Concentration of measure phenomenon and eigenvalues of Laplacian

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0. Introduction

Purpose.

Concentration of measure
(cf. Lévy and Milman)

$\overset{?}{\Longleftrightarrow}$

Eigenvalues of Laplacian

1. Concentration of measure

$X = (X, d_X, \mu_X)$: an **mm-space**

(a **m**etric **m**easure space)

$\overset{\text{def}}{\Longleftrightarrow}$ (X, d_X) : a cplt. sep. met. sp.,

μ_X : a Borel prob. meas. on X

Example.

M : a closed Riem. mfd.

d_M : the Riem. dist.

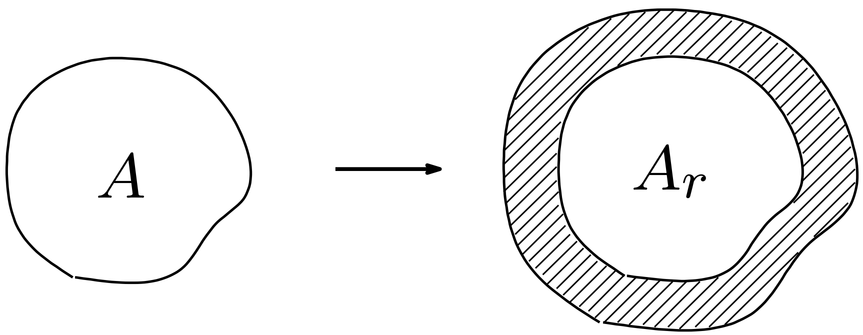
$\mu_M := vol(\cdot)/\, vol(M)$

Concentration of measure

For $\forall r, \kappa > 0$, estimate

$$\alpha_X(r;\kappa) := \sup\{\mu_X(X \setminus A_r) \mid A \subseteq X : \mu_X(A) \geq \kappa\}$$

from above, where A_r : the r-nbd. of A.



$\mu_X(A) \geq \kappa$ $\mu_X(X \setminus A_r)$?

$\{X_n\}_{n=1}^{\infty}$: mm-sps.

$\{X_n\}_{n=1}^{\infty}$ **is a Lévy family (cf. Gromov-V. Milman).**

$\overset{\text{def}}{\Longleftrightarrow} \lim_{n\to\infty} \alpha_{X_n}(r; 1/2) = 0$ for $\forall r > 0$.

$\Longleftrightarrow \lim_{n\to\infty} \alpha_{X_n}(r; \kappa) = 0$ for $\forall r > 0$ $\forall \kappa > 0$.

($\alpha_X(r;\kappa) = \sup\{\mu_X(X \setminus A_r) \mid A \subseteq X : \mu_X(A) \geq \kappa\}$)

$\Longleftrightarrow \forall r, \kappa > 0$ $\forall A_n \subseteq X_n : \mu_{X_n}(A_n) \geq \kappa$,

$$\lim_{n\to\infty} \mu_{X_n}((A_n)_r) = 1.$$

M : a closed Riem. mfd.

Eigenvalues of $\triangle_M$: $0 < \lambda_1(M) \le \lambda_2(M) \le \cdots$.

Theorem (Gromov-V. Milman, 83').

M : a closed Riem. mfd.

$$\Rightarrow \alpha_M(r; 1/2) \le 2 \exp(-\sqrt{\lambda_1(M)}\, r/3).$$

" $\lambda_1(M_n) \to +\infty \Rightarrow M_n$ **form a Lévy family"**

e.g.,

$\{\mathbb{S}^n\}_{n=1}^{\infty}, \{\mathbb{R}P^n\}_{n=1}^{\infty}, \{SO(n)\}_{n=1}^{\infty} \cdots$.

Theorem (E.Milman, '08-'09).

M : a closed Riem. mfd. s.t. $Ric_M \geq 0$

$\Rightarrow \exists C, \varepsilon_0 > 0$: univ. consts. s.t.

$$\alpha_M(r; 1/2) < \varepsilon_0 \Rightarrow \lambda_1(M) \geq C r^{-2}.$$

The proof relies on a consequence of Geometric measure theory and Riemannian geometry (concavity of isoperimetric profile under $Ric \geq 0$).

Corollary (Gromov-V. Milman, E. Milman).

M_n : closed Riem. mfds. s.t. $Ric_{M_n} \geq 0$.

Then, the following conditions (1) and (2) are equiv.

(1) $\{M_n\}_{n=1}^{\infty}$ is a Lévy family.

(2) $\lim_{n\to\infty} \lambda_1(M_n) = +\infty$.

2. Main theorem

Theorem.

$k \in \mathbb{N}$,

M_n : closed Riem. mfds. s.t.

$Ric_{M_n} \geq 0$ and $\sup_{n\in\mathbb{N}} \mathrm{Diam}\, M_n < +\infty$.

