Volume of L_p-zonotopes and best best constants in Brascamp-Lieb inequalities

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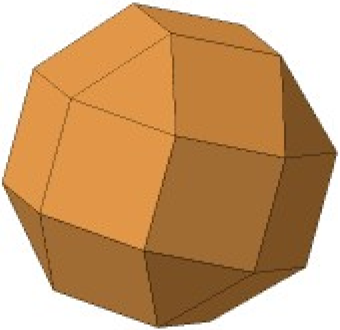
Convex bodies

- A subset $K \subset \mathbb{R}^n$ is said to be a convex body if it is convex, compact and has non-empty interior.



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  is a centrally symmetric convex body.
- Conversely, if K is a centrally symmetric convex body, it is the unit ball of the norm defined by

$$\|x\|_K := \min\{\lambda \geq 0 \,:\, x \in \lambda K\}$$

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Let K be a centrally symmetric convex body. Its polar body is defined by

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Ex:

$$B_p^n = \{x \in \mathbb{R}^n \,:\, \sum_{i=1}^n |x_i|^p \leq 1\}$$

$$(B_p^n)^\circ = B_{p'}^n \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$$

Convex bodies

Given two centrally symmetric convex sets K_1 and K_2, their l_p sum is defined

$$K_1 \oplus_p K_2 = \left\{ \tau_1 x_1 + \tau_2 x_2 \,:\, x_1 \in K_1, x_2 \in K_2 \,:\, \tau_1^{p\prime} + \tau_2^{p\prime} = 1 \right\}$$

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When $p = 1, \infty$

- $K_1 \oplus_1 K_2 = K_1 + K_2$ (Minkowski sum)
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l_p sums of segments are called p-zonotopes

p-zonotopes and duals

Let $a_1, \ldots, a_m$ be norm-one vectors spanning $\mathbb{R}^n$ and $\theta_1, \ldots, \theta_m$ positive numbers. We call

$$K_\infty := \{x \in \mathbb{R}^n \, : \, |\langle x, a_i \rangle| \leq \theta_i \, , \, i = 1, \ldots, m\}$$

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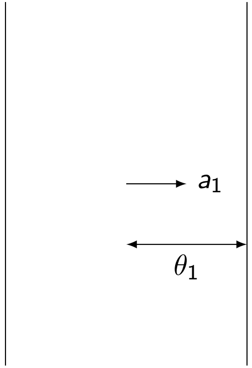

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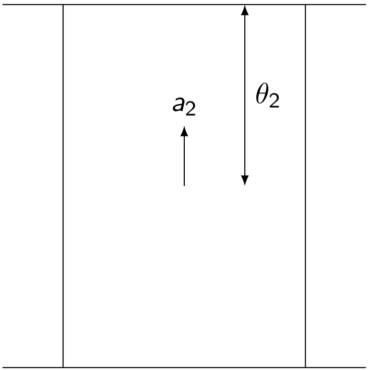

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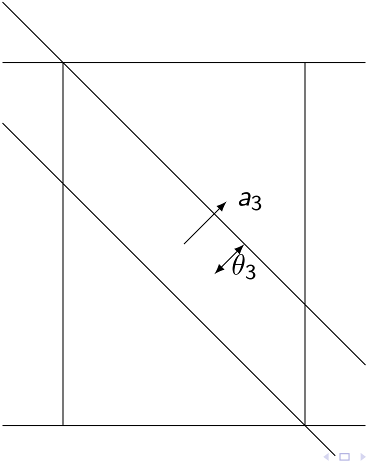

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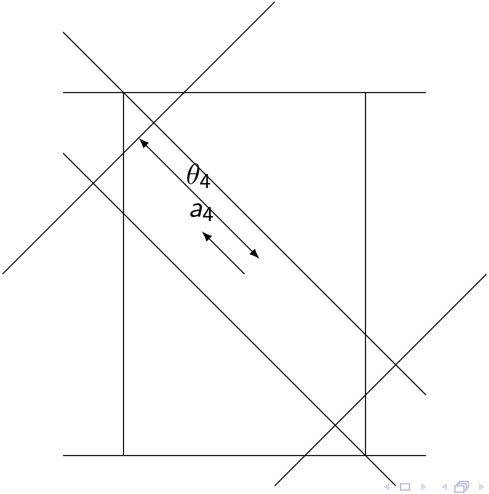

