The Chain Rule as a Functional Equation

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Fundamental operations in analysis and geometry like the Fourier transform, Legendre transform or polarity are often (almost) characterized by simple functional equations or monotonicity properties (in a non-degenerate setting), the Fourier transform e.g. by

$$\mathcal{F}(f \cdot g) = \mathcal{F}f * \mathcal{F}g$$

if acting bijectively on $\mathcal{S}$ and $\mathcal{S}'$ (Alesker, Artstein-Avidan, Faifman, Milman).

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if acting bijectively on $\mathcal{S}$ and $\mathcal{S}'$ (Alesker, Artstein-Avidan, Faifman, Milman).
We show that the derivative is (almost) characterized by the chain rule

$$D(f \circ g) = Df \circ g \cdot Dg \ .$$

Assume an operation $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ satisfies the chain rule

$$T(f \circ g) = Tf \circ g \cdot Tg \; ; \quad f, g \in C^1(\mathbb{R}) \; . \tag{1}$$

We do not assume that T is linear or continuous. What solutions does (1) have?

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(a) $H \in C(\mathbb{R})$, $H > 0$. $Tf := H \circ f / H$ satisfies (1).

(b) $Tf := \left\{ \begin{array}{ll} f' & f \text{ bijective} \\ 0 & \text{else} \end{array} \right\}$ satisfies (1).

(c) $p > 0$. $Tf := |f'|^p$ and $Tf := |f'|^p \operatorname{sgn}(f')$ satisfy (1).

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Definition

$C_b^1(\mathbb{R}) := \{f \in C^1(\mathbb{R}) \mid \; f \text{ bounded from below or above}\}$
$T : C^1(\mathbb{R}) \to C(\mathbb{R})$ is *non-degenerate* if $T\big|_{C_b^1(\mathbb{R})} \neq 0$.

Theorem 1

Assume $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ *satisfies the chain rule functional equation*

$$T(f \circ g) = Tf \circ g \cdot Tg \; ; \quad f, g \in C^1(\mathbb{R}) \; . \tag{1}$$

and is non-degenerate. Then there exists $p \geq 0$ *and* $H \in C(\mathbb{R})$ *with* $H > 0$ *such that*

$$Tf = \frac{H \circ f}{H} |f'|^p$$

or, with $p > 0$,

$$\left.\begin{aligned} Tf &= \frac{H \circ f}{H} |f'|^p \\ Tf &= \frac{H \circ f}{H} |f'|^p \mathrm{sgn}(f') \; . \end{aligned}\right\} \tag{2}$$

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Assume additionally that the image of T *contains functions with negative values. If then* $T(2\,\mathrm{Id})(0) = 2$, $Tf = H \circ f / H \cdot f'$ *; if stronger* $T(2\,\mathrm{Id}) = 2$ *holds,* $Tf = f'$ *is the only solution of* (1).

Remarks.

(i) For $p > 0$, let $G \in C^1(\mathbb{R})$ be such that $G' = H^{1/p}$. Then

$$Tf = \left| \frac{d(G \circ f)}{dG} \right|^p \left\{ \operatorname{sgn} \left(\frac{d(G \circ f)}{dG} \right) \right\} .$$

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(ii) The function H is determined completely by $T(2\,\mathrm{Id})$:

$$\begin{aligned} \text{letting} \quad \varphi(x) &= T(2\,\mathrm{Id})(x)/T(2\,\mathrm{Id})(0), \text{ we have} \\ H(x) &= \textstyle\prod_{n \in \mathbb{N}} \varphi\left(\frac{x}{2^n}\right) \; ; x \in \mathbb{R} \, . \end{aligned}$$

Cohomological interpretation

$G = (C^1(\mathbb{R}), \circ)$, $M = (C(\mathbb{R}), \cdot)$, $G \times M \to M$, $(f, H) \mapsto H \circ f$

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$d^n \ : \ F^n(G, M) \to F^{n+1}(G, M)$ coboundary operators

$\mathrm{Ker}(d^1) = \{$ Solutions of the chain rule$\}$ cocycles

$\mathrm{Im}(d^0) = \{f \mapsto H \circ f / H \mid H \in M\}$ coboundaries

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$H^1(G, M) = \mathrm{Ker}(d^1)/\mathrm{Im}(d^0)$ represented by powers of D (up to sign)

