# On the maximal distance between two convex bodies 

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(joint work with Márton Naszódi)

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## Banach-Mazur distance

## Definition

$K=-K, L=-L \subset \mathbb{R}^{n}$ convex o-symmetric bodies.

$$
d_{\mathrm{BM}}(K, L)=\inf \{\lambda>0: K \subset T(L) \subset \lambda K\},
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infimum: over all $T \in G L_{n}$.

## Question: What is the maximum? When is it attained?

John (1948)

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If $\mathscr{E}$ is the largest volume ellipsoid in $K$ then $K \subset \sqrt{n} \mathscr{E}$.
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There is a universal constant $c$ s.t. $\forall n$ there are $K, L \subset \mathbb{R}^{n}$ symmetric convex bodies with $d_{\mathrm{BM}}(K, L) \geq c n$.

## B-M distance non-symmetrical case

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d_{\mathrm{BM}}(K, L) \leq c n^{\frac{4}{3}} \log ^{\alpha} n .
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## When is it attained?

For which pair of bodies $K$ and $L$ does $d(K, L)=n$ hold?

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If $d(K, L)=n$ then $K$ or $L$ is a simplex.

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## John's Position

## Definition

$K$ is in John's position in $L$ if $K \subseteq L$ and

$$
\begin{gathered}
I_{n}=\sum_{i=1}^{m} a_{i} u_{i} \otimes v_{i} \\
0=\sum_{i=1}^{m} a_{i} u_{i}=\sum_{i=1}^{m} a_{i} v_{i} \\
\left\langle u_{i}, v_{i}\right\rangle=1 \text { for all } i=1, \ldots, m .
\end{gathered}
$$

for some

$$
\begin{aligned}
\left\{u_{i}\right\} & \subseteq \partial L \cap \partial K, \\
\left\{v_{i}\right\} & \subseteq \partial L^{\circ} \cap \partial K^{\circ}, \\
\left\{a_{i}\right\} & \subseteq \mathbb{R}^{+} .
\end{aligned}
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## Maximal distance

## Theorem (CHJ, M. Naszódi (2009))

Let $K$ be a convex body, and $D$ be a strictly convex body or a smooth convex body in $\mathbb{R}^{n}$, with $K \subseteq D$ in John's position. Assume that the smallest negative homothet of $K$ containing $D$ is $-n K$. Then $K$ is a simplex.

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## Corollary

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Can we drop the strict convexity or smoothness conditions on $D$ ? NO!

## Is strict convexity (or smoothness) necessary?

Can we drop the strict convexity or smoothness condition on $D$ ? NO!


## Idea behind the proof

Gordon, Litvak, Meyer, Pajor (2000)
If $K$ is in a position of maximal volume inside $L$ then there is a translation $z \in \mathbb{R}^{n}$ such that $K-z$ is in John's position in $L-z$.

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C.H.J., M.N. (2009)
$K, D \subset \mathbb{R}^{n}$ convex bodies, $D$ smooth or strictly convex.

- $K \subseteq D$ in John's position, and
- $D \subseteq-n K$ minimal homothet.

Then $K=\Delta^{n}$.

## From [GLMP]

$$
\begin{aligned}
\|-x\|_{K} & =\left\|-\sum_{i=1}^{m} a_{i}\left\langle x, v_{i}\right\rangle u_{i}\right\|_{K} \\
& =\left\|-\sum_{i=1}^{m} a_{i}\left\langle x, v_{i}\right\rangle u_{i}+\left(\sum_{i=1}^{m} a_{i} u_{i}\right) \max _{j=1}^{m}\left\langle x, v_{j}\right\rangle\right\|_{K} \\
& \leq \sum_{i=1}^{m}\left(\max _{j=1}^{m}\left\langle x, v_{j}\right\rangle-\left\langle x, v_{i}\right\rangle\right) a_{i}\left\|u_{i}\right\|_{K} \\
& =\max _{j=1}^{m}\left\langle x, v_{j}\right\rangle \sum_{i=1}^{m} a_{i}-\left\langle x, \sum_{i=1}^{m} a_{i} v_{i}\right\rangle \\
& =n \max _{j=1}^{m}\left\langle x, v_{j}\right\rangle \leq n
\end{aligned}
$$

## Geometric interpretation

For every $x \in \partial L \cap(-n \partial K)$ i.e. for every $x \in \partial L$ s.t. $\|-x\|_{K}=n$ we have that:

- $\frac{-x}{n} \in \operatorname{conv}\left\{u_{i}: i \in B\right\}, B \subset\{1, \ldots, m\}$
- $\left\langle x, v_{i}\right\rangle=1$ for all $i \in B^{C}$.

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Under strict convexity assumption:

- $x=u_{i}$ for some $1 \leq i \leq m$.
- $\partial D \cap(-n \partial K)=\left\{u_{i}: i=1, \ldots, r\right\}$ for some $1 \leq r \leq m$.
- $\left\langle u_{i}, v_{k}\right\rangle=-1 / n$ for every $i \neq k$.


## Thank you

