n v_i X_i\Big\|_p \le C\Big(\Big\|\sum_{i=1}^n v_i X_i\Big\|_1 + \sup_{\|\varphi\|_*\le 1}\Big\|\sum_{i=1}^n \varphi(v_i)\mathcal{E}_i\Big\|_p\Big)$$

or equivalently that

$$\Big\|\sum_{i=1}^n v_i X_i\Big\|_p \le C\Big(\Big\|\sum_{i=1}^n v_i X_i\Big\|_1 + \sup_{\|\varphi\|_*\le 1} p\|\varphi(v_i)\|_\infty + \sqrt{p}\|\varphi(v_i)\|_2\Big)?$$

This is related to the following question for isotropic 1-symmetric logconcave measures:

Is it true that

$$\mu(A+\sqrt{t}B_2^n + tB_1^n) \ge \min\Big\{\frac{1}{2}, e^{t/C}\mu(A)\Big\}?$$

Resent results of Klartag and E. Milman implies that

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# Concentration far away from the origin

**Proposition (L.,Wojtaszczyk)**

*Let $\mu$ be an isotropic unconditional, permutation invariant logconcave measures and $t \geq 1$. Then either*

$$\mu((A + tB_1^n) \cap C\sqrt{n}B_2^n) \geq \frac{1}{2}\mu(A)$$

*or*

$$\mu(A + tB_1^n) \geq e^{t/C}\mu(A).$$

# Sudakov minoration

How to estimate $\mathbb{E}\sup_{t\in T}\langle t, X\rangle$ for $T \subset \mathbb{R}^n$?

Suppose that $\#T \le e^p$ and $t_0$ is any vector then

$$\begin{aligned}\mathbb{E}\sup_{t\in T}\langle t, X\rangle &= \mathbb{E}\sup_{t\in T}\langle t - t_0, X\rangle \le \mathbb{E}\sup_{t\in T}|\langle t - t_0, X\rangle| \\ &\le (\mathbb{E}\sup_{t\in T}|\langle t - t_0, X\rangle|^p)^{1/p} \le (\mathbb{E}\sum_{t\in T}|\langle t - t_0, X\rangle|^p)^{1/p} \\ &\le e\sup_{t\in T}\|\langle t - t_0, X\rangle\|_p.\end{aligned}$$

May one in some way reverse this statement?

**Conjecture (Sudakov-type minoration)**

*Suppose that $T \subset \mathbb{R}^n$, $\#T \ge e^p$, $p \ge 2$ and for any $s, t \in A$, $s \ne t$ one has $\|\langle t - s, X\rangle\|_p \ge A$. Then $\mathbb{E}\sup_{t\in T}\langle t, X\rangle \ge \frac{1}{C}A$.*

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# Examples

If $G = (g_1, \ldots, g_n)$ is standard normal vector then $\langle t, G\rangle \sim \mathcal{N}(0, \|t\|_2^2)$ and $\|\langle t-s, G\rangle\|_p \sim \sqrt{p}\|t-s\|_2$. Hence the conjectured estimate in this situation is equivalent to $\mathbb{E}\sup_{t\in T}\langle t, G\rangle \geq \frac{1}{C}a\sqrt{\log N(T, aB_2^n)}$, that is Sudakov minoration for Gaussian processes.

More general if $X_i$ are symmetric independent logconcave then Sudakov minoration holds (Talagrand'94, L'97).

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# Two observations

Suppose that $T \subset \mathbb{R}^n$, $\#T \geq e^p$, $p \geq 2$ and for any $s, t \in A$, $s \neq t$ one has $\|\langle t - s, X\rangle\|_p \geq A$.

Let $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$, then $\|\langle t - s, \mathcal{E}\rangle\|_p \geq \frac{1}{C}A$ and by Talagrand's result $\mathbb{E}\sup_{t\in T}\langle t, \mathcal{E}\rangle \geq \frac{1}{C}A$. But

$$\mathbb{E}\sup_{t\in T}\langle t, X\rangle \geq \frac{1}{C}\mathbb{E}\sup_{t\in T}\sum_{i=1}^{n} t_i\varepsilon_i \geq \frac{1}{C\log n}\mathbb{E}\sup_{t\in T}\langle t, \mathcal{E}\rangle \geq \frac{1}{C\log n}A,$$

thus minoration conjecture holds up to logarithmic factor.

More delicate argument shows that

$$\mathbb{E}\sup_{t\in T}\langle t, X\rangle \geq \frac{1}{C\max\{1, \log\frac{n}{p}\}}A,$$

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Thank you for your attention.