If $\lim_{n\to\infty} \lambda_k(M_n) = +\infty$, then $\{M_n\}_{n=1}^{\infty}$ is a Lévy family.

Corollary.

M_n : closed Riem. mfds. s.t.

$Ric_{M_n} \geq 0$ and $\sup_{n \in \mathbb{N}} \mathrm{Diam}\, M_n < +\infty$.

Then, the following conditions (1),(2),(3) are equiv.

(1) $\{M_n\}_{n=1}^{\infty}$ is a Lévy family.

(2) $\lim_{n\to\infty} \lambda_k(M_n) = +\infty$ for some $k \in \mathbb{N}$.

(3) $\lim_{n\to\infty} \lambda_k(M_n) = +\infty$ for all $k \in \mathbb{N}$.

Observation

"$\lim_{n\to\infty} \lambda_k(M_n) = +\infty \Rightarrow \lim_{n\to\infty} \lambda_1(M_n) = +\infty$"

Theorem.

M : a closed Riem. mfd. s.t. $Ric_M \geq 0$

$$\Rightarrow \lambda_2(M) \leq C\lambda_1(M)\{\log(1+\lambda_1(M))\}^2,$$

where $C > 0$ depends only on $\operatorname{Diam} M$.

Proposition. If we can remove $\sup_{n\in\mathbb{N}} \mathrm{Diam}\, M_n < +\infty$ in Main Thm., then we have the following:

M : a closed Riem. mfd. s.t. $Ric_M \geq 0$

$$\Rightarrow \lambda_k(M) \leq C_k \lambda_1(M),$$

where $C_k > 0$ depends only on $k \in \mathbb{N}$.

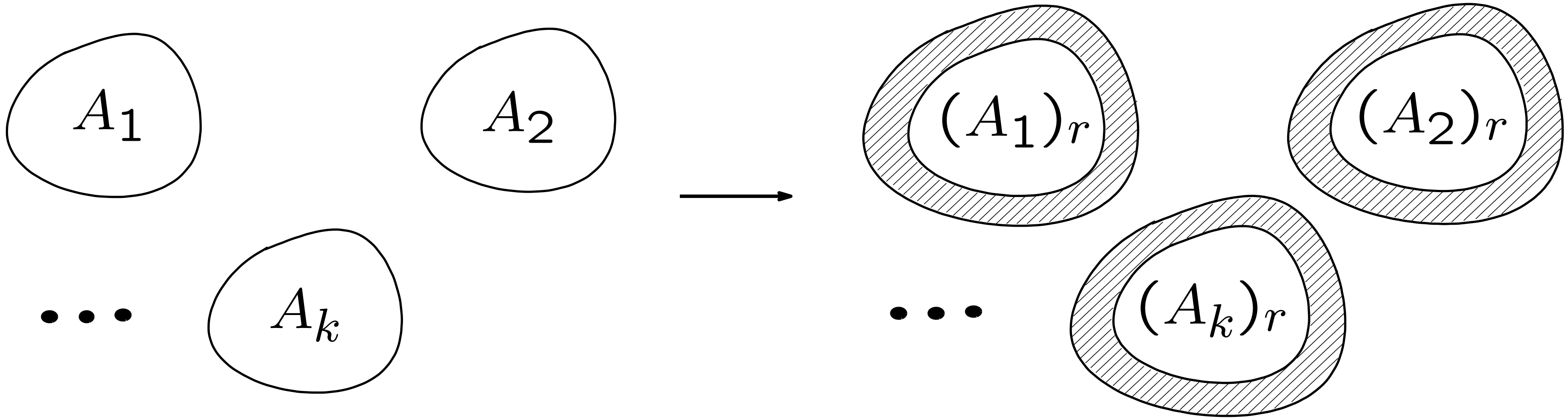
3. Generalization

Concentration around k-sets

For $\forall r, \kappa_1, \cdots, \kappa_k > 0$, estimate

$$\begin{aligned} &\alpha_X(r; \kappa_1, \cdots, \kappa_k) \\ := \ &\mathrm{sup}\Big\{\mu_X\Big(X \setminus \Big(\bigcup_{i=1}^{k} A_i\Big)_r\Big) \mid A_i \subseteq X : \mu_X(A_i) \geq \kappa_i \\ &\qquad\qquad\qquad\qquad\qquad d_X(A_i, A_j) \geq r \ (i \neq j)\Big\} \end{aligned}$$

from above.



$\mu_X(A_i) \geq \kappa_i$ $\qquad$ $\mu_X(X \setminus (\cup_{i=1}^k A_i)_r)$?

$\{X_n\}_{n=1}^\infty$: mm-sps. $\;k \in \mathbb{N}$.

