

Minkowski valuations intertwining the special linear group

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Valuation

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whenever $K, L, K \cup L \in \mathcal{K}_o^n$.

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What valuations “behave nicely” with respect to the action of the general linear group?

Homogeneity, Co-, and Contravariance

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There exists $q \in \mathbb{R}$ such that

$$Z(\lambda K) = \lambda^q Z K, \quad \lambda > 0.$$

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SL(n) Covariance

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$$Z(\phi K) = \phi Z K, \quad \phi \in \mathrm{SL}(n).$$

SL(n) Contravariance

$$Z(\phi K) = \phi^{-t} Z K, \quad \phi \in \mathrm{SL}(n).$$

Known Classification Results

What valuations are $\mathrm{SL}(n)$ co- or contravariant and homogeneous?

$\langle A, + \rangle$	Nontrivial Answer	
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$O(n)$ covariant valuations

Alesker, Bernig, Hadwiger, Hug, Kiderlen, McMullen,
Schneider, F. Schuster, R. Schuster, Wannerer, ...

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Question

Are the homogeneity assumptions necessitated only by the techniques used in the proofs?

Minkowski valuations

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Minkowski addition

$$K + L = \{x + y : x \in K, y \in L\}$$

$\text{SL}(n)$ covariant Minkowski valuations

Theorem (H., '10)

$Z : \mathcal{K}_o^n \rightarrow \langle \mathcal{K}_o^n, + \rangle$ is a continuous $\text{SL}(n)$ covariant Minkowski valuation
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$\exists c_1, \dots, c_4 \geq 0$ such that

$$ZK = c_1K + c_2(-K) + c_3\Gamma_+K + c_4\Gamma_+(-K).$$

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Support function

$$h_K(x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n.$$

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Asymmetric centroid body

$$h_{\Gamma_+ K}(x) = \int_K (x \cdot y)_+ dy,$$

where $(x \cdot y)_+ = \max\{x \cdot y, 0\}$.

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Counterexample

$$h_{\Gamma_o K}(x) = \int_{\text{conv}\{o, K\} \setminus K} |x \cdot y| dy$$

Wannerer '10 classified all continuous, homogeneous, $SL(n)$ covariant valuations on \mathcal{K}^n .

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- There exists a version without continuity assumptions for $Z : \mathcal{P}_o^n \rightarrow \langle \mathcal{K}_o^n, + \rangle$.

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$$Z K = c \Pi K.$$

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Projection body

$$h_{\Pi K}(u) = \text{vol}(K|u^\perp), \quad u \in S^{n-1}.$$

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$$\implies (\text{Induction}) \quad ZK = c_1K + c_2(-K) \text{ if } \dim K < n$$

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It suffices to prove

$$h_{Z(sT)} - c_1 h_{sT} - c_2 h_{-sT} = c_3 h_{\Gamma_+(sT)} + c_4 h_{\Gamma_+(-sT)}$$

for $s > 0$ and $T = \text{conv}\{o, e_1, \dots, e_n\}$.

Proof of the covariant case

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$$(sT) \cap H_\lambda = s\phi_\lambda(T \cap e_1^\perp)$$

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$$\implies Z(sT) + Z(s\phi_\lambda(T \cap e_1^\perp)) = Z(s\phi_\lambda T) + Z(s\psi_\lambda T).$$

Proof of the covariant case

Critical functional equation

$$f(s, x) = \lambda^{-\frac{1}{n}} f(s\lambda^{\frac{1}{n}}, \phi_\lambda^t x) + (1 - \lambda)^{-\frac{1}{n}} f(s(1 - \lambda)^{\frac{1}{n}}, \psi_\lambda^t x).$$

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$$g(s) := f(s^{\frac{1}{n}}, e_3)$$

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$$s = (x + y)^{\frac{1}{n}} \text{ and } \lambda = x(x + y)^{-1}$$

$$\implies (x + y)^{-\frac{1}{n}} g(x + y) = x^{-\frac{1}{n}} g(x) + y^{-\frac{1}{n}} g(y)$$

$$\implies g(s) = s^{1+\frac{1}{n}} g(1)$$

$$\implies f(s, e_3) = s^{n+1} f(1, e_3)$$

Proof of the covariant case

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- This homogeneity property is the key in the proof of

$$f(s, x) = c_3 h_{\Gamma_+(sT)}(x) + c_4 h_{\Gamma_+(-sT)}(x), \quad (s, x) \in \mathbb{R} \times (\mathbb{R}^2 \times \{0\}).$$

Proof of the covariant case

Critical functional equation

$$f(s, x) = \lambda^{-\frac{1}{n}} f(s\lambda^{\frac{1}{n}}, \phi_\lambda^t x) + (1 - \lambda)^{-\frac{1}{n}} f(s(1 - \lambda)^{\frac{1}{n}}, \psi_\lambda^t x).$$

- This homogeneity property is the key in the proof of

$$f(s, x) = c_3 h_{\Gamma_+(sT)}(x) + c_4 h_{\Gamma_+(-sT)}(x), \quad (s, x) \in \mathbb{R} \times (\mathbb{R}^2 \times \{0\}).$$

- Two functions which satisfy the above functional equation and are equal on $\mathbb{R} \times (\mathbb{R}^2 \times \{0\})$ coincide on $\mathbb{R} \times \mathbb{R}^n$.