Poisson Summation Formula Uniquely Characterizes the Fourier Transform

Dmitry Faifman

Tel-Aviv University

September 16, 2010

The Poisson Summation Formula

- $f : \mathbb{R} \to \mathbb{C}$, $f \in L^2(\mathbb{R})$
- Fourier Transform: $\mathcal{F}f(\omega) = \int_{-\infty}^{\infty} f(t)exp(-2\pi i \omega t)dt$
- Assume f is sufficiently smooth and fast decaying (for example, $f \in \mathcal{S}$). Then we have
  Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \mathcal{F}f(n)$$

- A uniqueness result:
  Theorem (Cordoba 88'). Suppose $\{x_k\}$ and $\{y_k\}$ are two discrete sets in $\mathbb{R}^n$, and for all $f \in \mathcal{S}$, $\sum_k f(x_k) = \sum_k \mathcal{F}f(y_k)$. Then $\{x_k\}$ and $\{y_k\}$ are dual lattices, i.e. there is $A \in SL(n)$ such that $\{x_k\} = A(\mathbb{Z}^n)$ and $\{y_k\} = (A^*)^{-1}(\mathbb{Z}^n)$.
- For $n = 1$, there is only one possibility: $\{x_k\} = \{y_k\} = \mathbb{Z}$

The Poisson Summation Formula

- $f : \mathbb{R} \to \mathbb{C}$, $f \in L^2(\mathbb{R})$
- Fourier Transform: $\mathcal{F}f(\omega) = \int_{-\infty}^{\infty} f(t) exp(-2\pi i \omega t) dt$
- Assume f is sufficiently smooth and fast decaying (for example, $f \in \mathcal{S}$). Then we have
  Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \mathcal{F}f(n)$$

- A uniqueness result:
  Theorem (Cordoba 88'). Suppose $\{x_k\}$ and $\{y_k\}$ are two discrete sets in $\mathbb{R}^n$, and for all $f \in \mathcal{S}$, $\sum_k f(x_k) = \sum_k \mathcal{F}f(y_k)$. Then $\{x_k\}$ and $\{y_k\}$ are dual lattices, i.e. there is $A \in SL(n)$ such that $\{x_k\} = A(\mathbb{Z}^n)$ and $\{y_k\} = (A^*)^{-1}(\mathbb{Z}^n)$.
- For $n = 1$, there is only one possibility: $\{x_k\} = \{y_k\} = \mathbb{Z}$

The Poisson Summation Formula

- $f : \mathbb{R} \to \mathbb{C}$, $f \in L^2(\mathbb{R})$
- Fourier Transform: $\mathcal{F}f(\omega) = \int_{-\infty}^{\infty} f(t) exp(-2\pi i \omega t) dt$
- Assume f is sufficiently smooth and fast decaying (for example, $f \in \mathcal{S}$). Then we have
  Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \mathcal{F}f(n)$$

- A uniqueness result:
  Theorem (Cordoba 88'). Suppose $\{x_k\}$ and $\{y_k\}$ are two discrete sets in $\mathbb{R}^n$, and for all $f \in \mathcal{S}$, $\sum_k f(x_k) = \sum_k \mathcal{F}f(y_k)$. Then $\{x_k\}$ and $\{y_k\}$ are dual lattices, i.e. there is $A \in SL(n)$ such that $\{x_k\} = A(\mathbb{Z}^n)$ and $\{y_k\} = (A^*)^{-1}(\mathbb{Z}^n)$.
- For $n = 1$, there is only one possibility: $\{x_k\} = \{y_k\} = \mathbb{Z}$

The Poisson Summation Formula

- $f : \mathbb{R} \to \mathbb{C}$, $f \in L^2(\mathbb{R})$
- Fourier Transform: $\mathcal{F}f(\omega) = \int_{-\infty}^{\infty} f(t) exp(-2\pi i \omega t) dt$
- Assume f is sufficiently smooth and fast decaying (for example, $f \in \mathcal{S}$). Then we have
  Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \mathcal{F}f(n)$$

- A uniqueness result:
  Theorem (Cordoba 88'). Suppose $\{x_k\}$ and $\{y_k\}$ are two discrete sets in $\mathbb{R}^n$, and for all $f \in \mathcal{S}$, $\sum_k f(x_k) = \sum_k \mathcal{F}f(y_k)$. Then $\{x_k\}$ and $\{y_k\}$ are dual lattices, i.e. there is $A \in SL(n)$ such that $\{x_k\} = A(\mathbb{Z}^n)$ and $\{y_k\} = (A^*)^{-1}(\mathbb{Z}^n)$.
- For $n = 1$, there is only one possibility: $\{x_k\} = \{y_k\} = \mathbb{Z}$

The Poisson Summation Formula

- $f : \mathbb{R} \to \mathbb{C}$, $f \in L^2(\mathbb{R})$
- Fourier Transform: $\mathcal{F}f(\omega) = \int_{-\infty}^{\infty} f(t) exp(-2\pi i \omega t) dt$
- Assume f is sufficiently smooth and fast decaying (for example, $f \in \mathcal{S}$). Then we have
  Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \mathcal{F}f(n)$$

- A uniqueness result:
  Theorem (Cordoba 88'). Suppose $\{x_k\}$ and $\{y_k\}$ are two discrete sets in $\mathbb{R}^n$, and for all $f \in \mathcal{S}$, $\sum_k f(x_k) = \sum_k \mathcal{F}f(y_k)$. Then $\{x_k\}$ and $\{y_k\}$ are dual lattices, i.e. there is $A \in SL(n)$ such that $\{x_k\} = A(\mathbb{Z}^n)$ and $\{y_k\} = (A^*)^{-1}(\mathbb{Z}^n)$.
- For $n = 1$, there is only one possibility: $\{x_k\} = \{y_k\} = \mathbb{Z}$

Some conventions

- $f(-x) = -f(x) \Rightarrow \mathcal{F}f(-\omega) = -\mathcal{F}f(\omega)$
  Thus $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \mathcal{F}f(n)$ is nontrivial only for even functions, and we consider from now on $f \in L^2[0,\infty)$.
- Assume further that $f(0) = \int_0^\infty f(t)dt = 0$.
  PSF now reads: $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \mathcal{F}f(n)$
- Introduce a scaling factor $x > 0$. Then PSF gives

$$\sum_{n=1}^{\infty} \mathcal{F}f(nx) = \frac{1}{x}\sum_{n=1}^{\infty} f(\frac{n}{x}) \qquad (1)$$

- **Question** (V. Milman): Does relation (1) uniquely characterize the Fourier Transform?

Some conventions

- $f(-x) = -f(x) \Rightarrow \mathcal{F}f(-\omega) = -\mathcal{F}f(\omega)$
  Thus $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \mathcal{F}f(n)$ is nontrivial only for even functions, and we consider from now on $f \in L^2[0,\infty)$.
- Assume further that $f(0) = \int_0^\infty f(t)dt = 0$.
  PSF now reads: $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \mathcal{F}f(n)$
- Introduce a scaling factor $x > 0$. Then PSF gives

$$\sum_{n=1}^{\infty} \mathcal{F}f(nx) = \frac{1}{x}\sum_{n=1}^{\infty} f\left(\frac{n}{x}\right) \tag{1}$$

- **Question** (V. Milman): Does relation (1) uniquely characterize the Fourier Transform?

Some conventions

- $f(-x) = -f(x) \Rightarrow \mathcal{F}f(-\omega) = -\mathcal{F}f(\omega)$
  Thus $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \mathcal{F}f(n)$ is nontrivial only for even functions, and we consider from now on $f \in L^2[0,\infty)$.
- Assume further that $f(0) = \int_0^\infty f(t)dt = 0$.
  PSF now reads: $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \mathcal{F}f(n)$
- Introduce a scaling factor $x > 0$. Then PSF gives

$$\sum_{n=1}^{\infty} \mathcal{F}f(nx) = \frac{1}{x}\sum_{n=1}^{\infty} f(\frac{n}{x}) \tag{1}$$

- **Question** (V. Milman): Does relation (1) uniquely characterize the Fourier Transform?