If T acts on smoother functions, we have:

Theorem 2

Take $k, \ell \in \mathbb{N}_0$ with $k > \ell$ and assume that $T : C^k(\mathbb{R}) \to C^\ell(\mathbb{R})$ satisfies the chain rule

$$T(f \circ g) = Tf \circ g \cdot Tg \; ; \quad f, g \in C^k(\mathbb{R}) \; . \tag{1}$$

and is non-degenerate on $C^k(\mathbb{R})$. Then T has the form

$$\frac{H \circ f}{H} |f'|^p \{ \operatorname{sgn}(f') \} \tag{2}$$

where $p > 0$ ($p \geq 0$) and $H \in C^\ell(\mathbb{R})$ is positive. In fact, $p \in \{0, ..., \ell\}$ or $p > \ell$ holds.
The result is also true for $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ ($k = \ell = \infty$).

On $C(\mathbb{R})$ there are no non degenerate examples:

Proposition 3

Assume $T : C(\mathbb{R}) \to C(\mathbb{R})$ *satisfies*

$$T(f \circ g) = (Tf) \circ g \cdot Tg \; ; \quad f, g \in C(\mathbb{R})$$

and that the image of T *contains functions having zeros. Then*

$$T \Big|_{C_b(\mathbb{R})} = 0 \, .$$

A weaker form of the chain rule equation admits almost the same conclusion:

Theorem 4

Assume $T, A : C^1(\mathbb{R}) \to C(\mathbb{R})$ *are operators such that*

$$T(f \circ g) = (Tf) \circ g \cdot Ag \ ; \quad f, g \in C^1(\mathbb{R})$$

holds. Under a somewhat stronger non-degeneracy condition on T, T *and* A *have the form*

$$\begin{aligned} Tf &= G_1 \circ f \cdot \frac{G_2}{G_1} \cdot |f'|^p \ \{\mathrm{sgn}\ (f')\} \\ Af &= Tf / G_2 \circ f = \frac{H \circ f}{H} |f'|^p \ \{\mathrm{sgn}\ (f')\} \end{aligned}$$

where $p > 0$, $G_1, G_2 \in C(\mathbb{R})$ *are positive,* $H = G_1 / G_2$.

Steps in the Proof of Theorems 1 and 2

I. Localization (on intervals)

a) T non-degenerate $\Rightarrow$ For open intervals $J \subset \mathbb{R}$, $y \in J$, $x \in \mathbb{R}$ find $g \in C^1(\mathbb{R})$ with $g(x) = y$, $\mathrm{Im}(g) \subset J$ and $(Tg)(x) \neq 0$.
(Shifts, scaling)

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b) For open intervals $J \subset \mathbb{R}$, $f|_J = \mathrm{Id}$ implies $(Tf)|_J = 1$.
($y \in J$, $\mathrm{Im}(g) \subset J$, $g(x) = y$, $(Tg)(x) \neq 0$. Then $f|_J = \mathrm{Id}$ yields
$f \circ g = g$, $\quad 0 \neq (Tg)(x) = T(f \circ g)(x) = (Tf)(y)(Tg)(x)$.
Therefore $(Tf)(y) = 1$; $(Tf)|_{\bar{J}} = 1$)

c) $f_1|_J = f_2|_J \Rightarrow (Tf_1)|_J = (Tf_2)|_J$.

II. Localization (pointwise), $T : C^1(\mathbb{R}) \to C(\mathbb{R})$

Proposition 5

There is $F : \mathbb{R}^3 \to \mathbb{R}$ *such that* $(Tf)(x) = F(x, f(x), f'(x))$, (3)

$x \in \mathbb{R},\ f \in C^1(\mathbb{R})$.

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There is $F : \mathbb{R}^3 \to \mathbb{R}$ *such that*
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$x \in \mathbb{R}, \ f \in C^1(\mathbb{R})$.

Proof.

