# On the location of roots of Steiner polynomials

María A. Hernández Cifre

(joint work with Martin Henk)

Universidad de Murcia

Toronto, September 2010

# Notation

- $\mathcal{K}^n$ = convex bodies (compact convex sets) in $\mathbb{R}^n$.

# Notation

- $\mathcal{K}^n$ = convex bodies (compact convex sets) in $\mathbb{R}^n$.
- $K$ = arbitrary convex body.

# Notation

- $\mathcal{K}^n$ = convex bodies (compact convex sets) in $\mathbb{R}^n$.
- $K$ = arbitrary convex body.
- $B_n$ = Euclidean unit ball.

# Notation

- $\mathcal{K}^n$ = convex bodies (compact convex sets) in $\mathbb{R}^n$.
- $K$ = arbitrary convex body.
- $B_n$ = Euclidean unit ball.
- $E$ = fixed convex body with non-empty interior.

# Notation

- $\mathcal{K}^n$ = convex bodies (compact convex sets) in $\mathbb{R}^n$.
- $K$ = arbitrary convex body.
- $B_n$ = Euclidean unit ball.
- $E$ = fixed convex body with non-empty interior.
- $\mathrm{vol}(K)$ = volume (Lebesgue measure) of $K$.

# Notation

- $\mathcal{K}^n$ = convex bodies (compact convex sets) in $\mathbb{R}^n$.
- $K$ = arbitrary convex body.
- $B_n$ = Euclidean unit ball.
- $E$ = fixed convex body with non-empty interior.
- $\mathrm{vol}(K)$ = volume (Lebesgue measure) of $K$.
- $+$ = Minkowski (vectorial) addition:

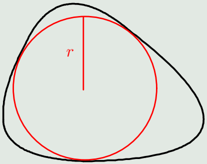
$$A + B = \{a + b : a \in A, b \in B\} = \bigcup_{b \in B} (A + b)$$

# Inradius and circumradius

- The **relative inradius** $\mathrm{r}(K;E)$ of $K$ with respect to $E$:

$$\mathrm{r}(K;E) = \max\big\{r \geq 0 : \text{some translate of } rE \subset K\big\}.$$

## The classical inradius and circumradius




If $E = B_n$: $\mathrm{r}(K)$

# Inradius and circumradius

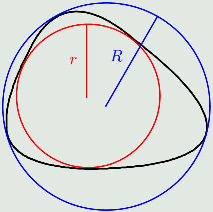
- The **relative inradius** $\mathrm{r}(K;E)$ of $K$ with respect to $E$:

$$\mathrm{r}(K;E) = \max\bigl\{r \geq 0 : \text{some translate of } rE \subset K\bigr\}.$$

- The **relative circumradius** $\mathrm{R}(K;E)$ of $K$ with respect to $E$:

$$\mathrm{R}(K;E) = \min\bigl\{R > 0 : \text{some translate of } K \subset RE\bigr\}$$

## The classical inradius and circumradius

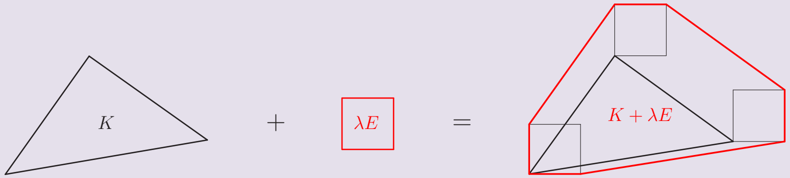



If $E = B_n$: $\mathrm{r}(K)$, $\mathrm{R}(K)$

# The (relative) outer parallel body

## The (relative) outer parallel body

$$\left.\begin{array}{l} K \text{ convex body} \\ \lambda \geq 0, \quad E \text{ fixed} \end{array}\right\} \rightsquigarrow \quad K + \lambda E = \textit{(relative) outer parallel body of } K \text{ at distance } \lambda$$



# The relative Steiner formula

## The relative Steiner formula

The volume of the Minkowski sum $K + \lambda E$ is a polynomial of degree $n$ in the parameter $\lambda$, the so called **relative Steiner polynomial** of $K$,

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)\lambda^i.$$

The coefficients $\mathrm{W}_i(K; E) = \mathrm{V}\big(K, \overset{(n-i)}{\ldots}, K, E, \overset{(i)}{\ldots}, E\big)$ are called the **relative quermassintegrals** of $K$.

# The relative Steiner formula

## The relative Steiner formula

The volume of the Minkowski sum $K+\lambda E$ is a polynomial of degree $n$ in the parameter $\lambda$, the so called **relative Steiner polynomial** of $K$,

$$\mathrm{vol}(K+\lambda E)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\lambda^i.$$

The coefficients $\mathrm{W}_i(K;E)=\mathrm{V}\left(K,\overset{(n-i)}{\ldots},K,E,\overset{(i)}{\ldots},E\right)$ are called the **relative quermassintegrals** of $K$.