Some conventions

- $f(-x) = -f(x) \Rightarrow \mathcal{F}f(-\omega) = -\mathcal{F}f(\omega)$
  Thus $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \mathcal{F}f(n)$ is nontrivial only for even functions, and we consider from now on $f \in L^2[0,\infty)$.
- Assume further that $f(0) = \int_0^\infty f(t)dt = 0$.
  PSF now reads: $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \mathcal{F}f(n)$
- Introduce a scaling factor $x > 0$. Then PSF gives

$$\sum_{n=1}^{\infty} \mathcal{F}f(nx) = \frac{1}{x} \sum_{n=1}^{\infty} f(\frac{n}{x}) \tag{1}$$

- **Question** (V. Milman): Does relation (1) uniquely characterize the Fourier Transform?

A Uniqueness Theorem

- **Theorem.** Assume $f \in C^2(0,\infty)$ and $f, f', f'' \in L_1[0,\infty)$. Also, assume $f(0) = \int_0^\infty f = 0$. Then $g = \mathcal{F}f$ is the unique $C(0,\infty)$ function satisfying
  (a) $g(x) = O(x^{-1-\epsilon})$, $x \to \infty$ for some $\epsilon > 0$
  (b) $\sum_{n=1}^\infty g(nx) = \frac{1}{x}\sum_{n=1}^\infty f(n/x)$.

Sequences

- Consider the space of sequences $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$.
- Associate to every sequence (a_n) its Dirichlet series

$$L(s; a_n) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- Given two sequences a_n, b_n, their convolution is

$$(a * b)_k = \sum_{mn=k} a_m b_n$$

- $L(s; a_n * b_n) = L(s; a_n) L(s; b_n)$
- $\delta_n = 1, 0, 0, 0, ... \Leftrightarrow L(s; \delta_n) = 1$ is the unit element in the ring of sequences with convolution.
- (a_n) is invertible precisely if $a_1 \neq 0$.

Sequences

- Consider the space of sequences $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$.
- Associate to every sequence (a_n) its Dirichlet series

$$L(s; a_n) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- Given two sequences a_n, b_n, their convolution is

$$(a * b)_k = \sum_{mn=k} a_m b_n$$

- $L(s; a_n * b_n) = L(s; a_n) L(s; b_n)$
- $\delta_n = 1, 0, 0, 0, \ldots \Leftrightarrow L(s; \delta_n) = 1$ is the unit element in the ring of sequences with convolution.
- (a_n) is invertible precisely if $a_1 \neq 0$.

Sequences

- Consider the space of sequences $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$.
- Associate to every sequence (a_n) its Dirichlet series

$$L(s;a_n) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- Given two sequences a_n, b_n, their convolution is

$$(a*b)_k = \sum_{mn=k} a_m b_n$$

- $L(s;a_n * b_n) = L(s;a_n)L(s;b_n)$
- $\delta_n = 1,0,0,0,... \Leftrightarrow L(s;\delta_n) = 1$ is the unit element in the ring of sequences with convolution.
- (a_n) is invertible precisely if $a_1 \neq 0$.

Sequences

- Consider the space of sequences $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$.
- Associate to every sequence (a_n) its Dirichlet series

$$L(s; a_n) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- Given two sequences a_n, b_n, their convolution is

$$(a * b)_k = \sum_{mn=k} a_m b_n$$

- $L(s; a_n * b_n) = L(s; a_n)L(s; b_n)$
- $\delta_n = 1, 0, 0, 0, ... \Leftrightarrow L(s; \delta_n) = 1$ is the unit element in the ring of sequences with convolution.
- (a_n) is invertible precisely if $a_1 \neq 0$.

Sequences

- Consider the space of sequences $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$.
- Associate to every sequence (a_n) its Dirichlet series

$$L(s; a_n) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- Given two sequences a_n, b_n, their convolution is

$$(a * b)_k = \sum_{mn=k} a_m b_n$$

- $L(s; a_n * b_n) = L(s; a_n) L(s; b_n)$
- $\delta_n = 1, 0, 0, 0, \ldots \Leftrightarrow L(s; \delta_n) = 1$ is the unit element in the ring of sequences with convolution.
- (a_n) is invertible precisely if $a_1 \neq 0$.

Sequences

- Consider the space of sequences $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$.
- Associate to every sequence (a_n) its Dirichlet series

$$L(s; a_n) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- Given two sequences a_n, b_n, their convolution is

$$(a * b)_k = \sum_{mn=k} a_m b_n$$

- $L(s; a_n * b_n) = L(s; a_n)L(s; b_n)$
- $\delta_n = 1, 0, 0, 0, ... \Leftrightarrow L(s; \delta_n) = 1$ is the unit element in the ring of sequences with convolution.
- (a_n) is invertible precisely if $a_1 \neq 0$.

Some operators on $L^2[0,\infty)$

- For a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ define
$$T(a_n)f(x) = \sum_{n=1}^{\infty} a_n f(nx)$$
- $T(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ is generally an unbounded operator.
- When all series involved are absolutely convergent, $T(a_n * b_n)f = T(a_n)T(b_n)f$.
- Define $S : L^2[0,\infty) \to L^2[0,\infty)$ by $Sf(x) = \frac{1}{x}f(\frac{1}{x})$
- S is a unitary involution: $\|Sf\| = \|f\|$ and $S = S^* = S^{-1}$.
- Let $e_n = 1$, $n \geq 1$. Then Poisson's formula $\sum_{n=1}^{\infty} \mathcal{F}f(nx) = \frac{1}{x}\sum_{n=1}^{\infty} f(\frac{n}{x})$ can be written as
$$T(e_n)\mathcal{F}f = ST(e_n)f$$

Some operators on $L^2[0,\infty)$

- For a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ define

$$T(a_n)f(x) = \sum_{n=1}^{\infty} a_n f(nx)$$

- $T(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ is generally an unbounded operator.
- When all series involved are absolutely convergent, $T(a_n * b_n)f = T(a_n)T(b_n)f$.
- Define $S : L^2[0,\infty) \to L^2[0,\infty)$ by $Sf(x) = \frac{1}{x} f(\frac{1}{x})$
- S is a unitary involution: $\|Sf\| = \|f\|$ and $S = S^* = S^{-1}$.
- Let $e_n = 1$, $n \geq 1$. Then Poisson's formula $\sum_{n=1}^{\infty} \mathcal{F}f(nx) = \frac{1}{x}\sum_{n=1}^{\infty} f(\frac{n}{x})$ can be written as

$$T(e_n)\mathcal{F}f = ST(e_n)f$$

Some operators on $L^2[0,\infty)$

- For a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ define

$$T(a_n)f(x) = \sum_{n=1}^{\infty} a_n f(nx)$$

- $T(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ is generally an unbounded operator.
- When all series involved are absolutely convergent, $T(a_n * b_n)f = T(a_n)T(b_n)f$.
- Define $S : L^2[0,\infty) \to L^2[0,\infty)$ by $Sf(x) = \frac{1}{x}f(\frac{1}{x})$
- S is a unitary involution: $\|Sf\| = \|f\|$ and $S = S^* = S^{-1}$.
- Let $e_n = 1$, $n \geq 1$. Then Poisson's formula $\sum_{n=1}^{\infty} \mathcal{F}f(nx) = \frac{1}{x}\sum_{n=1}^{\infty} f(\frac{n}{x})$ can be written as

$$T(e_n)\mathcal{F}f = ST(e_n)f$$

Some operators on $L^2[0,\infty)$

- For a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ define

$$T(a_n)f(x) = \sum_{n=1}^{\infty} a_n f(nx)$$

- $T(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ is generally an unbounded operator.
- When all series involved are absolutely convergent, $T(a_n * b_n)f = T(a_n)T(b_n)f$.
- Define $S : L^2[0,\infty) \to L^2[0,\infty)$ by $Sf(x) = \frac{1}{x}f(\frac{1}{x})$
- S is a unitary involution: $\|Sf\| = \|f\|$ and $S = S^* = S^{-1}$.
- Let $e_n = 1$, $n \geq 1$. Then Poisson's formula $\sum_{n=1}^{\infty} \mathcal{F}f(nx) = \frac{1}{x}\sum_{n=1}^{\infty} f(\frac{n}{x})$ can be written as

