Factoring Sobolev inequalities through classes of functions

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Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported $C^{1)}$ function.

Sobolev inequality

$$\|\nabla f\|_p \geq \mathbf{C_{p,n}} \|f\|_q \ , \qquad p \in [1, n), \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}$$

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where $\|\nabla f\|_p^p = \int_{\mathbb{R}^n} |\nabla f(x)|^p dx$

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Extreme case $p = n$ and $q = \infty$ not true.

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- **Improvements and extensions from Geometry (left hand side).** Lutwak, Yang, Zhang, Cianchi, Haberl, Schuster, Xiao...
- **Extensions and improvements from Analysis (right hand side)** Moser-Trudinger, Hanson, Brezis-Wainger, Maly-Pick, Tartar, Bastero-Milman-Ruiz, Martin...
- **The results**

Two remarks on Sobolev inequality

- The case $p = 1$ is equivalent to the isoperimetric inequality,

$$\|\nabla f\|_1 \geq n\, \omega_n^{\frac{1}{n}} \|f\|_{\frac{n}{n-1}} \iff S(\partial K) \geq n\, \omega_n^{\frac{1}{n}} |K|_n^{\frac{n-1}{n}}$$

- Sobolev inequality follows from Polya-Szegö rearrangement inequality

$$\|\nabla f\|_p \geq \|\nabla f^\circ\|_p \qquad\qquad p \geq 1$$

where $f^\circ(x) := f^*(\omega_n |x|^n)$, is the radial extension to $\mathbb{R}^n$ of the nonincreasing rearrangement f^*.

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$$\|\nabla f\|_p \geq \|\nabla f^\circ\|_p \geq (\text{Hardy ineq.}) \geq \mathbf{C_{p,n}} \|f\|_q, \qquad p \in [1, n)$$

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Improvements and extensions (Geometry)

Consider the space

$$\mathcal{E}_p(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R}; \mathcal{E}_p(f) := \frac{1}{I_p} \left(\int_{S^{n-1}} \|D_u f\|_p^{-n} du \right)^{-\frac{1}{n}} < \infty \right\}$$

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Zhang ($p = 1$, 1999), Lutwak-Yang-Zhang (general case, 2002)

$$\mathcal{E}_p(f) \geq \mathcal{E}_p(f^\circ) \ , \qquad 1 \leq p < \infty$$

Remark and Observation

- The case $p = 1$ (Zhang) is equivalent to the Petty projection inequality.
- The case $p > 1$ (Lutwak-Yang-Zhang) uses involved L_p-Brunn-Minkowsky theory.

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Using Zhang's original ideas ($p = 1$) and techniques from the usual proof of the Polya-Szegö inequality, one has (penalty on the constants)

Proposition (Alonso-Gutiérrez, Bastero, B.)

Let $1 \leq p < \infty$ then

$$\mathcal{E}_p(f^\circ) \leq \frac{I_p}{I_1}\mathcal{E}_p(f)$$

Improvements and extensions (Geometry)

Corollary

Let $p \in [1, n)$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$

$$\|\nabla f\|_p \geq \mathcal{E}_p(f) \geq \mathcal{E}_p(f^\circ) \geq \mathbf{c_{n,p}} \|f\|_q$$

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The first inequality is valid for all $p \geq 1$.

Haberl, Schuster, Xiao, $\geq$ 2009, stated the asymmetric case $\mathcal{E}_p^+(\mathbb{R}^n)$.

Improvements and extensions (Geometry)

Cianchi, Lutwak, Yang, Zhang,(symmetric case) and Haberl, Schuster, Xiao, (asymmetric case) in $\geq$ 2009:

The case p=n

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The case p>n (and therefore negative q!)

Back

$$\mathcal{E}_p(f) \geq \left(\frac{p'}{|q|}\right)^{\frac{1}{p'}} n\omega_n^{1/n}|\text{supp}\, f|_n^{1/q}\|f\|_\infty \qquad \text{where} \quad \frac{1}{p}+\frac{1}{p'}=1$$

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and the constants *depending on the size of the support of f* are sharp.

Extensions and improvements (Analysis)

Sobolev inequality

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$$\|\nabla f\|_n \geq \mathbf{c_n} \|f\|_{\mathcal{MT}}$$

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$$\|\nabla f\|_n \geq c_n \|f\|_{H_n} \geq c_n' \|f\|_{\mathcal{MT}}$$

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Dependence on the support of f

Extensions and improvements (Analysis)

Tartar, Maly-Pick, and Bastero-Milman-Ruiz, 1998-2003 introduced *classes of functions.* For $1 \leq p < \infty$ denote

$$\mathcal{A}_{\infty,p}(\mathbb{R}^n) = \{f; \|f\|_{\infty,p} = \left(\int_0^\infty (f^{**}(t) - f^*(t))^p \frac{dt}{t^{p/n}}\right)^{1/p} < \infty\}$$

where f^* is the decreasing rearrangement of f and f^{**} is its Hardy transform f^* defined by $f^{**}(t) = \dfrac{1}{t}\displaystyle\int_0^t f^*(s)ds$.

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Bastero-Milman-Ruiz, 2003

$$f^{**}(t) - f^*(t) \leq c_n t^{1/n} |\nabla f|^{**}(t), \quad a.e. \quad t \geq 0$$

Extensions and improvements (Analysis)

As a Corollary,

Bastero-Milman-Ruiz, 2003

$$\|\nabla f\|_n \geq (n-1)\,\omega_n^{\frac{1}{n}}\,\|f\|_{\infty,n} \geq c_n\|f\|_{H_n}$$

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Once classes of functions are allowed,

Martin-Milman, 2010

$$\|\nabla f\|_p \geq c_{n,p}\|f\|_{\infty,p} \geq c'_{n,p}\|f\|_q \qquad 1 \leq p < n$$

No dependence on the support of f.

Results

Proposition

Let $1 \leq p < \infty$ and $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{1}{n}$. Then

$$\|\nabla f\|_p \geq \mathcal{E}_p(f) \geq \left(1 - \frac{1}{q}\right) n\, \omega_n^{1/n} \|f\|_{\infty,p}$$

and the constant is sharp...

▸ Sobolev

...by considering truncations of

- $f^*(t) = t^{-1/q}$, whenever $p < n$,
- $f^*(t) = \log(1/t)$, for $p = n$ and
- $f^*(t) = (1 - t^{-1/q})\chi_{[0,1]}$, whenever $p > n$.

Results

Proposition

Let $p > n$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and f a compactly supported $C^{1)}$ function. Then,

$$\|f\|_\infty |\text{supp}\, f|_n^{1/q} \leq \sup_{t>0}\{(\|f\|_\infty - f^*(t))\, t^{1/q}\} \leq c_{n,p}\|f\|_{\infty,p}$$

for some $c_{n,p} > 0$ (independent of the support of f).

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Sobolev

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- The proof gives $c_{p,n} = \left(\left(p(1 - \frac{1}{q})\right)^{p'/p} + \frac{|q|}{p'}\right)^{\frac{1}{p'}}$.
- $f^*(t) = (1 - t^{-1/q})\chi_{[0,1]}(t)$ verifies $\sup_{t>0}\{(\|f\|_\infty - f^*(t))\, t^{1/q}\} = 1$ while $\|f\|_{\infty,p} = \infty$.