$\{X_n\}_{n=1}^\infty$ **concentrates around** k**-sets.**

$\overset{\text{def}}{\iff} \forall r, \kappa_1, \cdots, \kappa_k > 0$, $\lim_{n\to\infty} \alpha_{X_n}(r; \kappa_1, \cdots, \kappa_k) = 0$.

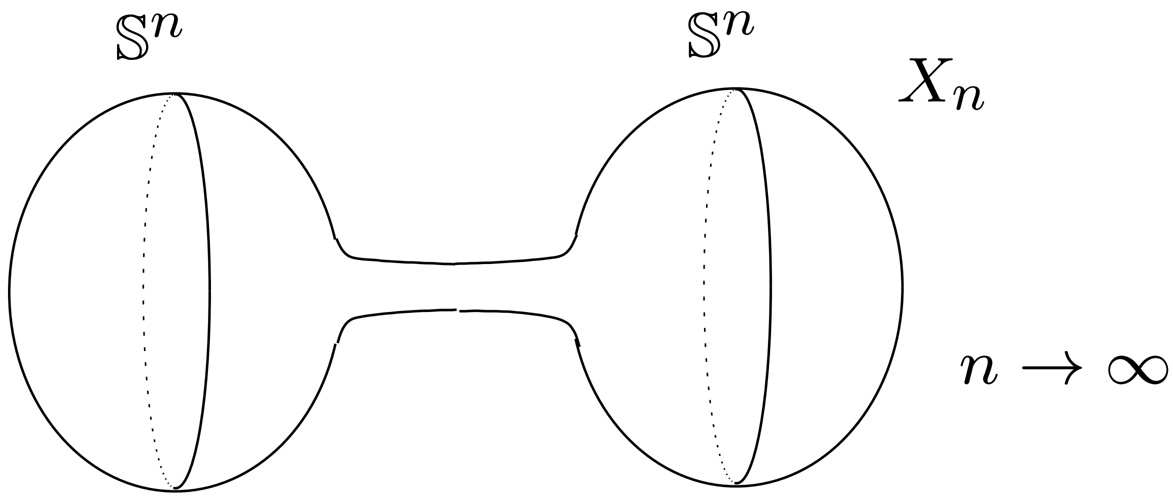
Easy observation

Concentration of measure ($k = 1$)

$\Rightarrow$ Concentration around k-sets

"No converse!"

Example. ($k = 2$)



Theorem (Chung-Grigor'yan-Yau, '97)**.**

$k \in \mathbb{N}$, M : a closed Riem. mfd.

$$\Rightarrow \alpha_M(r; \underbrace{\kappa, \cdots, \kappa}_{k}) \leq \kappa^{-1} e^{1/2} \cdot \exp(-2^{-1}\sqrt{\lambda_k(M)}r)$$

$$\text{for } \forall r, \kappa > 0.$$

Based on the eigenfunctions expansion of the heat kernel.

Corollary (Chung-Grigor'yan-Yau).

$k \in \mathbb{N}$,

M_n : closed Riem. mfds. s.t.

$$\lim_{n\to\infty} \lambda_k(M_n) = +\infty.$$

$\Rightarrow M_n$ conc. around k-sets.

Remark. The above corollary is also mentioned by Gromov in his Green book.

3. Proof of the main thm.

Key Theorem.

$k \in \mathbb{N}$,

X_n : mm-sps. s.t.

BM$(0, \infty)$ ($\Leftarrow Ric \geq 0$) and $\sup_{n \in \mathbb{N}} \operatorname{Diam} X_n < +\infty$.

Then, the following conditions (1) and (2) are equiv.

(1) $\{X_n\}_{n=1}^{\infty}$ is a Lévy family.

(2) $\{X_n\}_{n=1}^{\infty}$ conc. around k-sets.

Idea of the proof of the key thm.