$$T(e_n)\mathcal{F}f = ST(e_n)f$$

Some operators on $L^2[0,\infty)$

- For a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ define

$$T(a_n)f(x) = \sum_{n=1}^{\infty} a_n f(nx)$$

- $T(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ is generally an unbounded operator.
- When all series involved are absolutely convergent, $T(a_n * b_n)f = T(a_n)T(b_n)f$.
- Define $S : L^2[0,\infty) \to L^2[0,\infty)$ by $Sf(x) = \frac{1}{x} f(\frac{1}{x})$
- S is a unitary involution: $\|Sf\| = \|f\|$ and $S = S^* = S^{-1}$.
- Let $e_n = 1$, $n \geq 1$. Then Poisson's formula $\sum_{n=1}^{\infty} \mathcal{F}f(nx) = \frac{1}{x}\sum_{n=1}^{\infty} f(\frac{n}{x})$ can be written as

$$T(e_n)\mathcal{F}f = ST(e_n)f$$

Some operators on $L^2[0,\infty)$

- For a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ define

$$T(a_n)f(x) = \sum_{n=1}^{\infty} a_n f(nx)$$

- $T(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ is generally an unbounded operator.
- When all series involved are absolutely convergent, $T(a_n * b_n)f = T(a_n)T(b_n)f$.
- Define $S : L^2[0,\infty) \to L^2[0,\infty)$ by $Sf(x) = \frac{1}{x}f(\frac{1}{x})$
- S is a unitary involution: $\|Sf\| = \|f\|$ and $S = S^* = S^{-1}$.
- Let $e_n = 1$, $n \geq 1$. Then Poisson's formula $\sum_{n=1}^{\infty} \mathcal{F}f(nx) = \frac{1}{x}\sum_{n=1}^{\infty} f(\frac{n}{x})$ can be written as

$$T(e_n)\mathcal{F}f = ST(e_n)f$$

Proof of uniqueness theorem

- Recall the Möbius function

$$\mu(n) = \begin{cases} (-1)^{\sharp\{p \text{ prime} | p \text{ divides } n\}}, & n \text{ square-free} \\ 0, & d^2 | n \end{cases}$$

- For $e_n = 1$, $e * \mu = \delta$, so we expect $T(\mu_n) = T(e_n)^{-1}$.
- **Lemma.** Suppose $g \in C(0,\infty)$ and satisfies $g(x) = O(x^{-1-\epsilon})$ for some $\epsilon > 0$. Then $T(e_n)g = O(x^{-1-\epsilon})$ and $T(\mu_n)g = O(x^{-1-\epsilon})$, and these are inverse transforms: $T(e_n)T(\mu_n)g = T(\mu_n)T(e_n)g = g$.
- So, the equation $T(e_n)\mathcal{F}f = ST(e_n)f$ can be explicitly inverted under certain conditions:

$$\mathcal{F}f(x) = T(\mu_n)ST(e_n)f(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{nx} \sum_{m=1}^{\infty} f\left(\frac{m}{nx}\right)$$

Proof of uniqueness theorem

- Recall the Möbius function

$$\mu(n) = \begin{cases} (-1)^{\sharp\{p \text{ prime} | p \text{ divides } n\}}, & n \text{ square-free} \\ 0, & d^2 | n \end{cases}$$

- For $e_n = 1$, $e * \mu = \delta$, so we expect $T(\mu_n) = T(e_n)^{-1}$.
- **Lemma.** Suppose $g \in C(0,\infty)$ and satisfies $g(x) = O(x^{-1-\epsilon})$ for some $\epsilon > 0$. Then $T(e_n)g = O(x^{-1-\epsilon})$ and $T(\mu_n)g = O(x^{-1-\epsilon})$, and these are inverse transforms: $T(e_n)T(\mu_n)g = T(\mu_n)T(e_n)g = g$.
- So, the equation $T(e_n)\mathcal{F}f = ST(e_n)f$ can be explicitly inverted under certain conditions:

$$\mathcal{F}f(x) = T(\mu_n)ST(e_n)f(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{nx} \sum_{m=1}^{\infty} f\left(\frac{m}{nx}\right)$$

Proof of uniqueness theorem

- Recall the Möbius function

$$\mu(n) = \begin{cases} (-1)^{\sharp\{p \text{ prime} | p \text{ divides } n\}}, & n \text{ square-free} \\ 0, & d^2 | n \end{cases}$$

- For $e_n = 1$, $e * \mu = \delta$, so we expect $T(\mu_n) = T(e_n)^{-1}$.
- **Lemma.** Suppose $g \in C(0,\infty)$ and satisfies $g(x) = O(x^{-1-\epsilon})$ for some $\epsilon > 0$. Then $T(e_n)g = O(x^{-1-\epsilon})$ and $T(\mu_n)g = O(x^{-1-\epsilon})$, and these are inverse transforms: $T(e_n)T(\mu_n)g = T(\mu_n)T(e_n)g = g$.
- So, the equation $T(e_n)\mathcal{F}f = ST(e_n)f$ can be explicitly inverted under certain conditions:

$$\mathcal{F}f(x) = T(\mu_n)ST(e_n)f(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{nx} \sum_{m=1}^{\infty} f\left(\frac{m}{nx}\right)$$

Proof of uniqueness theorem

- Recall the Möbius function

$$\mu(n) = \begin{cases} (-1)^{\sharp\{p \text{ prime} | p \text{ divides } n\}}, & n \text{ square-free} \\ 0, & d^2 | n \end{cases}$$

- For $e_n = 1$, $e * \mu = \delta$, so we expect $T(\mu_n) = T(e_n)^{-1}$.
- **Lemma.** Suppose $g \in C(0,\infty)$ and satisfies $g(x) = O(x^{-1-\epsilon})$ for some $\epsilon > 0$. Then $T(e_n)g = O(x^{-1-\epsilon})$ and $T(\mu_n)g = O(x^{-1-\epsilon})$, and these are inverse transforms: $T(e_n)T(\mu_n)g = T(\mu_n)T(e_n)g = g$.
- So, the equation $T(e_n)\mathcal{F}f = ST(e_n)f$ can be explicitly inverted under certain conditions:

$$\mathcal{F}f(x) = T(\mu_n)ST(e_n)f(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{nx} \sum_{m=1}^{\infty} f\left(\frac{m}{nx}\right)$$

Davenport's Theorem

- The equality $\mathcal{F}f = T(\mu_n)ST(e_n)f$ holds in other cases too. For example, consider
- **Theorem (Davenport 37')** $\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\{nx\} = -\frac{1}{\pi}\sin(2\pi x)$ (here $\{t\} = t - \lfloor t \rfloor$ denotes the fractional part)
- This can be used to show that the Fourier Transform of a mean-zero step function such as $f(x) = \sum_k \alpha_k \chi_{[a_k,b_k]}$ $(\sum \alpha_k(b_k - a_k) = 0)$, when symmetrically extended to the real axis, is given by

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{nx} \sum_{m=1}^{\infty} f\left(\frac{m}{nx}\right) = \sum_k \frac{\alpha_k}{\pi x}(\sin(2\pi b_k x) - \sin(2\pi a_k x))$$

Davenport's Theorem

- The equality $\mathcal{F}f = T(\mu_n)ST(e_n)f$ holds in other cases too. For example, consider
- **Theorem (Davenport 37')** $\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\{nx\} = -\frac{1}{\pi}\sin(2\pi x)$ (here $\{t\} = t - \lfloor t\rfloor$ denotes the fractional part)
- This can be used to show that the Fourier Transform of a mean-zero step function such as $f(x) = \sum_k \alpha_k \chi_{[a_k,b_k]}$ ($\sum \alpha_k(b_k - a_k) = 0$), when symmetrically extended to the real axis, is given by

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{nx}\sum_{m=1}^{\infty} f\left(\frac{m}{nx}\right) = \sum_k \frac{\alpha_k}{\pi x}(\sin(2\pi b_k x) - \sin(2\pi a_k x))$$

Davenport's Theorem

- The equality $\mathcal{F}f = T(\mu_n)ST(e_n)f$ holds in other cases too. For example, consider
- **Theorem (Davenport 37')** $\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\{nx\} = -\frac{1}{\pi}\sin(2\pi x)$ (here $\{t\} = t - \lfloor t \rfloor$ denotes the fractional part)
- This can be used to show that the Fourier Transform of a mean-zero step function such as $f(x) = \sum_k \alpha_k \chi_{[a_k,b_k]}$ ($\sum \alpha_k(b_k - a_k) = 0$), when symmetrically extended to the real axis, is given by

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{nx}\sum_{m=1}^{\infty} f\left(\frac{m}{nx}\right) = \sum_k \frac{\alpha_k}{\pi x}(\sin(2\pi b_k x) - \sin(2\pi a_k x))$$

Generalizing Poisson's summation formula

- We are led naturally to the following
  Question. Given a sequence (a_n), does there exist a linear operator $\mathcal{F}(a_n)$ on $L^2[0,\infty)$ such that for nice functions f the generalized Poisson formula
  $$T(a_n)\mathcal{F}(a_n)f = ST(a_n)f$$
  holds? Is it unique? Is it unitary?

