

# Random embedding of $\ell_p^n$ into $\ell_r^N$

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# Local theory of Banach spaces.

Banach Mazur distance :

$$d(X, Y) = \inf \{ \|T\| \|T^{-1}\|, \quad T : X \rightarrow Y \text{ isomorphism} \}$$

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*Let  $E$  be a normed space of dimension  $N$ ,  $(\mathbb{R}^N, \|\cdot\|)$  such that  $\|x\| \leq |x|_2$ , and*

$$M = \int_{S^{N-1}} \|\theta\| d\sigma(\theta).$$

*$\forall \varepsilon \in (0, 1)$ , if*

$$n \leq c N M^2 \varepsilon^2 / \log(3/\varepsilon) \quad \text{then} \quad \ell_2^n \stackrel{1+\varepsilon}{\hookrightarrow} E.$$

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Proofs : random methods that can be described through the use of Gaussian operators,

$$G = (g_{ij}) : \ell_2^n \rightarrow \ell_1^N \text{ where } g_{ij} \sim \mathcal{N}(0, 1).$$

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Main properties :

1) if  $\theta, \theta_1, \dots, \theta_n$  are i.i.d. standard  $p$ -stable then for every  $\alpha_1, \dots, \alpha_n$ ,

$$\sum \alpha_i \theta_i \sim \left( \sum |\alpha_i|^p \right)^{1/p} \theta$$

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Consequence :  $\ell_p^n \xrightarrow{1} L_1$

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$X$  is of stable type  $p$  iff for some (every)  $r < p$ , there exists  $C > 0$  such that for every finite collection of vectors

$x_1, \dots, x_n$

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- **Maurey-Pisier** [’76] *Let  $X$  be a Banach space of infinite dimension,  $\forall n \in \mathbb{N}, \forall \varepsilon > 0, \ell_p^n \xrightarrow{1+\varepsilon} X$  iff  $X$  is **not** of stable type  $p$ .*

# Other embeddings.

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| $\ell_2^n$                                                                                                                                                    | $\ell_p^n$ for $1 < p < 2$                                                                                                                                                                                    |
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*Let  $0 < r < p < 2$  and  $\frac{2p}{p+2} < r \leq 1$ ,*

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Definition of  $S$  independent of  $r$ . Remarks.

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Second idea :

$Y = \pm e_i$  with probability  $1/2N$ ,  $Y_j$  independent copies of  $Y$ ,  $\lambda_j$  independent exponential r.v. and  $\Gamma_j = \sum_{i=1}^j \lambda_i$ , let

$$\tilde{\Theta} = \sum_{j \geq 1} \Gamma_j^{-1/p} Y_j$$

**Theorem** [LePage, Woodroffe, Zinn '81]  $\tilde{\Theta}$  is a  $p$ -stable random vector and has the same distribution than

$$\frac{s_p}{N^{1/p}} \sum_{\ell=1}^N \theta_\ell e_\ell$$

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Let  $\tilde{\Theta}_1, \dots, \tilde{\Theta}_n$  independent copies of  $\tilde{\Theta}$  and

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$$T : \ell_p^n \rightarrow \ell_1^N$$
$$\alpha \mapsto \frac{\sigma_p}{N^{1/q}} \sum_{i=1}^n \sum_{j \geq 1} \alpha_i j^{-1/p} Y_{ij}$$

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**Key properties :**

1)  $\mathbb{E}|T\alpha|_1 \simeq \mathbb{E}|\tilde{T}\alpha|_1 = |\alpha|_p$

2) **Concentration** of  $|T\alpha|_1$  around its mean ?

**YES**

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**Key properties :**

- 1)  $|\mathbb{E}|T\alpha|_1 - |\alpha|_p| \leq D_p \left(\frac{n}{N}\right)^{1/q} |\alpha|_p \rightarrow \text{P[’83]}$
- 2) **Concentration** of  $|T\alpha|_1$  around its mean

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Independence of the columns of the matrix  $T$ , but not of all entries.

