Integral functionals verifying a Brunn-Minkowski type inequality

Andrea Colesanti in collaboration with
Daniel Hug and Eugenia Saorín–Gomez

Asymptotic Geometric Analysis and Convexity

Toronto, 13–17 September 2010

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By a result of McMullen (1980), (1) $\Leftrightarrow$ (2).

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Note that in dimension two $\mathcal{M}$ is *linear* and the Brunn–Minkowski inequality becomes an equality, <u>for every choice of f</u>.

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Then what can be said about f? In particular, does it follow that f is a support function?

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  for every $K_0, K_1 \in \mathcal{K}^n$ and $t \in [0,1]$. This allows to remove the assumption $\mathcal{M} \geq 0$.

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As proved by McMullen (1990), the same result holds, without symmetry assumption, if (3) is replaced by monotonicity w.r.t. inclusion.

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Brunn–Minkowski inequality for $\mathcal{M} \Rightarrow g^{1/(n-1)}$ *concave*.

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- (4) becomes a functional inequality involving f, h and ϕ;
- in particular, its validity for every choice of h and ϕ reveals to be a powerful condition, from which one can deduce that f is a support function.

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A special case

- $n = 3$, f even and smooth.
- Choose $h \equiv 1$ (i.e. K is the unit ball of $\mathbb{R}^3$); inequality (4) implies
$$\int_{\mathbb{S}^2} f\phi^2 \, d\mathcal{H}^2 \leq \int_{\mathbb{S}^2} \langle H\nabla\phi, \nabla\phi\rangle \, d\mathcal{H}^2 \,, \tag{5}$$
for every $\phi \in C^\infty(\mathbb{S}^2)$, supported in a hemisphere, where
$$H = \text{cofactor matrix of } (f_{ij} + f\delta_{ij}).$$
- (5) forces $H \geq 0$ on $\mathbb{S}^2$;
- $$H \geq 0 \; \Rightarrow \; (f_{ij} + f\delta_{ij}) \geq 0 \,.$$
- The last condition is equivalent to say that f is a support function.