Bounded operators

- **Lemma.** Assume
$$\sum \frac{|a_n|}{\sqrt{n}} < \infty \tag{2}$$
holds. Then
  1. $T(a_n)$ extends to a bounded operator on $L^2[0,\infty)$, and $\|T(a_n)\| \leq \sum \frac{|a_n|}{\sqrt{n}}$.
  2. For $f \in C(0,\infty)$ and $f = O(x^{-1-\epsilon})$ for some $\epsilon > 0$, the formula
  $$T(a_n)f(x) = \sum_{n=1}^{\infty} a_n f(nx)$$
  is valid.
- Note that when condition (2) holds, $L(s; a_n)$ is absolutely convergent for $\Re s \geq 1/2$

Bounded operators

- **Lemma.** Assume
$$\sum \frac{|a_n|}{\sqrt{n}} < \infty \tag{2}$$
holds. Then
  1. $T(a_n)$ extends to a bounded operator on $L^2[0,\infty)$, and $\|T(a_n)\| \leq \sum \frac{|a_n|}{\sqrt{n}}$.
  2. For $f \in C(0,\infty)$ and $f = O(x^{-1-\epsilon})$ for some $\epsilon > 0$, the formula
$$T(a_n)f(x) = \sum_{n=1}^{\infty} a_n f(nx)$$
is valid.
- Note that when condition (2) holds, $L(s; a_n)$ is absolutely convergent for $\Re s \geq 1/2$

Bounded operators

- **Lemma.** Assume
$$\sum \frac{|a_n|}{\sqrt{n}} < \infty \tag{2}$$
holds. Then
  1. $T(a_n)$ extends to a bounded operator on $L^2[0,\infty)$, and $\|T(a_n)\| \leq \sum \frac{|a_n|}{\sqrt{n}}$.
  2. For $f \in C(0,\infty)$ and $f = O(x^{-1-\epsilon})$ for some $\epsilon > 0$, the formula
$$T(a_n)f(x) = \sum_{n=1}^{\infty} a_n f(nx)$$
is valid.
- Note that when condition (2) holds, $L(s; a_n)$ is absolutely convergent for $\Re s \geq 1/2$

Bounded operators

- **Lemma.** Assume
$$\sum \frac{|a_n|}{\sqrt{n}} < \infty \tag{2}$$
holds. Then
  1. $T(a_n)$ extends to a bounded operator on $L^2[0,\infty)$, and $\|T(a_n)\| \leq \sum \frac{|a_n|}{\sqrt{n}}$.
  2. For $f \in C(0,\infty)$ and $f = O(x^{-1-\epsilon})$ for some $\epsilon > 0$, the formula
$$T(a_n)f(x) = \sum_{n=1}^{\infty} a_n f(nx)$$
is valid.
- Note that when condition (2) holds, $L(s; a_n)$ is absolutely convergent for $\Re s \geq 1/2$

Generalized operator - direct approach

- **Theorem.** Assume (a_n) is real, $\sum |a_n| n^\epsilon < \infty$ for some $\epsilon > 0$, and (b_n) satisfies $\sum |b_n|/\sqrt{n} < \infty$. Then

$$\mathcal{F}(a_n) = T(b_n) S T(a_n)$$

is a unitary operator.

- **Corollary.** Take a continuous f satisfying $f(x) = O(x^{-1-\epsilon})$ as $x \to \infty$ and $f(x) = O(x^\epsilon)$ as $x \to 0$ for some $\epsilon > 0$. Then
  1. $\mathcal{F}(a_n) f$ is continuous and $\mathcal{F}(a_n) f(x) = O(x^{-1-\epsilon})$.
  2. The formula $\sum a_n \mathcal{F}(a_n) f(nx) = (1/x) \sum a_n f(n/x)$ holds pointwise.

Generalized operator - direct approach

- **Theorem.** Assume (a_n) is real, $\sum |a_n| n^\epsilon < \infty$ for some $\epsilon > 0$, and (b_n) satisfies $\sum |b_n|/\sqrt{n} < \infty$. Then

$$\mathcal{F}(a_n) = T(b_n) S T(a_n)$$

is a unitary operator.

- **Corollary.** Take a continuous f satisfying $f(x) = O(x^{-1-\epsilon})$ as $x \to \infty$ and $f(x) = O(x^\epsilon)$ as $x \to 0$ for some $\epsilon > 0$. Then

1. $\mathcal{F}(a_n) f$ is continuous and $\mathcal{F}(a_n) f(x) = O(x^{-1-\epsilon})$.
2. The formula $\sum a_n \mathcal{F}(a_n) f(nx) = (1/x) \sum a_n f(n/x)$ holds pointwise.

Generalized operator - direct approach

- **Theorem.** Assume (a_n) is real, $\sum |a_n| n^\epsilon < \infty$ for some $\epsilon > 0$, and (b_n) satisfies $\sum |b_n|/\sqrt{n} < \infty$. Then

$$\mathcal{F}(a_n) = T(b_n) S T(a_n)$$

  is a unitary operator.
- **Corollary.** Take a continuous f satisfying $f(x) = O(x^{-1-\epsilon})$ as $x \to \infty$ and $f(x) = O(x^\epsilon)$ as $x \to 0$ for some $\epsilon > 0$. Then
  1. $\mathcal{F}(a_n)f$ is continuous and $\mathcal{F}(a_n)f(x) = O(x^{-1-\epsilon})$.
  2. The formula $\sum a_n \mathcal{F}(a_n) f(nx) = (1/x) \sum a_n f(n/x)$ holds pointwise.

Generalized operator - direct approach

- **Theorem.** Assume (a_n) is real, $\sum |a_n| n^\epsilon < \infty$ for some $\epsilon > 0$, and (b_n) satisfies $\sum |b_n|/\sqrt{n} < \infty$. Then

$$\mathcal{F}(a_n) = T(b_n) S T(a_n)$$

is a unitary operator.

- **Corollary.** Take a continuous f satisfying $f(x) = O(x^{-1-\epsilon})$ as $x \to \infty$ and $f(x) = O(x^\epsilon)$ as $x \to 0$ for some $\epsilon > 0$. Then
  1. $\mathcal{F}(a_n)f$ is continuous and $\mathcal{F}(a_n)f(x) = O(x^{-1-\epsilon})$.
  2. The formula $\sum a_n \mathcal{F}(a_n) f(nx) = (1/x) \sum a_n f(n/x)$ holds pointwise.

Change of space

- By the change of variable $x = e^y$, we get the isometric isomorphism $u : L^2([0,\infty), dm) \to L^2(\mathbb{R}, e^y dy),\ f(x) \mapsto g(y) = f(e^y)$.
- How to apply Fourier Transform in such space?
- Suggestion of B. Klartag: Define $w(f)(\omega) = \widehat{u(f)}(\omega + i/2)$. Then $w : L^2(\mathbb{R}, e^y dy) \to (\mathbb{R}, dm)$ is an isometric isomorphism, where

$$\widehat{h}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

  denotes the Fourier Transform.
- Alternatively, define the isometry $v : L^2(\mathbb{R}, e^y dy) \to L^2(\mathbb{R}, dm)$ by $v(g)(y) = e^{y/2} g(y)$. Then $w(f) = \widehat{v(u(f))}$.

Change of space

- By the change of variable $x = e^y$, we get the isometric isomorphism $u : L^2([0,\infty), dm) \to L^2(\mathbb{R}, e^y dy)$, $f(x) \mapsto g(y) = f(e^y)$.
- How to apply Fourier Transform in such space?
- Suggestion of B. Klartag: Define $w(f)(\omega) = \widehat{u(f)}(\omega + i/2)$. Then $w : L^2(\mathbb{R}, e^y dy) \to (\mathbb{R}, dm)$ is an isometric isomorphism, where

$$\widehat{h}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

denotes the Fourier Transform.
- Alternatively, define the isometry $v : L^2(\mathbb{R}, e^y dy) \to L^2(\mathbb{R}, dm)$ by $v(g)(y) = e^{y/2} g(y)$. Then $w(f) = \widehat{v(u(f))}$.