- $\mathrm{W}_0(K;E)=\mathrm{vol}(K)$, $\mathrm{W}_n(K;E)=\mathrm{vol}(E)$.

# The relative Steiner formula

## The relative Steiner formula

The volume of the Minkowski sum $K + \lambda E$ is a polynomial of degree $n$ in the parameter $\lambda$, the so called **relative Steiner polynomial** of $K$,

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)\lambda^i.$$

The coefficients $\mathrm{W}_i(K; E) = \mathrm{V}\big(K, \overset{(n-i)}{\ldots}, K, E, \overset{(i)}{\ldots}, E\big)$ are called the **relative quermassintegrals** of $K$.

- $\mathrm{W}_0(K; E) = \mathrm{vol}(K)$, $\mathrm{W}_n(K; E) = \mathrm{vol}(E)$.

## The (classical) Steiner formula, 1840

$$\mathrm{vol}(K + \lambda B_n) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K)\lambda^i.$$

# The relative Steiner formula

## The relative Steiner formula

The volume of the Minkowski sum $K+\lambda E$ is a polynomial of degree $n$ in the parameter $\lambda$, the so called **relative Steiner polynomial** of $K$,

$$\mathrm{vol}(K+\lambda E)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\lambda^i.$$

The coefficients $\mathrm{W}_i(K;E)=\mathrm{V}\big(K,\overset{(n-i)}{\ldots},K,E,\overset{(i)}{\ldots},E\big)$ are called the **relative quermassintegrals** of $K$.

## Notation: we denote the Steiner polynomial by

$$f_{K;E}(\lambda)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\lambda^i$$

# The relative Steiner formula

## The relative Steiner formula

The volume of the Minkowski sum $K + \lambda E$ is a polynomial of degree $n$ in the parameter $\lambda$, the so called **relative Steiner polynomial** of $K$,

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)\lambda^i.$$

The coefficients $\mathrm{W}_i(K; E) = \mathrm{V}\big(K, \overset{(n-i)}{\ldots}, K, E, \overset{(i)}{\ldots}, E\big)$ are called the **relative quermassintegrals** of $K$.

## Notation: we denote the Steiner polynomial by

$$f_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E) z^i, \qquad z \in \mathbb{C}$$

- Considering $f_{K;E}(z)$ as a formal polynomial in a complex variable $z \in \mathbb{C}$, what can we say about its roots?

# Bonnesen's inequality and Teissier's problem

# Bonnesen's inequality and Teissier's problem

**Bonnesen (Blaschke)'s inequality: In the plane...**

$$\mathrm{A}(K) + 2\mathrm{W}_1(K;E)\lambda + \mathrm{A}(E)\lambda^2 \leq 0 \quad \text{if} \quad -\mathrm{R}(K;E) \leq \lambda \leq -\mathrm{r}(K;E)$$

# Bonnesen's inequality and Teissier's problem

## Bonnesen (Blaschke)'s inequality: In the plane...

$$\underbrace{\mathrm{A}(K) + 2\mathrm{W}_1(K;E)\lambda + \mathrm{A}(E)\lambda^2}_{\text{Steiner polynomial}} \leq 0 \quad \text{if} \quad -\mathrm{R}(K;E) \leq \lambda \leq -\mathrm{r}(K;E)$$

# Bonnesen's inequality and Teissier's problem

**Bonnesen (Blaschke)'s inequality: In the plane...**

$$\mathrm{A}(K) + 2\mathrm{W}_1(K;E)\lambda + \mathrm{A}(E)\lambda^2 \leq 0 \quad \text{if} \quad -\mathrm{R}(K;E) \leq \lambda \leq -\mathrm{r}(K;E)$$

Bonnesen's inequality $\Longrightarrow \lambda_1 \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \lambda_2 \leq 0$

# Bonnesen's inequality and Teissier's problem

**Bonnesen (Blaschke)'s inequality: In the plane...**

$$\mathrm{A}(K)+2\mathrm{W}_1(K;E)\lambda+\mathrm{A}(E)\lambda^2\leq 0 \quad \text{if} \quad -\mathrm{R}(K;E)\leq\lambda\leq-\mathrm{r}(K;E)$$

Bonnesen's inequality $\Longrightarrow \lambda_1\leq-\mathrm{R}(K;E)\leq-\mathrm{r}(K;E)\leq\lambda_2\leq 0$

B. Teissier: Bonnesen-type inequalities in algebraic geometry I. Introduction to the problem. *Seminar on Differential Geometry*, Princeton Univ. Press, Princeton, N. J., 1982, 85–105.

- Let $\gamma_1,\dots,\gamma_n$ be the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1)\leq\cdots\leq\mathrm{Re}(\gamma_n)$. For which convex bodies it holds

$$\mathrm{Re}(\gamma_1)\leq-\mathrm{R}(K;E)\leq-\mathrm{r}(K;E)\leq\mathrm{Re}(\gamma_n)\leq 0\,?$$

# Teissier's problem and Sangwine-Yager's conjecture

**Conjecture:** For any $K \in \mathcal{K}^n$,

if $\gamma_1, \dots, \gamma_n$ are the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$, then

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0.$$

J. R. Sangwine-Yager: Bonnesen-style inequalities for Minkowski relative geometry. *Trans. Amer. Math. Soc.* **307** (1) (1988), 373–382.