Take $x_0 \in \mathbb{R}$, $f \in C^1(\mathbb{R})$. Let $J_1 = (x_0, \infty)$, $J_2 = (-\infty, x_0)$.
Let $g(x) = f(x_0) + f'(x_0)(x - x_0)$ be the tangent line and

$$h(x) = \left\{ \begin{array}{ll} g(x) & x \in J_1 \\ f(x) & x \in \overline{J_2} \ . \end{array} \right\}$$

Since $h|_{J_1} = g|_{J_1}$, $\quad h|_{J_2} = f|_{J_2}$, by c)
$(Tg)|_{\overline{J_1}} = (Th)|_{\overline{J_1}}$, $\quad (Th)|_{\overline{J_2}} = (Tf)_{\overline{J_2}}$. Since $x_0 \in \overline{J_1} \cap \overline{J_2}$, $(Tg)(x_0) = (Tf)(x_0)$.
But g only depends on $x_0, f(x_0)$ and $f'(x_0)$;

$$(Tf)(x_0) = F(x_0, f(x_0), f'(x_0)) \ .$$

□

III. Structural form of F (in (3))

There are $H : \mathbb{R} \to \mathbb{R}_{>0}$ and $K : \mathbb{R} \to \mathbb{R}$ with

$$K(uv) = K(u)K(v)\,, \quad K(u) = 0 \Leftrightarrow u = 0$$

such that

$$(Tf)(x) = H(f(x))/H(x)\ K(f'(x))\,; \quad f \in C^1(\mathbb{R}),\ x \in \mathbb{R}\,. \tag{4}$$

Proof.

For $x_0, y_0 \in \mathbb{R}$, $\quad f, g \in C^1(\mathbb{R})$ with $f(y_0) = x_0, \; g(x_0) = y_0$

$$T(f \circ g)(x_0) = (Tf)(y_0)(Tg)(x_0) = (Tg)(x_0)(Tf)(y_0) = T(g \circ f)(y_0) \quad (5)$$

means $\quad F(x_0, x_0, f'(y_0)g'(x_0)) = F(y_0, y_0, g'(x_0)f'(y_0))$.
Hence $\quad F(x_0, x_0, u) = F(y_0, y_0, u) =: K(u)$ for all $u \in \mathbb{R}$,
independently of $x_0, y_0 \in \mathbb{R}$.

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independently of $x_0, y_0 \in \mathbb{R}$.
$x_0 = y_0$ yields $\quad K(uv) = K(u)K(v)$. $\quad$ By (5) also

$$\begin{aligned} K(f'(y_0)g'(x_0)) &= F(x_0, y_0, g'(x_0))F(y_0, x_0, f'(y_0)) , \\ F(x_0, y_0, u) &= \tfrac{K(uv)}{F(y_0, x_0, v)} = \tfrac{K(u)}{F(y_0, x_0, 1)} . \end{aligned}$$

Proof.

For $x_0, y_0 \in \mathbb{R}$, $f, g \in C^1(\mathbb{R})$ with $f(y_0) = x_0,\ g(x_0) = y_0$

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Let $G(x_0, y_0) = 1/F(y_0, x_0, 1)$ and $H(y) := G(0, y)$. Then
$G(x, y) = G(x, 0)G(0, y) = H(y)/H(x)$. In fact,

$$H(x_0) = G(0, x_0) = F(0, x_0, 1) = T(\mathrm{Id} + x_0)(0) .$$

IV. Smoothness of K and H

Sierpinski, Banach: Assume $K : \mathbb{R} \to \mathbb{R}$ is measurable, $\not\equiv 0$,
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Then $K(u) = |u|^p\{\mathrm{sgn}(u)\}$ for a suitable p.

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We use this in

$$(Tf)(x) = H(f(x))/H(x)\ K(f'(x)) \tag{4}$$

$$\varphi(x) := \frac{H(2x)}{H(x)} = K(2)^{-1}\ T(2\,\mathrm{Id})(x) \qquad \text{is continuous on } \mathbb{R}\ ,$$

$$\frac{H(b)}{H(1)} = \frac{H\left(\frac{b}{2^k}\right)}{H\left(\frac{1}{2^k}\right)} \prod_{i=1}^{k} \left(\frac{\varphi\left(\frac{b}{2^i}\right)}{\varphi\left(\frac{1}{2^i}\right)} \right) .$$

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H is pointwise limit of continuous functions, hence measurable. Take $f(x) = x^2/2$. Then $K(x) = H(x)/H(x^2/2)(Tf)(x)$ is measurable and thus $K(u) = |u|^p\{\mathrm{sgn}(u)\}$.