Change of space

- By the change of variable $x = e^y$, we get the isometric isomorphism $u : L^2([0,\infty), dm) \to L^2(\mathbb{R}, e^y dy),\ f(x) \mapsto g(y) = f(e^y)$.
- How to apply Fourier Transform in such space?
- Suggestion of B. Klartag: Define $w(f)(\omega) = \widehat{u(f)}(\omega + i/2)$. Then $w : L^2(\mathbb{R}, e^y dy) \to (\mathbb{R}, dm)$ is an isometric isomorphism, where

$$\widehat{h}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

denotes the Fourier Transform.
- Alternatively, define the isometry $v : L^2(\mathbb{R}, e^y dy) \to L^2(\mathbb{R}, dm)$ by $v(g)(y) = e^{y/2} g(y)$. Then $w(f) = \widehat{v(u(f))}$.

Change of space

- By the change of variable $x = e^y$, we get the isometric isomorphism $u : L^2([0,\infty), dm) \to L^2(\mathbb{R}, e^y dy),\ f(x) \mapsto g(y) = f(e^y)$.
- How to apply Fourier Transform in such space?
- Suggestion of B. Klartag: Define $w(f)(\omega) = \widehat{u(f)}(\omega + i/2)$. Then $w : L^2(\mathbb{R}, e^y dy) \to (\mathbb{R}, dm)$ is an isometric isomorphism, where

$$\widehat{h}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

  denotes the Fourier Transform.
- Alternatively, define the isometry $v : L^2(\mathbb{R}, e^y dy) \to L^2(\mathbb{R}, dm)$ by $v(g)(y) = e^{y/2} g(y)$. Then $w(f) = \widehat{v(u(f))}$.

Some diagonal forms

- For an operator $A : L^2[0,\infty) \to L^2[0,\infty)$, we write $\tilde{A} = wAw^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ for its conjugate by w.
- $Sf(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$ becomes $\tilde{S}h(\omega) = h(-\omega)$.
- Assuming $\sum \frac{|a_n|}{\sqrt{n}} < \infty$, we get for $g \in L^2(\mathbb{R}, e^y dy)$

$$(uT(a_n)u^{-1}g)(y) = \sum a_n g(y + \log n) = g * \nu(y)$$

where $\nu(y) = \sum a_n \delta_{-\log n}(y)$.
-

$$\sqrt{2\pi}\hat{\nu}(z) = \sum a_n e^{iz \log n} = \sum a_n n^{iz} = L(-iz; a_n)$$

which converges for $\Im z \geq 1/2$.
- Thus for $h = \widehat{vg}$, the diagonal form of $T(a_n)$ is

$$\widetilde{T(a_n)}h(\omega) = \widehat{g * \nu}(\omega + i/2) = L(1/2 - i\omega; a_n)h(\omega)$$

Some diagonal forms

- For an operator $A : L^2[0,\infty) \to L^2[0,\infty)$, we write $\tilde{A} = wAw^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ for its conjugate by w.
- $Sf(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$ becomes $\tilde{S}h(\omega) = h(-\omega)$.
- Assuming $\sum \frac{|a_n|}{\sqrt{n}} < \infty$, we get for $g \in L^2(\mathbb{R}, e^y dy)$

$$(uT(a_n)u^{-1}g)(y) = \sum a_n g(y + \log n) = g * \nu(y)$$

where $\nu(y) = \sum a_n \delta_{-\log n}(y)$.
- $$\sqrt{2\pi}\hat{\nu}(z) = \sum a_n e^{iz \log n} = \sum a_n n^{iz} = L(-iz; a_n)$$

which converges for $\Im z \geq 1/2$.
- Thus for $h = \widehat{vg}$, the diagonal form of $T(a_n)$ is

$$\widetilde{T(a_n)} h(\omega) = \widehat{g * \nu}(\omega + i/2) = L(1/2 - i\omega; a_n) h(\omega)$$

Some diagonal forms

- For an operator $A : L^2[0,\infty) \to L^2[0,\infty)$, we write $\tilde{A} = wAw^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ for its conjugate by w.
- $Sf(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$ becomes $\tilde{S}h(\omega) = h(-\omega)$.
- Assuming $\sum \frac{|a_n|}{\sqrt{n}} < \infty$, we get for $g \in L^2(\mathbb{R}, e^y dy)$

$$(uT(a_n)u^{-1}g)(y) = \sum a_n g(y + \log n) = g * \nu(y)$$

  where $\nu(y) = \sum a_n \delta_{-\log n}(y)$.
-
$$\sqrt{2\pi}\hat{\nu}(z) = \sum a_n e^{iz\log n} = \sum a_n n^{iz} = L(-iz; a_n)$$

  which converges for $\Im z \geq 1/2$.
- Thus for $h = \widehat{vg}$, the diagonal form of $T(a_n)$ is

$$\widetilde{T(a_n)}h(\omega) = \widehat{g * \nu}(\omega + i/2) = L(1/2 - i\omega; a_n)h(\omega)$$

Some diagonal forms

- For an operator $A : L^2[0,\infty) \to L^2[0,\infty)$, we write $\tilde{A} = wAw^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ for its conjugate by w.
- $Sf(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$ becomes $\tilde{S}h(\omega) = h(-\omega)$.
- Assuming $\sum \frac{|a_n|}{\sqrt{n}} < \infty$, we get for $g \in L^2(\mathbb{R}, e^y dy)$
$$(uT(a_n)u^{-1}g)(y) = \sum a_n g(y + \log n) = g * \nu(y)$$
where $\nu(y) = \sum a_n \delta_{-\log n}(y)$.
- $$\sqrt{2\pi}\hat{\nu}(z) = \sum a_n e^{iz \log n} = \sum a_n n^{iz} = L(-iz; a_n)$$
which converges for $\Im z \geq 1/2$.
- Thus for $h = \widehat{vg}$, the diagonal form of $T(a_n)$ is
$$\widetilde{T(a_n)}h(\omega) = \widehat{g * \nu}(\omega + i/2) = L(1/2 - i\omega; a_n)h(\omega)$$

Some diagonal forms

- For an operator $A : L^2[0,\infty) \to L^2[0,\infty)$, we write $\tilde{A} = wAw^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ for its conjugate by w.
- $Sf(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$ becomes $\tilde{S}h(\omega) = h(-\omega)$.
- Assuming $\sum \frac{|a_n|}{\sqrt{n}} < \infty$, we get for $g \in L^2(\mathbb{R}, e^y dy)$
$$(uT(a_n)u^{-1}g)(y) = \sum a_n g(y + \log n) = g * \nu(y)$$
where $\nu(y) = \sum a_n \delta_{-\log n}(y)$.
- $$\sqrt{2\pi}\hat{\nu}(z) = \sum a_n e^{iz\log n} = \sum a_n n^{iz} = L(-iz; a_n)$$
which converges for $\Im z \geq 1/2$.
- Thus for $h = \widehat{vg}$, the diagonal form of $T(a_n)$ is
$$\widetilde{T(a_n)}h(\omega) = \widehat{g * \nu}(\omega + i/2) = L(1/2 - i\omega; a_n)h(\omega)$$

Existence and uniqueness theorem

- The generalized Poisson summation formula becomes

$$L(1/2 - i\omega; a_n)\widetilde{\mathcal{F}(a_n)}h(\omega) = L(1/2 + i\omega; a_n)h(-\omega)$$

- **Theorem.** Assume $\sum |a_n| n^{-1/2} < \infty$, and $a_n \in e^{i\theta}\mathbb{R}$ for some fixed θ. Then
  1. There exists a unitary involution $\mathcal{F}(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ satisfying the generalized PSF in operator form: $T(a_n)\mathcal{F}(a_n) = ST(a_n)$.
  2. If for some $\epsilon > 0$, $\sum |a_n| n^{-1/2+\epsilon} < \infty$, then a bounded $\mathcal{F}(a_n)$ satisfying $T(a_n)\mathcal{F}(a_n) = ST(a_n)$ is unique.
- $\mathcal{F}(a_n)$ has the skew-diagonal form

$$\widetilde{\mathcal{F}(a_n)}h(\omega) = e^{2i(\arg L(1/2+i\omega;a_n)-\theta)}h(-\omega)$$

- Note that PSF need not hold pointwise (pointed out by F. Nazarov).