**Theorem** Let  $1 < p < 2$ ,

$$\mathbb{P} \left\{ \forall \alpha \in S_p^{n-1}, c(p)^{1/\eta} \leq |T\alpha|_1 \leq C(p) \right\} \geq 1 - c \exp(-c(p)n)$$

# Net argument

- Fix  $\alpha \in \mathcal{S}_p^{n-1}$ ,

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$$1 - t \leq |T\alpha|_1 \leq 1 + t$$

$$t = \varepsilon \in (0, 1), \quad N = C(\varepsilon) n$$

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- But  $\forall t > 0$

$$|T\alpha|_1 \leq 1 + t$$

→ Concentration around the mean is **useful** for **almost isometric result** and for **upper bound** of  $\|T\|$ .

# Small ball

**Theorem**  $\forall \eta > 0$  and integers  $N = (1 + \eta)n$ ,

$$\mathbb{P} \{ \exists \alpha \in \mathcal{S}_p^{n-1}, |T\alpha|_1 \leq c(p)^{1/\eta} \} \leq c \exp(-c_p n)$$

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**Remarks :**

- In the case  $p = 2$  and  $|\cdot|_2$ , we would be studying the **smallest singular value** of  $T : \ell_2^n \rightarrow \ell_2^N$
- **Random matrices with independent entries :**  
Schechtman [’04] :  $\pm 1$  entries,  
Litvak, Pajor, Rudelson, Tomczak-Jaegermann [’05] :  
general strategy, **finite moments** of the entries  
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- Decomposition of the sphere into two subsets on which you prove differently the small ball estimate

# Almost sparse vectors

$$\mathbb{P} \left\{ \left| |T\alpha|_1 - |\alpha|_p \right| \geq t \right\} \leq 2 \exp(-b_p N t^q)$$

- Assume  $\alpha \in S_p^{n-1}$  is sparse i.e.  $|\text{supp}(\alpha)| \leq \delta n$ .

$$\begin{array}{l} \text{ALMOST ISOMETRIC} \\ C n \times n \\ (1 + \eta) n \times \delta n \end{array}$$

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→ the case of  $AS(\delta, \rho)$

# Non-Almost sparse vectors

Basic property :

$\exists I \subset \{1, \dots, n\}$  such that  $|I| \geq \frac{1}{2} \delta n \rho^p$  and

$$\forall k \in I, \frac{\rho}{(2n)^{1/p}} \leq |\alpha_k| \leq \frac{1}{(\delta n)^{1/p}}$$

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**Example :** take  $\alpha = (\frac{1}{n^{1/p}}, \dots, \frac{1}{n^{1/p}})$  then

$$\begin{aligned} T\alpha &= \frac{\sigma_p}{N^{1/q}} \sum_{i=1}^n \sum_{j \geq 1} \alpha_i j^{-1/p} Y_{ij} \\ &\simeq \frac{1}{N} \sum_{i=1}^n \sum_{j \geq 1} j^{-1/p} Y_{ij} \end{aligned}$$

We would like to understand  $\mathbb{P}\{|T\alpha|_1 \leq t\}$  ?

# Multi-dimensional Esseen type inequality

**Theorem** Let  $X$  be a random vector in  $\mathbb{R}^N$ , such that the function

$$\xi \mapsto \mathbb{E} \exp(i\langle \xi, X \rangle) \in L_1(\mathbb{R}^N).$$

Let  $K$  be a compact, star-shape subset of  $\mathbb{R}^N$ .

Then for any  $t > 0$

$$\mathbb{P} \{ \|X\|_K \leq t \} \leq |K| \left( \frac{t}{2\pi} \right)^N \int_{\mathbb{R}^N} |\mathbb{E} \exp(i\langle \xi, X \rangle)| d\xi.$$

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- We generalize the classical Esseen inequality to the multi-dimensional case and to any norms.
- The proof is an application of Fourier analysis.

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Apply it to

$$X = N T \alpha = \sum_{i=1}^n \sum_{j \geq 1} j^{-1/p} Y_{ij}$$

and  $K = N B_1^N$  :  $\rightarrow \|X\|_K = |T \alpha|_1$  and  $|K| \simeq C^N$

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$$\mathbb{P}\{|T\alpha|_1 \leq t\} \leq C^N \left(\frac{t}{2\pi}\right)^N \int_{\mathbb{R}^N} |\mathbb{E} \exp(i\langle \xi, X \rangle)| d\xi.$$

Study

$$\int_{\mathbb{R}^N} |\mathbb{E} \exp(i\langle \xi, X \rangle)| d\xi$$

$\rightarrow$  This is delicate but doable :  $\leq C^N$

# Random embedding of $\ell_p^n$ into $\ell_r^N$

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