Continuity of H follows from the one of

$$H \circ f(x)/H(x) = (Tf)(x)/K(f'(x))$$

for $f \in C^1(\mathbb{R})$ with $f'(x) \neq 0$: If H would be discontinuous somewhere, it would be "uniformly discontinuous everywhere",

$$\overline{\lim_{y \to c}} H(y)/H(c) \qquad \text{and} \qquad \underline{\lim_{x \to c}} H(x)/H(c)$$

would be independent of c (f = translation from one c to another).

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would be independent of c (f = translation from one c to another). Yields a contradiction for a sequence $x_n \to 0$ with suitably defined function f , $f(x_n) = y_n \to 0$, $H(y_n)/H(x_n) \not\to 1$.

V. C^k-Localization for $T : C^k(\mathbb{R}) \to C^\ell(\mathbb{R})$

Replace f on $x > x_0$ by a Taylor polynomial g of f of degree k to get a $C^k(\mathbb{R})$-function. Localization on intervals then gives

$$(Tf)(x) = F(x, f(x), ..., f^{(k)}(x)) \; ; \quad x \in \mathbb{R}, \; f \in C^k(\mathbb{R}) \; .$$

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gives with $K := F(x_0, x_0, ...)$ and $g^{(k)}(x_0) = t_k \ , \quad f^{(k)}(y_0) = s_k$

$$K(s_1 t_1, s_1 t_2 + t_1^2 s_2, ...) = K(s_1 t_1, s_1^2 t_2 + t_1 s_2, ...) \ .$$

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$$K(s_1 t_1, s_1 t_2 + t_1^2 s_2, ...) = K(s_1 t_1, s_1^2 t_2 + t_1 s_2, ...) \ .$$

K is independent of the second and further variables: for arbitrary a_2, b_2 and given values s_1, t_1 (first derivatives) solve

$$s_1 t_2 + t_1^2 s_2 = a_2 \ , \quad s_1^2 t_2 + t_1 s_2 = b_2$$

for (t_2, s_2): possible if $s_1 t_1 \notin \{0, 1, -1\}$. Get as before

$$\begin{aligned} (Tf)(x) &= \tilde{F}(x, f(x), f'(x)) \\ &= H(f(x))/H(x) \ K(f'(x)) \ . \end{aligned}$$

VI. Higher Smoothness of H $(\mathrm{Im}\, T \subset C^\ell(\mathbb{R}))$

Take $f = 2\,\mathrm{Id}$ then $H(2x)/H(x) = K(2)^{-1}(Tf)(x)$ is in $C^\ell(\mathbb{R})$. Show $H \in C^\ell(\mathbb{R})$. Take logarithm and apply the

Lemma 6

For $0 < a \leq 1$, $L \in C(\mathbb{R})$ s. th. $\psi(x) := L(x) - aL(x/2)$ is in $C^1(\mathbb{R})$. Then $L \in C^1(\mathbb{R})$.

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Iteration technique for proof:

$$\sum_{j=0}^{n-1} a^j \psi\left(\frac{x}{2^j}\right) = L(x) - a^n L\left(\frac{x}{2^n}\right)$$

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yields with $a^n L\left(\frac{x}{2^n}\right) \to 0$ for $a < 1$ and $\to L(0)$ for $a = 1$ that

$$\lim_{x \to 0} \frac{L(x) - L(0)}{x} = \frac{\psi'(0)}{1 - a/2} .$$

An n-dimensional analogue

$C_b^1(\mathbb{R}^n,\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R}^n \mid f \in C^1,\ \mathrm{Im}\, f \subseteq H,\ H \subseteq R^n$ open half-space$\}$

A map $T : C^1(\mathbb{R}^n,\mathbb{R}^n) \to C(\mathbb{R}^n, L(\mathbb{R}^n,\mathbb{R}^n))$ is *non-degenerate* if

$$\exists\ x_1 \in \mathbb{R}^n\ ,\ f \in C_b^1(\mathbb{R}^n,\mathbb{R}^n) \quad \det\,(Tf)(x_1) \neq 0\ .$$