Existence and uniqueness theorem

- The generalized Poisson summation formula becomes

$$L(1/2 - i\omega; a_n)\widetilde{\mathcal{F}(a_n)}h(\omega) = L(1/2 + i\omega; a_n)h(-\omega)$$

- **Theorem.** Assume $\sum |a_n| n^{-1/2} < \infty$, and $a_n \in e^{i\theta}\mathbb{R}$ for some fixed θ. Then
  1. There exists a unitary involution $\mathcal{F}(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ satisfying the generalized PSF in operator form: $T(a_n)\mathcal{F}(a_n) = ST(a_n)$.
  2. If for some $\epsilon > 0$, $\sum |a_n| n^{-1/2+\epsilon} < \infty$, then a bounded $\mathcal{F}(a_n)$ satisfying $T(a_n)\mathcal{F}(a_n) = ST(a_n)$ is unique.
- $\mathcal{F}(a_n)$ has the skew-diagonal form

$$\widetilde{\mathcal{F}(a_n)}h(\omega) = e^{2i(\arg L(1/2+i\omega; a_n) - \theta)}h(-\omega)$$

- Note that PSF need not hold pointwise (pointed out by F. Nazarov).

Existence and uniqueness theorem

- The generalized Poisson summation formula becomes

$$L(1/2 - i\omega; a_n)\widetilde{\mathcal{F}(a_n)}h(\omega) = L(1/2 + i\omega; a_n)h(-\omega)$$

- **Theorem.** Assume $\sum |a_n| n^{-1/2} < \infty$, and $a_n \in e^{i\theta}\mathbb{R}$ for some fixed θ. Then
  1. There exists a unitary involution $\mathcal{F}(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ satisfying the generalized PSF in operator form: $T(a_n)\mathcal{F}(a_n) = ST(a_n)$.
  2. If for some $\epsilon > 0$, $\sum |a_n| n^{-1/2+\epsilon} < \infty$, then a bounded $\mathcal{F}(a_n)$ satisfying $T(a_n)\mathcal{F}(a_n) = ST(a_n)$ is unique.
- $\mathcal{F}(a_n)$ has the skew-diagonal form

$$\widetilde{\mathcal{F}(a_n)}h(\omega) = e^{2i(\arg L(1/2+i\omega;a_n)-\theta)}h(-\omega)$$

- Note that PSF need not hold pointwise (pointed out by F. Nazarov).

Existence and uniqueness theorem

- The generalized Poisson summation formula becomes

$$L(1/2 - i\omega; a_n)\widetilde{\mathcal{F}(a_n)}h(\omega) = L(1/2 + i\omega; a_n)h(-\omega)$$

- **Theorem.** Assume $\sum |a_n| n^{-1/2} < \infty$, and $a_n \in e^{i\theta}\mathbb{R}$ for some fixed θ. Then
  1. There exists a unitary involution $\mathcal{F}(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ satisfying the generalized PSF in operator form: $T(a_n)\mathcal{F}(a_n) = ST(a_n)$.
  2. If for some $\epsilon > 0$, $\sum |a_n| n^{-1/2+\epsilon} < \infty$, then a bounded $\mathcal{F}(a_n)$ satisfying $T(a_n)\mathcal{F}(a_n) = ST(a_n)$ is unique.
- $\mathcal{F}(a_n)$ has the skew-diagonal form

$$\widetilde{\mathcal{F}(a_n)}h(\omega) = e^{2i(\arg L(1/2+i\omega;a_n)-\theta)}h(-\omega)$$

- Note that PSF need not hold pointwise (pointed out by F. Nazarov).

Existence and uniqueness theorem

- The generalized Poisson summation formula becomes

$$L(1/2 - i\omega; a_n)\widetilde{\mathcal{F}(a_n)}h(\omega) = L(1/2 + i\omega; a_n)h(-\omega)$$

- **Theorem.** Assume $\sum |a_n| n^{-1/2} < \infty$, and $a_n \in e^{i\theta}\mathbb{R}$ for some fixed θ. Then
  1. There exists a unitary involution $\mathcal{F}(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ satisfying the generalized PSF in operator form: $T(a_n)\mathcal{F}(a_n) = ST(a_n)$.
  2. If for some $\epsilon > 0$, $\sum |a_n| n^{-1/2+\epsilon} < \infty$, then a bounded $\mathcal{F}(a_n)$ satisfying $T(a_n)\mathcal{F}(a_n) = ST(a_n)$ is unique.
- $\mathcal{F}(a_n)$ has the skew-diagonal form

$$\widetilde{\mathcal{F}(a_n)}h(\omega) = e^{2i(\arg L(1/2+i\omega;a_n)-\theta)}h(-\omega)$$

- Note that PSF need not hold pointwise (pointed out by F. Nazarov).

Existence and uniqueness theorem

- The generalized Poisson summation formula becomes

$$L(1/2 - i\omega; a_n)\widetilde{\mathcal{F}(a_n)}h(\omega) = L(1/2 + i\omega; a_n)h(-\omega)$$

- **Theorem.** Assume $\sum |a_n| n^{-1/2} < \infty$, and $a_n \in e^{i\theta}\mathbb{R}$ for some fixed θ. Then
  1. There exists a unitary involution $\mathcal{F}(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ satisfying the generalized PSF in operator form: $T(a_n)\mathcal{F}(a_n) = ST(a_n)$.
  2. If for some $\epsilon > 0$, $\sum |a_n| n^{-1/2+\epsilon} < \infty$, then a bounded $\mathcal{F}(a_n)$ satisfying $T(a_n)\mathcal{F}(a_n) = ST(a_n)$ is unique.
- $\mathcal{F}(a_n)$ has the skew-diagonal form

$$\widetilde{\mathcal{F}(a_n)}h(\omega) = e^{2i(\arg L(1/2+i\omega;a_n)-\theta)}h(-\omega)$$

- Note that PSF need not hold pointwise (pointed out by F. Nazarov).

Existence and uniqueness theorem

- The generalized Poisson summation formula becomes

$$L(1/2 - i\omega; a_n)\widetilde{\mathcal{F}(a_n)}h(\omega) = L(1/2 + i\omega; a_n)h(-\omega)$$

- **Theorem.** Assume $\sum |a_n| n^{-1/2} < \infty$, and $a_n \in e^{i\theta}\mathbb{R}$ for some fixed θ. Then
  1. There exists a unitary involution $\mathcal{F}(a_n) : L^2[0,\infty) \to L^2[0,\infty)$ satisfying the generalized PSF in operator form: $T(a_n)\mathcal{F}(a_n) = ST(a_n)$.
  2. If for some $\epsilon > 0$, $\sum |a_n| n^{-1/2+\epsilon} < \infty$, then a bounded $\mathcal{F}(a_n)$ satisfying $T(a_n)\mathcal{F}(a_n) = ST(a_n)$ is unique.
- $\mathcal{F}(a_n)$ has the skew-diagonal form

$$\widetilde{\mathcal{F}(a_n)}h(\omega) = e^{2i(\arg L(1/2+i\omega;a_n)-\theta)}h(-\omega)$$

- Note that PSF need not hold pointwise (pointed out by F. Nazarov).

What about differentiation?

- Denote $Df = df/dx$, and $Xf = x \cdot f$. The Fourier transform satisfies for nice functions f the identitiy $\mathcal{F}(Df) = iX\mathcal{F}(f)$.
- For an even f, Df is odd. Thus we shouldn't expect to have such a formula in our setting.
- However, for an even function we can also write $\mathcal{F}(Xf) = iD\mathcal{F}(f)$. Those can be combined together into $(\star)$ $XD\mathcal{F} + \mathcal{F}XD + \mathcal{F} = 0$, where $\mathcal{F}$ is applied only to even functions.
- Denote $B = i(XD + Id/2)$ - a symmetric operator in any reasonable domain. Then $(\star)$ reads $\mathcal{F}B + B\mathcal{F} = 0$.

What about differentiation?