# Teissier's problem and Sangwine-Yager's conjecture

**Conjecture: For any $K \in \mathcal{K}^n$,**

if $\gamma_1, \dots, \gamma_n$ are the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \le \cdots \le \mathrm{Re}(\gamma_n)$, then

$$\mathrm{Re}(\gamma_1) \le -\mathrm{R}(K;E) \le -\mathrm{r}(K;E) \le \mathrm{Re}(\gamma_n) \le 0.$$

- Bonnesen's inequality $\Longrightarrow$ Conjecture is true if $n = 2$.

J. R. Sangwine-Yager: Bonnesen-style inequalities for Minkowski relative geometry. *Trans. Amer. Math. Soc.* **307** (1) (1988), 373–382.

# Teissier's problem and Sangwine-Yager's conjecture

**Conjecture:** For any $K \in \mathcal{K}^n$,

if $\gamma_1, \dots, \gamma_n$ are the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \le \cdots \le \mathrm{Re}(\gamma_n)$, then

$$\mathrm{Re}(\gamma_1) \le -\mathrm{R}(K;E) \le -\mathrm{r}(K;E) \le \mathrm{Re}(\gamma_n) \le 0.$$

- Bonnesen's inequality $\Longrightarrow$ Conjecture is true if $n = 2$.

H.C., M. Henk: Notes on the roots of Steiner polynomials. *Rev. Mat. Iberoamericana* **24** (2) (2008), 631–644.

**The full conjecture is false:**

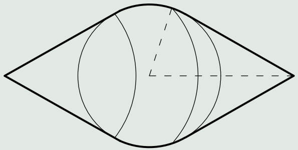
# Teissier's problem and Sangwine-Yager's conjecture

**Conjecture: For any $K \in \mathcal{K}^n$,**

if $\gamma_1, \dots, \gamma_n$ are the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$, then

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0.$$

A cap-body of $E$ is the convex hull of $E$ and countably many points such that the line segment joining any pair of these points intersects $E$.



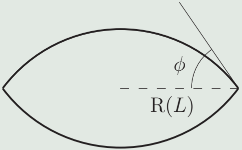
**The full conjecture is false:**

# Teissier's problem and Sangwine-Yager's conjecture

**Conjecture: For any $K \in \mathcal{K}^n$,**

if $\gamma_1, \dots, \gamma_n$ are the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \dots \leq \mathrm{Re}(\gamma_n)$, then

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0.$$




**The full conjecture is false:**

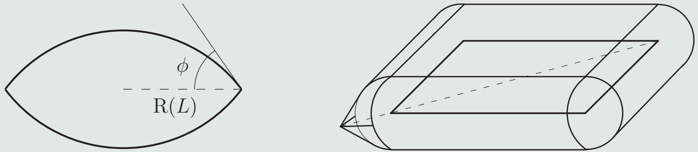
- Circumradius part. Counterexample: A symmetric lens in $\mathbb{R}^3$.

# Teissier's problem and Sangwine-Yager's conjecture

**Conjecture: For any $K \in \mathcal{K}^n$,**

if $\gamma_1, \dots, \gamma_n$ are the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \le \cdots \le \mathrm{Re}(\gamma_n)$, then

$$\mathrm{Re}(\gamma_1) \le -\mathrm{R}(K;E) \le -\mathrm{r}(K;E) \le \mathrm{Re}(\gamma_n) \le 0.$$




**The full conjecture is false:**

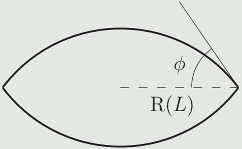
- Circumradius part. Counterexample: A symmetric lens in $\mathbb{R}^3$.
- Inradius part. Counterexample: A cap-body of an outer parallel body of a square in $\mathbb{R}^3$.

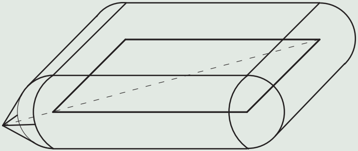
# Teissier's problem and Sangwine-Yager's conjecture

**Conjecture: For any $K \in \mathcal{K}^n$,**

if $\gamma_1, \ldots, \gamma_n$ are the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$, then

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0.$$



**The full conjecture is false:**

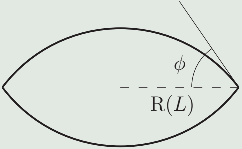
- Circumradius part. Counterexample: A symmetric lens in $\mathbb{R}^3$.
- Inradius part. Counterexample: A cap-body of an outer parallel body of a square in $\mathbb{R}^3$.
- Negativity part: Counterexample in $\mathbb{R}^{12}$.