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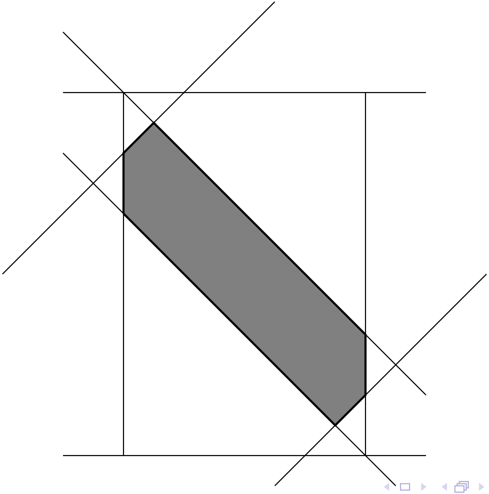

p-zonotopes and duals

- K_∞ is the unit ball of the norm

$$\|x\|_{K_\infty} = \max_{i=1,\ldots,m} |\langle x, \frac{a_i}{\theta_i} \rangle|$$

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$$\|x\|_{K_\infty} = \max_{i=1,\dots,m} |\langle x, \frac{a_i}{\theta_i} \rangle|$$

- For $p \geq 1$, define K_p as the unit ball of the norm given by

$$\|x\|_{K_p}^p = \sum_{i=1}^{m} |\langle x, \frac{a_i}{\theta_i} \rangle|^p$$

p-zonotopes and duals

- The polar body of K_p is the p-zonotope

$$K_p^\circ = \sum_{i=1}^{m} \oplus_p \frac{1}{\theta_i}[-a_i, a_i] = \left\{ x = \sum_{i=1}^{m} \tau_i \frac{a_i}{\theta_i} \in \mathbb{R}^n \ : \ \|(\tau_1, \ldots, \tau_m)\|_{p'} \leq 1 \right\}$$

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- It is the unit ball of the norm

$$\|x\|_{K_p^\circ} = \inf_{x = \sum_{i=1}^{m} \tau_i \frac{a_i}{\theta_i}} \|(\tau_1, \dots, \tau_m)\|_{p'}.$$

Notation

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- $a_i \otimes a_i(x) = \langle a_i, x \rangle a_i$
- If $S = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_m \end{pmatrix}$ then $ASA^t = \sum_{i=1}^m s_i a_i \otimes a_i$

Brascamp-Lieb inequality and its reverse

Theorem

For every $0 < c_1, \dots, c_m < 1$ and for every $f_1, \dots, f_m : \mathbb{R} \to \mathbb{R}^+$ such that $f_i^{\frac{1}{c_i}} \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle a_i, x\rangle)dx \le D(A,c) \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i^{\frac{1}{c_i}}(x)dx \right)^{c_i} \qquad \text{(BL)}$$

$$\int_{\mathbb{R}^n}^* \sup_{x=\sum_{i=1}^m c_i y_i a_i} \prod_{i=1}^m f_i(y_i)dx \ge \frac{1}{D(A,c)} \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i^{\frac{1}{c_i}}(x)dx \right)^{c_i}. \qquad \text{(RBL)}$$

where

$$D(A,c) = \sup_{s_1,\dots,s_m>0} \left\{ \frac{\int_{\mathbb{R}^n} \prod_{i=1}^m g_i(\langle a_i, x\rangle)dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}} g_i^{\frac{1}{c_i}}(x)dx \right)^{c_i}} : g_i(x) = e^{-\frac{s_i^2 x^2}{2}} \right\},$$

Brascamp-Lieb inequality and its reverse

- $\sum_{i=1}^{m} c_i = n$ is a necessary condition for $D(A, c)$ to be finite. In this case

$$D(A, c) = \sup_{s_1, \ldots, s_m > 0} \sqrt{\frac{\prod_{i=1}^{m} \left(\frac{s_i^2}{c_i}\right)^{c_i}}{\det(AS^2A^t)}},$$

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- $f_i = \chi_{[-\theta_i,\theta_i]}$ or $f_i = e^{(-\frac{|x|}{\theta_i})^p}$ in BL
$$|K_p| \le |B_p^n| D(A,c) \prod_{i=1}^m (\theta_i c_i^{\frac{1}{p}})^{c_i}$$

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- $f_i = \chi_{[-\frac{1}{c_i\theta_i},\frac{1}{c_i\theta_i}]}$ or $f_i = e^{-(c_i|x|\theta_i)^{p'}}$ in RBL
$$|K_p^\circ| \ge \frac{|B_{p'}^n|}{D(A,c)} \prod_{i=1}^m \left(\frac{1}{c_i^{\frac{1}{p}}\theta_i}\right)^{c_i}$$
for every $0 < c_1,\dots,c_m < 1$ that $\sum_{i=1}^m c_i = n$.