- Denote $Df = df/dx$, and $Xf = x \cdot f$. The Fourier transform satisfies for nice functions f the identitiy $\mathcal{F}(Df) = iX\mathcal{F}(f)$.
- For an even f, Df is odd. Thus we shouldn't expect to have such a formula in our setting.
- However, for an even function we can also write $\mathcal{F}(Xf) = iD\mathcal{F}(f)$. Those can be combined together into $(\star)$ $XD\mathcal{F} + \mathcal{F}XD + \mathcal{F} = 0$, where $\mathcal{F}$ is applied only to even functions.
- Denote $B = i(XD + Id/2)$ - a symmetric operator in any reasonable domain. Then $(\star)$ reads $\mathcal{F}B + B\mathcal{F} = 0$.

What about differentiation?

- Denote $Df = df/dx$, and $Xf = x \cdot f$. The Fourier transform satisfies for nice functions f the identitiy $\mathcal{F}(Df) = iX\mathcal{F}(f)$.
- For an even f, Df is odd. Thus we shouldn't expect to have such a formula in our setting.
- However, for an even function we can also write $\mathcal{F}(Xf) = iD\mathcal{F}(f)$. Those can be combined together into $(\star)$ $XD\mathcal{F} + \mathcal{F}XD + \mathcal{F} = 0$, where $\mathcal{F}$ is applied only to even functions.
- Denote $B = i(XD + Id/2)$ - a symmetric operator in any reasonable domain. Then $(\star)$ reads $\mathcal{F}B + B\mathcal{F} = 0$.

What about differentiation?

- Denote $Df = df/dx$, and $Xf = x \cdot f$. The Fourier transform satisfies for nice functions f the identitiy $\mathcal{F}(Df) = iX\mathcal{F}(f)$.
- For an even f, Df is odd. Thus we shouldn't expect to have such a formula in our setting.
- However, for an even function we can also write $\mathcal{F}(Xf) = iD\mathcal{F}(f)$. Those can be combined together into $(\star)$ $XD\mathcal{F} + \mathcal{F}XD + \mathcal{F} = 0$, where $\mathcal{F}$ is applied only to even functions.
- Denote $B = i(XD + Id/2)$ - a symmetric operator in any reasonable domain. Then $(\star)$ reads $\mathcal{F}B + B\mathcal{F} = 0$.

A general formula involving derivative

- **Theorem.** Assume $a_n \in \mathbb{R}$ satisfies $\sum |a_n| n^{\epsilon} < \infty$ for some $\epsilon > 0$, and the convolution inverse (b_n) satisfies $\sum |b_n|/\sqrt{n} < \infty$.
- Note that
$$\frac{L(1-z;a_n)}{L(z;a_n)}$$
admits an analytic extension to the strip $-\epsilon < \Re z < 1+\epsilon$.
- Assume further that there exists N such that
$$|L(1-z;a_n)/L(z;a_n)| \leq C|y|^N$$
- Let $f \in \mathcal{S}_0$, where
$$\mathcal{S}_0 = \{f \in C^{\infty} : \sup |x|^n |f^{(k)}(x)| < \infty \ \forall k \geq 0 \ , \forall n \in \mathbb{Z}\}$$
- Then $\mathcal{F}(a_n)Bf + B\mathcal{F}(a_n)f = 0$.

A general formula involving derivative

- **Theorem.** Assume $a_n \in \mathbb{R}$ satisfies $\sum |a_n| n^{\epsilon} < \infty$ for some $\epsilon > 0$, and the convolution inverse (b_n) satisfies $\sum |b_n|/\sqrt{n} < \infty$.
- Note that

$$\frac{L(1-z;a_n)}{L(z;a_n)}$$

admits an analytic extension to the strip $-\epsilon < \Re z < 1+\epsilon$.
- Assume further that there exists N such that

$$|L(1-z;a_n)/L(z;a_n)| \le C|y|^N$$

- Let $f \in \mathcal{S}_0$, where

$$\mathcal{S}_0 = \{f \in C^{\infty} : \sup |x|^n |f^{(k)}(x)| < \infty \ \forall k \ge 0 \ , \forall n \in \mathbb{Z}\}$$

- Then $\mathcal{F}(a_n)Bf + B\mathcal{F}(a_n)f = 0$.

A general formula involving derivative

- **Theorem.** Assume $a_n \in \mathbb{R}$ satisfies $\sum |a_n| n^{\epsilon} < \infty$ for some $\epsilon > 0$, and the convolution inverse (b_n) satisfies $\sum |b_n|/\sqrt{n} < \infty$.
- Note that

$$\frac{L(1-z; a_n)}{L(z; a_n)}$$

admits an analytic extension to the strip $-\epsilon < \Re z < 1 + \epsilon$.
- Assume further that there exists N such that

$$|L(1-z; a_n)/L(z; a_n)| \leq C|y|^N$$

- Let $f \in \mathcal{S}_0$, where

$$\mathcal{S}_0 = \{f \in C^{\infty} : \sup |x|^n |f^{(k)}(x)| < \infty \ \forall k \geq 0 \ , \forall n \in \mathbb{Z}\}$$

- Then $\mathcal{F}(a_n)Bf + B\mathcal{F}(a_n)f = 0$.

A general formula involving derivative

- **Theorem.** Assume $a_n \in \mathbb{R}$ satisfies $\sum |a_n| n^{\epsilon} < \infty$ for some $\epsilon > 0$, and the convolution inverse (b_n) satisfies $\sum |b_n|/\sqrt{n} < \infty$.
- Note that
$$\frac{L(1-z; a_n)}{L(z; a_n)}$$
admits an analytic extension to the strip $-\epsilon < \Re z < 1 + \epsilon$.
- Assume further that there exists N such that
$$|L(1-z; a_n)/L(z; a_n)| \leq C|y|^N$$
- Let $f \in \mathcal{S}_0$, where
$$\mathcal{S}_0 = \{f \in C^{\infty} : \sup |x|^n |f^{(k)}(x)| < \infty \ \forall k \geq 0 \ , \forall n \in \mathbb{Z}\}$$
- Then $\mathcal{F}(a_n)Bf + B\mathcal{F}(a_n)f = 0$.

A general formula involving derivative

- **Theorem.** Assume $a_n \in \mathbb{R}$ satisfies $\sum |a_n| n^\epsilon < \infty$ for some $\epsilon > 0$, and the convolution inverse (b_n) satisfies $\sum |b_n|/\sqrt{n} < \infty$.
- Note that
$$\frac{L(1-z;a_n)}{L(z;a_n)}$$
admits an analytic extension to the strip $-\epsilon < \Re z < 1+\epsilon$.
- Assume further that there exists N such that
$$|L(1-z;a_n)/L(z;a_n)| \leq C|y|^N$$
- Let $f \in \mathcal{S}_0$, where
$$\mathcal{S}_0 = \{f \in C^\infty : \sup |x|^n |f^{(k)}(x)| < \infty \ \forall k \geq 0 \ , \forall n \in \mathbb{Z}\}$$
- Then $\mathcal{F}(a_n)Bf + B\mathcal{F}(a_n)f = 0$.

An integral formula

- It is not hard to see that the general form of $\mathcal{F}(a_n)$ with $\sum |a_n|/\sqrt{n} < \infty$ is given by

$$\mathcal{F}(a_n)f(x) = \int_0^\infty A(xs)f(s)ds$$

  Where A is some generalized function depending on a_n.
- For example, for $(a_n) = (\delta_n) = 1, 0, 0, \ldots$ we get $A(s) = \delta_1(s)$
- Though not fitting into our discussion, the ordinary Fourier transform corresponds to $a_n = 1, 1, 1, \ldots$ and $A(s) = 2\cos s$.
- Any such operator formally satisfies the identity $\mathcal{F}B + B\mathcal{F} = 0$. The difficulty lies in verifying that $\mathcal{F}f$ is sufficiently regular for regular functions f.

An integral formula

- It is not hard to see that the general form of $\mathcal{F}(a_n)$ with $\sum |a_n|/\sqrt{n} < \infty$ is given by

$$\mathcal{F}(a_n)f(x) = \int_0^\infty A(xs)f(s)ds$$

  Where A is some generalized function depending on a_n.
- For example, for $(a_n) = (\delta_n) = 1, 0, 0, \ldots$ we get $A(s) = \delta_1(s)$
- Though not fitting into our discussion, the ordinary Fourier transform corresponds to $a_n = 1, 1, 1, \ldots$ and $A(s) = 2\cos s$.
- Any such operator formally satisfies the identity $\mathcal{F}B + B\mathcal{F} = 0$. The difficulty lies in verifying that $\mathcal{F}f$ is sufficiently regular for regular functions f.