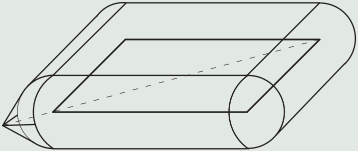
# Teissier's problem and Sangwine-Yager's conjecture

**Conjecture:** For any $K \in \mathcal{K}^n$,

if $\gamma_1, \ldots, \gamma_n$ are the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$, then

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0.$$



**The full conjecture is false:**

- Circumradius part. Counterexample: A symmetric lens in $\mathbb{R}^3$.
- Inradius part. Counterexample: A cap-body of an outer parallel body of a square in $\mathbb{R}^3$.
- Negativity part: Counterexample in $\mathbb{R}^{12}$. True for $n = 3, \ldots, 9$.

# Solutions for Teissier's problem?

- The problem arises to find solutions (particular sets/families of sets) for Teissier's problem:

**Let $\gamma_1, \dots, \gamma_n$ be the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \le \cdots \le \mathrm{Re}(\gamma_n)$.**

For which convex bodies it holds

$$\mathrm{Re}(\gamma_1) \le -\mathrm{R}(K;E) \le -\mathrm{r}(K;E) \le \mathrm{Re}(\gamma_n) \le 0\,?$$

# Solutions for Teissier's problem?

- The problem arises to find solutions (particular sets/families of sets) for Teissier's problem:

**Let $\gamma_1,\ldots,\gamma_n$ be the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1)\leq\cdots\leq\mathrm{Re}(\gamma_n)$.**

For which convex bodies it holds

$$\mathrm{Re}(\gamma_1)\leq -\mathrm{R}(K;E)\leq -\mathrm{r}(K;E)\leq \mathrm{Re}(\gamma_n)\leq 0\,?$$

- All the sets in $\mathbb{R}^2$ provide a solution for Teissier's problem.
- $\mathbb{R}^3$: it is easy to find solutions (cylinders, cones, boxes...).

# Solutions for Teissier's problem?

- The problem arises to find solutions (particular sets/families of sets) for Teissier's problem:

**Let $\gamma_1, \dots, \gamma_n$ be the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$.**

For which convex bodies it holds

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0\,?$$

- All the sets in $\mathbb{R}^2$ provide a solution for Teissier's problem.
- $\mathbb{R}^3$: it is easy to find solutions (cylinders, cones, boxes...).
- $\mathbb{R}^n$: even to find particular sets as solutions is not easy...

# Solutions for Teissier's problem?

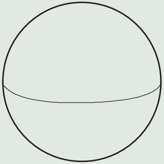
- The problem arises to find solutions (particular sets/families of sets) for Teissier's problem:

**Let $\gamma_1, \ldots, \gamma_n$ be the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$.**

For which convex bodies it holds

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0\,?$$

- All the sets in $\mathbb{R}^2$ provide a solution for Teissier's problem.
- $\mathbb{R}^3$: it is easy to find solutions (cylinders, cones, boxes...).
- $\mathbb{R}^n$: even to find particular sets as solutions is not easy...



- Negativity part: YES
- Inradius part: YES
- Circumradius part: YES

# Solutions for Teissier's problem?

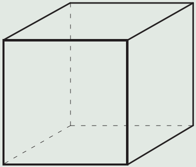
- The problem arises to find solutions (particular sets/families of sets) for Teissier's problem:

**Let $\gamma_1,\dots,\gamma_n$ be the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1)\le\dots\le\mathrm{Re}(\gamma_n)$.**

For which convex bodies it holds

$$\mathrm{Re}(\gamma_1)\le -\mathrm{R}(K;E)\le -\mathrm{r}(K;E)\le \mathrm{Re}(\gamma_n)\le 0\,?$$

- All the sets in $\mathbb{R}^2$ provide a solution for Teissier's problem.
- $\mathbb{R}^3$: it is easy to find solutions (cylinders, cones, boxes...).
- $\mathbb{R}^n$: even to find particular sets as solutions is not easy...



- Negativity part: YES (Katsnelson, 2009)
- Inradius part: YES
- Circumradius part: ?

# Solutions for Teissier's problem?

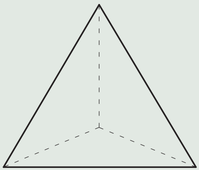
- The problem arises to find solutions (particular sets/families of sets) for Teissier's problem:

**Let $\gamma_1, \ldots, \gamma_n$ be the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$.**

For which convex bodies it holds

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0\,?$$

- All the sets in $\mathbb{R}^2$ provide a solution for Teissier's problem.
- $\mathbb{R}^3$: it is easy to find solutions (cylinders, cones, boxes...).
- $\mathbb{R}^n$: even to find particular sets as solutions is not easy...



- Negativity part: ?
- Inradius part: YES
- Circumradius part: ?