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A map $T : C^1(\mathbb{R}^n, \mathbb{R}^n) \to C(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^n))$ is *non-degenerate* if

$$\exists\ x_1 \in \mathbb{R}^n\ ,\ f \in C_b^1(\mathbb{R}^n, \mathbb{R}^n) \quad \det\,(Tf)(x_1) \neq 0\ .$$

It is *locally surjective* if

$$\exists\ x_2 \in \mathbb{R}^n\ \{(Tf)(x_2) \mid f \in C^1(\mathbb{R}^n, \mathbb{R}^n),\ f(x_2) = x_2,\ \det f'(x_2) \neq 0\} \supseteq GL(n).$$

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$$\text{We denote} \quad GL(n)_+ = \left\{ \begin{array}{cc} GL(n) & n \text{ odd} \\ \{u \in GL(n) \mid \det u > 0\} & n \text{ even} \end{array} \right\}.$$

Theorem 7

Assume that $T : C^1(\mathbb{R}^n, \mathbb{R}^n) \to C(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^n))$ is non-degenerate and locally surjective and satisfies the chain rule equation

$$T(f \circ g)(x) = ((Tf) \circ g)(x) \circ Tg(x) \; ; \; f, g \in C^1(\mathbb{R}^n, \mathbb{R}^n), \; x \in \mathbb{R}^n \, .$$

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Then there is $p \geq 0$ and $H \in C(\mathbb{R}^n, GL(n))$ and – for $n \in \mathbb{N}$ even – a diagonal matrix J with diagonal entries ± 1 such that for all $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\det f'(x) \neq 0$ $(Tf)(x)$ is given by either

$$\{\mathrm{sgn}(\det f'(x))\} |\det f'(x)|^p \; (H \circ f)(x) \circ f'(x) \circ H(x)^{-1}$$

or

$$\left\{ \begin{array}{ll} \{\mathrm{sgn}(\det f'(x))\} |\det f'(x)|^p \; (H \circ f)(x) \circ H(x)^{-1} & \text{if } f'(x) \in GL(n)_+ \\ \{\mathrm{sgn}(\det f'(x))\} |\det f'(x)|^p \; (H \circ f)(x) \circ J \circ H(x)^{-1}, & \text{if } f'(x) \notin GL(n)_+ \end{array} \right\}$$

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Conversely, any such operator – with $p > 0$ if $\{\mathrm{sgn}(\det f'(x))\}$ is present – satisfies the chain rule.
If additionally $T(2\,\mathrm{Id}) = 2\,\mathrm{Id}$ is constant, $H = 1$ and $p = 0$ or $p = 1/n$ so that $Tf = \pm f'$.

Localization Step: $(Tf)(x_0) = F(x_0, f(x_0), f'(x_0))$

For $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $k \in \{0, ..., n\}$ define

$$h_k(x) = f(x_{01}, ..., x_{0k}, x_{k+1}, ..., x_n) + \sum_{j=1}^{k} (x_j - x_{0j}) \frac{\partial f}{\partial x_j}(x_{01}, ..., x_{0k}, x_{k+1}, ..., x_n) \ ,$$

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$$g_k(x) = \left\{ \begin{array}{ll} h_{k-1}(x) & x_k < x_{0k} \\ h_k(x) & x_k \geq x_{0k} \end{array} \right\}$$

is in C^1 and

$$(Tf)(x_0) = (Th_0)(x_0) = (Th_1)(x_0) = \ldots = (Th_n)(x_0) = (Th)(x_0) .$$

The chain rule means for F

$$F(x_0, z_0, u \circ v) = F(y_0, z_0, u) \circ F(x_0, y_0, v) \; .$$

Let $K_{x_0}(u) := F(x_0, x_0, u)$. For $u = \lambda \,\mathrm{Id}$, $K(\lambda \,\mathrm{Id}) = K_{x_0}(\lambda \,\mathrm{Id})$ is independent of x_0 and measurable in λ. Further

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If $K_{x_0} \mid_{SL(n)}$ is trivial, get second form of T in Theorem 7.
Else K_{x_0} is an automorphism of $GL(n)$, and of the form

$$K_{x_0}(u) = \chi(u) \quad H(x_0) \circ u \circ H(x_0)^{-1} ,$$

where $\chi : GL(n) \to \mathbb{R}$ is a character independent of x_0.