An integral formula

- It is not hard to see that the general form of $\mathcal{F}(a_n)$ with $\sum |a_n|/\sqrt{n} < \infty$ is given by

$$\mathcal{F}(a_n)f(x) = \int_0^\infty A(xs)f(s)ds$$

  Where A is some generalized function depending on a_n.
- For example, for $(a_n) = (\delta_n) = 1, 0, 0, \ldots$ we get $A(s) = \delta_1(s)$
- Though not fitting into our discussion, the ordinary Fourier transform corresponds to $a_n = 1, 1, 1, \ldots$ and $A(s) = 2\cos s$.
- Any such operator formally satisfies the identity $\mathcal{F}B + B\mathcal{F} = 0$. The difficulty lies in verifying that $\mathcal{F}f$ is sufficiently regular for regular functions f.

An integral formula

- It is not hard to see that the general form of $\mathcal{F}(a_n)$ with $\sum |a_n|/\sqrt{n} < \infty$ is given by

$$\mathcal{F}(a_n)f(x) = \int_0^\infty A(xs)f(s)ds$$

Where A is some generalized function depending on a_n.

- For example, for $(a_n) = (\delta_n) = 1, 0, 0, \dots$ we get $A(s) = \delta_1(s)$
- Though not fitting into our discussion, the ordinary Fourier transform corresponds to $a_n = 1, 1, 1, \dots$ and $A(s) = 2\cos s$.
- Any such operator formally satisfies the identity $\mathcal{F}B + B\mathcal{F} = 0$. The difficulty lies in verifying that $\mathcal{F}f$ is sufficiently regular for regular functions f.

A family of unitary operators

- Let us determine when the operators $T(a_n)$ are unitary.
- Call an operator T C-unitary if $\frac{1}{C}T$ is unitary.
- **Corollary.** Assume a_n satisfies $\sum |a_n| n^{-1/2} < \infty$. Then the following are equivalent:
  (a) $|L(1/2+ix; a_n)| = C$
  (b) $T(a_n)$ is C-unitary on $L^2[0,\infty)$
  (c) (a_n) satisfies

$$\sum_{k=1}^{\infty} \frac{a_{m_0 k}\overline{a_{n_0 k}}}{k} = \begin{cases} C^2, & (m_0, n_0) = (1,1) \\ 0, & gcd(m_0, n_0) = 1, m_0 \neq n_0 \end{cases} \tag{3}$$

A family of unitary operators

- Let us determine when the operators $T(a_n)$ are unitary.
- Call an operator T C-unitary if $\frac{1}{C}T$ is unitary.
- **Corollary.** Assume a_n satisfies $\sum |a_n| n^{-1/2} < \infty$. Then the following are equivalent:
  (a) $|L(1/2 + ix; a_n)| = C$
  (b) $T(a_n)$ is C-unitary on $L^2[0, \infty)$
  (c) (a_n) satisfies

$$\sum_{k=1}^{\infty} \frac{a_{m_0 k} \overline{a_{n_0 k}}}{k} = \begin{cases} C^2, & (m_0, n_0) = (1, 1) \\ 0, & gcd(m_0, n_0) = 1, m_0 \neq n_0 \end{cases} \quad (3)$$

A family of unitary operators

- Let us determine when the operators $T(a_n)$ are unitary.
- Call an operator T C-unitary if $\frac{1}{C}T$ is unitary.
- **Corollary.** Assume a_n satisfies $\sum |a_n| n^{-1/2} < \infty$. Then the following are equivalent:
  (a) $|L(1/2 + ix; a_n)| = C$
  (b) $T(a_n)$ is C-unitary on $L^2[0, \infty)$
  (c) (a_n) satisfies

$$\sum_{k=1}^{\infty} \frac{a_{m_0 k} \overline{a_{n_0 k}}}{k} = \begin{cases} C^2, & (m_0, n_0) = (1, 1) \\ 0, & gcd(m_0, n_0) = 1, m_0 \neq n_0 \end{cases} \tag{3}$$

Unitary operators - examples

- A curious example of such an operator is $T(a_n)f(x) = f(x) + f(2x) - f(4x) + f(8x) - f(16x) + \dots$ associated with $a_n = 1, 1, 0, -1, 0, 0, 0, 1, \dots$ and $L(s; a_n) = \frac{2+2^s}{1+2^s}$. $T(a_n)$ is $\sqrt{2}$-unitary on $L^2[0, \infty)$.
- Note that in this case, the convolution-inverse of a_n is

$$(b_n) = (a_n)^{-1} = 1, -1, 0, 2, 0, 0, 0, -4, 0, 0, 0, 0, 0, 0, 0, 8, \dots$$

And so the inverse of $T(a_n)$ is not $T(b_n)$ (which is unbounded) but rather $T(a_n)^{-1}f = T(a_n)^*f = \sum_{n=1}^{\infty} \frac{a_n}{n} f\left(\frac{x}{n}\right)$

- A similar example is

$$Tf(x) = f(x) - f(2x) - f(4x) - f(8x) - \dots$$

Unitary operators - examples

- A curious example of such an operator is $T(a_n)f(x) = f(x) + f(2x) - f(4x) + f(8x) - f(16x) + \dots$ associated with $a_n = 1, 1, 0, -1, 0, 0, 0, 1, \dots$ and $L(s; a_n) = \frac{2+2^s}{1+2^s}$. $T(a_n)$ is $\sqrt{2}$-unitary on $L^2[0, \infty)$.
- Note that in this case, the convolution-inverse of a_n is

$$(b_n) = (a_n)^{-1} = 1, -1, 0, 2, 0, 0, 0, -4, 0, 0, 0, 0, 0, 0, 0, 8, \dots$$

And so the inverse of $T(a_n)$ is not $T(b_n)$ (which is unbounded) but rather $T(a_n)^{-1}f = T(a_n)^* f = \sum_{n=1}^{\infty} \frac{a_n}{n} f\left(\frac{x}{n}\right)$

- A similar example is

$$Tf(x) = f(x) - f(2x) - f(4x) - f(8x) - \dots$$

Unitary operators - examples

- A curious example of such an operator is $T(a_n)f(x) = f(x) + f(2x) - f(4x) + f(8x) - f(16x) + \dots$ associated with $a_n = 1, 1, 0, -1, 0, 0, 0, 1, \dots$ and $L(s; a_n) = \frac{2+2^s}{1+2^s}$. $T(a_n)$ is $\sqrt{2}$-unitary on $L^2[0, \infty)$.
- Note that in this case, the convolution-inverse of a_n is

$$(b_n) = (a_n)^{-1} = 1, -1, 0, 2, 0, 0, 0, -4, 0, 0, 0, 0, 0, 0, 0, 8, \dots$$

And so the inverse of $T(a_n)$ is not $T(b_n)$ (which is unbounded) but rather $T(a_n)^{-1}f = T(a_n)^*f = \sum_{n=1}^{\infty} \frac{a_n}{n} f\left(\frac{x}{n}\right)$

- A similar example is

$$Tf(x) = f(x) - f(2x) - f(4x) - f(8x) - \dots$$

Unitary operators - more examples

- For a natural number m, take $b^{(m)}_{m^k} = \left(\frac{m}{2}\right)^{k/2} a_{2^k}$ and $b^{(m)}_n = 0$ for $n \neq m^k$. Then
$$T_m f(x) = \sum b^{(m)}_n f(nx)$$
is again a $\sqrt{2}$-isometry.
- For unitary $T(a_n)$, $T(b_n)$, we have a new unitary operator $T(a_n * b_n) = T(a_n)T(b_n)$. Thus we can construct sequences a_n having larger support with $T(a_n)$ C-unitary.

Unitary operators - more examples

- For a natural number m, take $b_{m^k}^{(m)} = \left(\frac{m}{2}\right)^{k/2} a_{2^k}$ and $b_n^{(m)} = 0$ for $n \neq m^k$. Then
$$T_m f(x) = \sum b_n^{(m)} f(nx)$$
is again a $\sqrt{2}$-isometry.
- For unitary $T(a_n)$, $T(b_n)$, we have a new unitary operator $T(a_n * b_n) = T(a_n)T(b_n)$. Thus we can construct sequences a_n having larger support with $T(a_n)$ C-unitary.

The end

Thank you!