# Solutions for Teissier's problem?

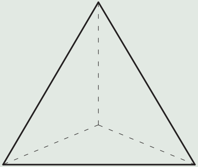
- The problem arises to find solutions (particular sets/families of sets) for Teissier's problem:

Let $\gamma_1, \dots, \gamma_n$ be the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$.

For which convex bodies it holds

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0\,?$$

- All the sets in $\mathbb{R}^2$ provide a solution for Teissier's problem.
- $\mathbb{R}^3$: it is easy to find solutions (cylinders, cones, boxes...).
- $\mathbb{R}^n$: even to find particular sets as solutions is not easy...



- Negativity part: ?...Real roots?
- Inradius part: YES
- Circumradius part: ?

# Solutions for Teissier's problem?

- The problem arises to find solutions (particular sets/families of sets) for Teissier's problem:

**Let $\gamma_1, \dots, \gamma_n$ be the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$.**

For which convex bodies it holds

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0\,?$$

- Are there solutions for arbitrary dimension $n$?

# Solutions for Teissier's problem?

- The problem arises to find solutions (particular sets/families of sets) for Teissier's problem:

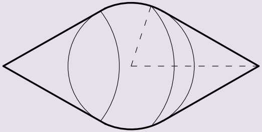
**Let $\gamma_1, \ldots, \gamma_n$ be the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$.**

For which convex bodies it holds

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0\,?$$

- Are there solutions for arbitrary dimension $n$?
- Only 2 families of convex bodies are known to be solution of Teissier's problem:

Cap-bodies and *sausages* provide a solution



# Locating the roots of the Steiner polynomial

H.C., M. Henk: On the location of roots of Steiner polynomials. *To appear in B. Braz. Math. Soc.*

# Locating the roots of the Steiner polynomial

Can be a root of the Steiner polynomial pure imaginary complex?

# Locating the roots of the Steiner polynomial

Can be a root of the Steiner polynomial pure imaginary complex?

## Pure imaginary roots

- Not all roots can be imaginary pure complex numbers.

$$f_{K;E}(z) \neq \begin{cases} \mathrm{vol}(E)\displaystyle\prod_{j=1}^{n/2}(z \pm b_j \mathrm{i}), & n \text{ even} \\ \mathrm{vol}(E)(z-a)\displaystyle\prod_{j=1}^{(n-1)/2}(z \pm b_j \mathrm{i}), & n \text{ odd} \end{cases}$$

# Locating the roots of the Steiner polynomial

Can be a root of the Steiner polynomial pure imaginary complex?

## Pure imaginary roots

- Not all roots can be imaginary pure complex numbers.
- A pure complex number $b\mathrm{i}$, $b \in \mathbb{R}$ and $b \neq 0$, cannot be a root of the Steiner polynomial for any convex body $K \in \mathcal{K}^n$ with $n \leq 9$.

$$f_{K;E}(z) \neq \begin{cases} \mathrm{vol}(E) \displaystyle\prod_{j=1}^{n/2} (z \pm b_j \mathrm{i}), & n \text{ even} \\ \mathrm{vol}(E)(z-a) \displaystyle\prod_{j=1}^{(n-1)/2} (z \pm b_j \mathrm{i}), & n \text{ odd} \end{cases}$$

# Locating the roots of the Steiner polynomial

Can be a root of the Steiner polynomial pure imaginary complex?

## Pure imaginary roots

- Not all roots can be imaginary pure complex numbers.
- A pure complex number $b\mathrm{i}$, $b \in \mathbb{R}$ and $b \neq 0$, cannot be a root of the Steiner polynomial for any convex body $K \in \mathcal{K}^n$ with $n \leq 9$.
- However, in arbitrary dimension (at least in dim $n \geq 12$) the Steiner polynomial may have pure imaginary roots (but not all of them!).

$$f_{K;E}(z) \neq \begin{cases} \mathrm{vol}(E) \displaystyle\prod_{j=1}^{n/2} (z \pm b_j \mathrm{i}), & n \text{ even} \\ \mathrm{vol}(E)(z-a) \displaystyle\prod_{j=1}^{(n-1)/2} (z \pm b_j \mathrm{i}), & n \text{ odd} \end{cases}$$

# Locating the roots of the Steiner polynomial

$$f_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E) z^i$$

Which complex numbers can be roots of the Steiner polynomial?

# Locating the roots of the Steiner polynomial

$$f_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E) z^i$$

Which complex numbers can be roots of the Steiner polynomial?

$$\mathcal{R}(n,E) := \left\{ z \in \mathbb{C}^+ : f_{K;E}(z) = 0 \text{ for some } K \in \mathcal{K}^n \right\}$$

# Locating the roots of the Steiner polynomial

$$f_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E) z^i$$

Which complex numbers can be roots of the Steiner polynomial?

$$\Updownarrow$$

**Problem:** to determine $\mathcal{R}(n,E)$

$$\mathcal{R}(n,E) := \left\{ z \in \mathbb{C}^+ : f_{K;E}(z) = 0 \text{ for some } K \in \mathcal{K}^n \right\}$$

# Locating the roots of the Steiner polynomial

$$f_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E) z^i$$

Which complex numbers can be roots of the Steiner polynomial?

$$\Updownarrow$$

**Problem:** to determine $\mathcal{R}(n,E)$

$$\mathcal{R}(n,E) := \left\{ z \in \mathbb{C}^+ : f_{K;E}(z) = 0 \text{ for some } K \in \mathcal{K}^n \right\}$$

**Theorem**

$\mathcal{R}(n,E)$ is a convex cone containing the non-positive real axis.