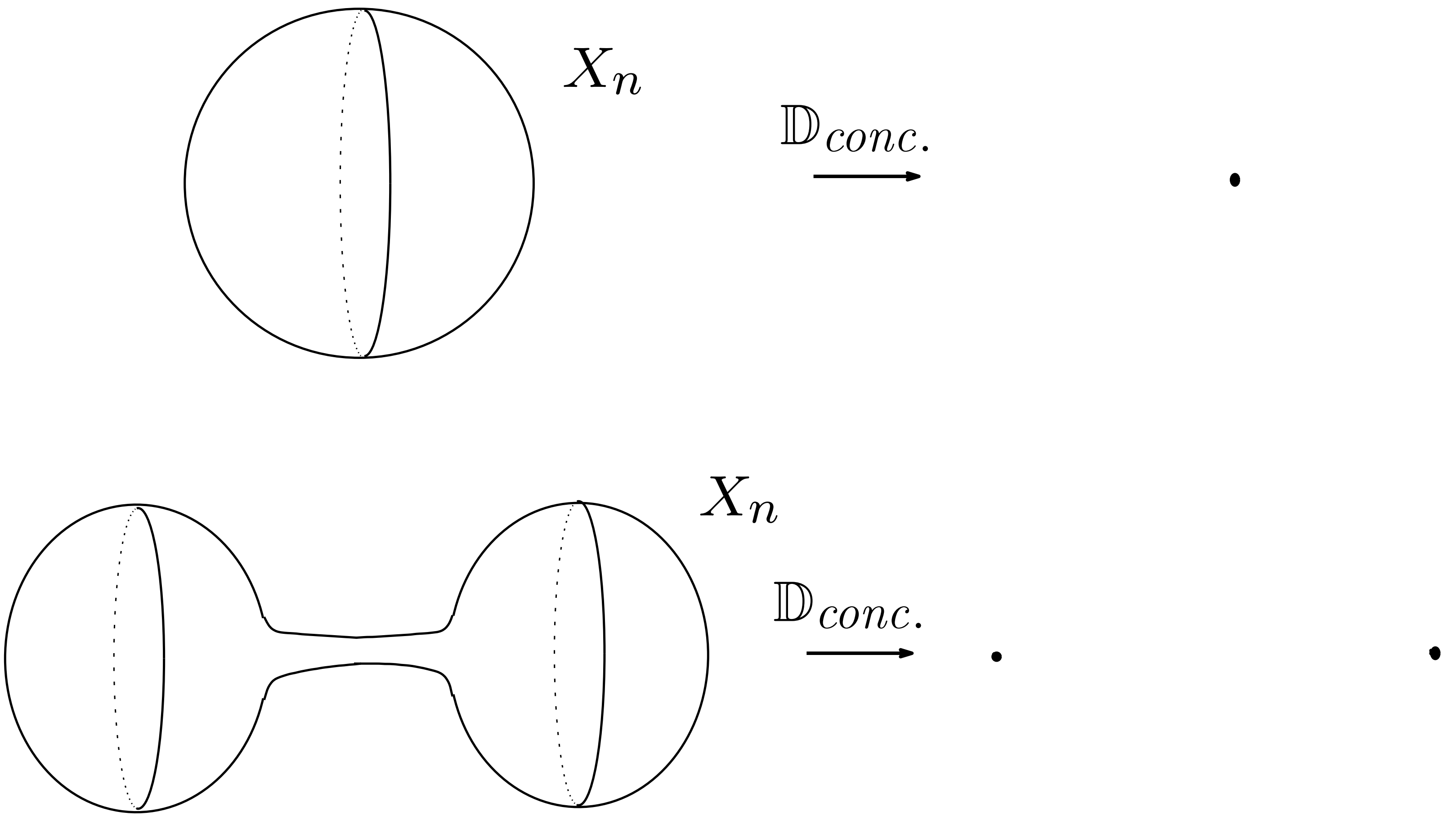
"Convergence of mm-sps. $\Rightarrow$ Conc. of meas.".

We use the dist. $\mathbb{D}_{conc.}(X,Y)$ between mm-sps. introduced by Gromov.

(1) $\{X_n\}_{n=1}^{\infty}$ is a Lévy family. $\iff$ $X_n \xrightarrow{\mathbb{D}_{conc.}} \{*\}$.

(2) Under the diam. assump., Conc. around k-sets $\Rightarrow$ Conv. to a finite sp. X w.r.t. $\mathbb{D}_{conc.}$ s.t. $\#X \leq k$ (cf. Gromov).

(3) Under BM$(0,\infty)$ and the diam. assump., we prove the connectivity of the limit sp.



X_n
$\mathbb{D}_{conc.}$
X_n
$\mathbb{D}_{conc.}$

Question . 1. Can we remove $\sup_{n \in \mathbb{N}} \operatorname{Diam} M_n < +\infty$ in Main Thm.?

2. How about the case for graphs ? (cf. Alon-V. Milman)

3. Can $\mathbb{D}_{conc.}$ capture the asymp. behavior of λ_k ?

Theorem (Fukaya, Cheeger-Colding).

M, M_n : closed Riem. mfds. s.t. $M_n \xrightarrow{\text{mGH}} M$,

$\sup_{n \in \mathbb{N}} \dim M_n < +\infty$, and $Ric_{M_n} \geq K \in \mathbb{R}$.

$\Rightarrow \lim_{n \to \infty} \lambda_k(M_n) = \lambda_k(M)$ for any $k \in \mathbb{N}$.

Remark.

1. mGH topology $>$ topology determined by $\mathbb{D}_{conc.}$

2. Under $\sup_{n \in \mathbb{N}} \dim M_n < +\infty$ and $Ric_{M_n} \geq K \in \mathbb{R}$,

$M_n \xrightarrow{\text{mGH}} M \iff M_n \xrightarrow{\mathbb{D}_{conc.}} M$