# Locating the roots of the Steiner polynomial

$$f_{K;E}(z)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)z^i$$

Which complex numbers can be roots of the Steiner polynomial?

$$\Updownarrow$$

**Problem:** to determine $\mathcal{R}(n,E)$

$$\mathcal{R}(n,E):=\left\{z\in\mathbb{C}^+ : f_{K;E}(z)=0 \text{ for some } K\in\mathcal{K}^n\right\}$$

**Theorem**

$\mathcal{R}(n,E)$ is a convex cone containing the non-positive real axis.

- By the isoperimetric inequality, for $n=2$ the Steiner polynomial has only real roots. Therefore, $\mathcal{R}(2,E)=\left\{x+y\mathrm{i}\in\mathbb{C}^+ : x\leq 0, y=0\right\}$.

# Locating the roots of the Steiner polynomial

**Theorem: $n = 3$**

If $E \in \mathcal{K}_0^3$ is a cap-body of a planar convex body then

$$\mathcal{R}(3, E) = \left\{x + y\mathrm{i} \in \mathbb{C}^+ : x + \sqrt{3}\, y \leq 0\right\},$$

otherwise

$$\mathcal{R}(3, E) = \left\{x + y\mathrm{i} \in \mathbb{C}^+ : x + \sqrt{3}\, y < 0\right\} \cup \{0\}.$$

# Locating the roots of the Steiner polynomial

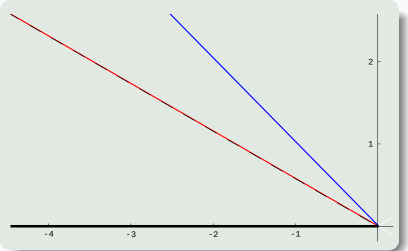
**Theorem: $n=4$**

If $E\in\mathcal{K}_0^4$ is a cap-body of a 3-dimensional convex body then

$$\mathcal{R}(4,E)=\Big\{x+y\mathrm{i}\in\mathbb{C}^+ : x+y\leq 0\Big\},$$

otherwise

$$\mathcal{R}(4,E)=\Big\{x+y\mathrm{i}\in\mathbb{C}^+ : x+y<0\Big\}\cup\{0\}.$$



# Locating the roots of the Steiner polynomial

**Theorem: $n = 4$**

If $E \in \mathcal{K}_0^4$ is a cap-body of a 3-dimensional convex body then

$$\mathcal{R}(4, E) = \left\{x + y\mathrm{i} \in \mathbb{C}^+ : x + y \leq 0\right\},$$

otherwise

$$\mathcal{R}(4, E) = \left\{x + y\mathrm{i} \in \mathbb{C}^+ : x + y < 0\right\} \cup \{0\}.$$

Moreover, if $n \geq 12$ then

$\{z \in \mathbb{C}^+ : \mathrm{Re}(z) \leq 0\} \subset \mathcal{R}(n, E)$.

# Locating the roots of the Steiner polynomial

## Open questions:

- $\mathcal{R}(n,E) \subseteq \mathcal{R}(n+1,E)$ for all $n$?

Moreover, if $n \geq 12$ then

$\{z \in \mathbb{C}^+ : \mathrm{Re}(z) \leq 0\} \subset \mathcal{R}(n,E)$.

# Locating the roots of the Steiner polynomial

## Open questions:

- $\mathcal{R}(n,E) \subseteq \mathcal{R}(n+1,E)$ for all $n$?
- For high dimensions, is the cone $\mathcal{R}(n,E)$ closer and closer to the upper half plane?, i.e.,

$$\mathcal{R}(n,E) \stackrel{n\to\infty}{\longrightarrow} \mathbb{C}^+?$$

Moreover, if $n \geq 12$ then

$\{z \in \mathbb{C}^+ : \mathrm{Re}(z) \leq 0\} \subset \mathcal{R}(n,E)$.

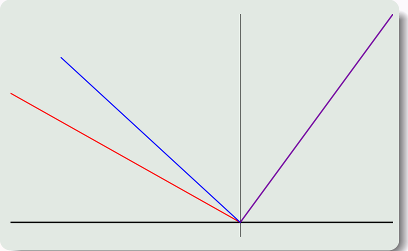
# Locating the roots of the Steiner polynomial

## Open questions:

- $\mathcal{R}(n,E) \subseteq \mathcal{R}(n+1,E)$ for all $n$?
- For high dimensions, is the cone $\mathcal{R}(n,E)$ closer and closer to the upper half plane?, i.e.,

$$\mathcal{R}(n,E) \overset{n\to\infty}{\longrightarrow} \mathbb{C}^+?$$

- Is the closure $\mathrm{cl}\,\mathcal{R}(n,E)$ independent of the gauge body $E$?



Moreover, if $n \geq 12$ then

$\{z \in \mathbb{C}^+ : \mathrm{Re}(z) \leq 0\} \subset \mathcal{R}(n,E)$.

# Characterizing the gauge body $E$

Is it possible to characterize families of convex bodies by properties of the roots of the Steiner polynomial?

# Characterizing the gauge body $E$

Is it possible to characterize families of convex bodies by properties of the roots of the Steiner polynomial?

## A first characterization: the gauge body $E$

Let $K \in \mathcal{K}^n$, let $\gamma_i$, $i = 1, \dots, n$, be the roots of $f_{K;E}(z)$, and let $a > 0$. Then $\left|\mathrm{Re}(\gamma_i)\right| = a > 0$ for all $i = 1, \dots, n$ if and only if $K = aE$ (up to translations).

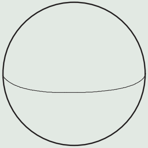
# Characterizing the gauge body $E$

Is it possible to characterize families of convex bodies by properties of the roots of the Steiner polynomial?

**A first characterization: the gauge body $E$**

Let $K \in \mathcal{K}^n$, let $\gamma_i$, $i = 1, \dots, n$, be the roots of $f_{K;E}(z)$, and let $a > 0$. Then $\left|\mathrm{Re}(\gamma_i)\right| = a > 0$ for all $i = 1, \dots, n$ if and only if $K = aE$ (up to translations).

- Thus, the Euclidean ball $aB_n$ is characterized by an $n$-fold real root, $-a$, of the Steiner polynomial $f_{K;B_n}(z)$.



$$\Longleftrightarrow \qquad f_{K;B_n}(z) = \mathrm{vol}(B_n)(z+a)^n$$

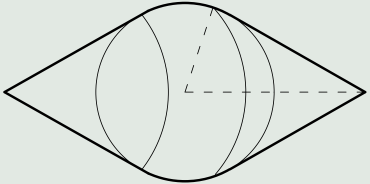
# Characterizing cap-bodies

## Characterizing cap-bodies of $E$

Let $K, E \in \mathcal{K}_0^n$, $E \subset K$, and let $\gamma_1, \dots, \gamma_n$ be the roots of $f_{K;E}(z)$. Then $K$ is a cap-body of $E$ if and only if there exists $\alpha \in (0,1)$ such that $(1/\alpha)^{1/n}(1 + 1/\gamma_k)$, $k = 1, \dots, n$, are the $n$ roots of unity, i.e.,

$$\gamma_k^{-1} = \alpha^{1/n}\, \mathrm{e}^{\frac{2\pi(k-1)}{n}\mathrm{i}} - 1, \qquad \text{for} \quad k = 1, \dots, n$$

## Cap-body of $B_3$



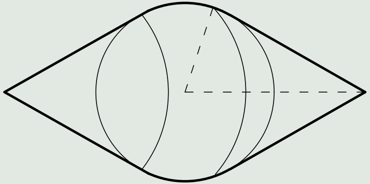
# Characterizing cap-bodies

Explicitly, the roots of the Steiner polynomial of a cap-body $K$ of $E$ are

$$\gamma_k = \frac{-1+\beta\cos\frac{2\pi(k-1)}{n}}{1+\beta\Bigl(\beta-2\cos\frac{2\pi(k-1)}{n}\Bigr)} - \frac{\beta\sin\frac{2\pi(k-1)}{n}}{1+\beta\Bigl(\beta-2\cos\frac{2\pi(k-1)}{n}\Bigr)}\,\mathrm{i},$$

with $\beta=\bigl(1-\mathrm{vol}(E)/\mathrm{vol}(K)\bigr)^{1/n}$, $k=1,\dots,n$.

## Cap-body of $B_3$



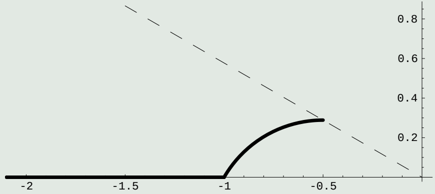
# Characterizing cap-bodies

Explicitly, the roots of the Steiner polynomial of a cap-body $K$ of $E$ are

$$\gamma_k = \frac{-1+\beta\cos\frac{2\pi(k-1)}{n}}{1+\beta\left(\beta-2\cos\frac{2\pi(k-1)}{n}\right)} - \frac{\beta\sin\frac{2\pi(k-1)}{n}}{1+\beta\left(\beta-2\cos\frac{2\pi(k-1)}{n}\right)}\,\mathrm{i},$$

with $\beta = \bigl(1-\mathrm{vol}(E)/\mathrm{vol}(K)\bigr)^{1/n}$, $k=1,\dots,n$.

**Complex numbers $z$ which are roots of $f_{K;B_3}(z)$ for any cap-body $K$ of $B_3$:**

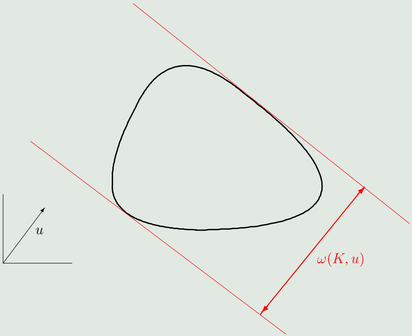


# Characterizing constant width sets

## Constant width sets

A convex body $K \in \mathcal{K}^n$ is a **constant width set** if the width keeps constant in any direction.

## The width in a given direction



# Characterizing constant width sets

**Theorem**

Let $\mathrm{b} > 0$. A convex body $K \in \mathcal{K}^n$ verifies the relations

$$\mathrm{W}_{n-k}(K;E) = \sum_{i=0}^{k}(-1)^i \binom{k}{i} \mathrm{b}^{k-i}\mathrm{W}_{n-i}(K;E) \qquad (\dagger)$$

for $k = 0, 1, \ldots, n$, if and only if all the roots of its Steiner polynomial are symmetric with respect to $-\mathrm{b}/2$.

# Characterizing constant width sets

## Theorem

Let $\mathrm{b} > 0$. A convex body $K \in \mathcal{K}^n$ verifies the relations

$$\mathrm{W}_{n-k}(K; E) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \mathrm{b}^{k-i} \mathrm{W}_{n-i}(K; E) \qquad (\dagger)$$

for $k = 0, 1, \ldots, n$, if and only if all the roots of its Steiner polynomial are symmetric with respect to $-\mathrm{b}/2$.

In particular, if $(\dagger)$ holds and $n$ is odd then $-\mathrm{b}/2$ is a root of $f_{K;E}(z)$.

# Characterizing constant width sets

## Theorem

Let $\mathrm{b} > 0$. A convex body $K \in \mathcal{K}^n$ verifies the relations

$$\mathrm{W}_{n-k}(K;E) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \mathrm{b}^{k-i} \mathrm{W}_{n-i}(K;E) \qquad (\dagger)$$

for $k = 0, 1, \ldots, n$, if and only if all the roots of its Steiner polynomial are symmetric with respect to $-\mathrm{b}/2$.

In particular, if $(\dagger)$ holds and $n$ is odd then $-\mathrm{b}/2$ is a root of $f_{K;E}(z)$.

## Remark:

Equations $(\dagger)$ are not independent: they are equivalent to the $(n+1)/2$ relations obtained for only odd values of $k$.

# Characterizing constant width sets

## Theorem

Let $\mathrm{b} > 0$. A convex body $K \in \mathcal{K}^n$ verifies the relations

$$\mathrm{W}_{n-k}(K;E) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \mathrm{b}^{k-i} \mathrm{W}_{n-i}(K;E) \qquad (\dagger)$$

for $k = 0, 1, \ldots, n$, if and only if all the roots of its Steiner polynomial are symmetric with respect to $-\mathrm{b}/2$.

In particular, if $(\dagger)$ holds and $n$ is odd then $-\mathrm{b}/2$ is a root of $f_{K;E}(z)$.

## Constant width sets:

When $E = B_n$, **constant width sets** verify the relations $(\dagger)$. Thus, a constant width set $K$ with breath $\mathrm{b}$ verifies that all the roots of $f_{K;B_n}(z)$ are symmetric with respect to $-\mathrm{b}/2$.

# Characterizing sausages

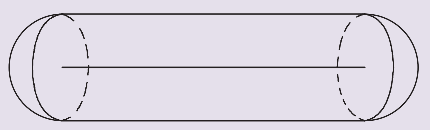
## Conjecture

Let $K \in \mathcal{K}^n$, $n \geq 2$. Then $K$ is a sausage with inradius $\mathrm{r}(K)$ if and only if $-\mathrm{r}(K)$ is an $(n-1)$-fold root of $f_{K;B_n}(z)$.

# Characterizing sausages

**Conjecture**

Let $K \in \mathcal{K}^n$, $n \geq 2$. Then $K$ is a sausage with inradius $\mathrm{r}(K)$ if and only if $-\mathrm{r}(K)$ is an $(n-1)$-fold root of $f_{K;B_n}(z)$.



**Theorem**

Let $K \in \mathcal{K}^n$, $n \geq 2$. Then $K$ is a sausage with inradius $\mathrm{r}(K)$ if and only if $-\mathrm{r}(K)$ is an $(n-1)$-fold root of $f_{K;B_n}(z)$ and all its 2-dimensional projections onto any 2-dimensional linear subspace have inradius $\mathrm{r}(K)$.

# On the location of roots of Steiner polynomials

María A. Hernández Cifre

(joint work with Martin Henk)

Universidad de Murcia

Toronto, September 2010