Volume estimates

- $|K_p| \leq |B_p^n| D(A,c) \prod_{i=1}^m (\theta_i c_i^{\frac{1}{p}})^{c_i}$

  $|K_p^\circ| \geq \frac{|B_{p'}^n|}{D(A,c)} \prod_{i=1}^m \left(\frac{1}{\theta_i c_i^{\frac{1}{p}}} \right)^{c_i}$
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  for every $0 < c_1, \ldots, c_m < 1$ that $\sum_{i=1}^m c_i = n$.
- Our purpose is to find $f(c_1, \ldots, c_m)$ verifying

$$\inf_{c_1,\ldots,c_m} \{D(A,c) \prod_{i=1}^m (\theta_i c_i^{\frac{1}{p}})^{c_i}\} \leq f(c_1, \ldots, c_m) \leq D(A,c) \prod_{i=1}^m (\theta_i c_i^{\frac{1}{p}})^{c_i}.$$

New estimates

Theorem

For every $p \geq 2$ and for any $c_1, \ldots, c_m$ such that $\sum c_i = n$ the following inequalities hold

$$|K_p| \leq \frac{|B_p^n|}{\sqrt{\det\left(\sum_{i=1}^m \frac{c_i^{1-\frac{2}{p}}}{\theta_i^2} a_i \otimes a_i\right)}}$$

$$|K_p^\circ| \geq |B_{p'}^n| \sqrt{\det\left(\sum_{i=1}^m \frac{c_i^{1-\frac{2}{p}}}{\theta_i^2} a_i \otimes a_i\right)}$$

Proof

$$2 \log D(A, c) \quad = \quad \sup_{s_1,\ldots,s_m>0} \left\{ \sum_{i=1}^{m} c_i \log s_i^2 - \log\det(AS^2A^t) \right\} - \sum_{i=1}^{m} c_i \log c_i$$

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&= \sup_{t_1,\ldots,t_m} \left\{ \sum_{i=1}^{m} c_i t_i - \log\det(Ae^{T}A^t) \right\} - \sum_{i=1}^{m} c_i \log c_i
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$\phi_A(t_1,\dots,t_m) = \log\det Ae^T A^t$ is convex and

$$2\log D(A,c) + \sum_{i=1}^m c_i \log c_i = \phi_A^*(c_1,\dots,c_m)$$

Proof

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\begin{aligned}
\phi_A(t_1, \ldots, t_m) &= \phi_A^{**}(t_1, \ldots, t_m) \\
&= \sup_{c_1, \ldots, c_m > 0} \left\{ \sum_{i=1}^{m} c_i t_i - 2 \log D(A, c) - \sum_{i=1}^{m} c_i \log c_i \right\}
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\end{aligned}
$$

Equivalently

$$
-\phi_A(t_1, \dots, t_m) = \inf_{c_1, \dots, c_m > 0} \left\{ 2 \log D(A, c) + \sum_{i=1}^{m} c_i \log c_i - \sum_{i=1}^{m} c_i t_i \right\}
$$

Proof

- $\frac{\partial}{\partial t_i}\left(\sum_{i=1}^{m} c_i t_i - \phi_A\right)(t_1, \ldots, t_m) = c_i - e^{t_i} a_i \cdot \left(Ae^T A^t\right)^{-1} a_i$

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- $2 \log D(A, c) =$
  $\sup_{t_1, \ldots, t_m} \{\sum_{i=1}^m c_i t_i - \phi_A(t_1, \ldots, t_m)\} - \sum_{i=1}^m c_i \log c_i$

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- $c_i = e^{t_i} a_i \cdot \left(A e^T A^t \right)^{-1} a_i \Rightarrow$
  $-\phi_A(t_1, \ldots, t_m) = 2 \log D(A, c) + \sum_{i=1}^m c_i \log c_i - \sum_{i=1}^m c_i t_i$
- $-\phi_A(t_1, \ldots, t_m) =$
  $\min_{c_1, \ldots, c_m > 0} \{ 2 \log D(A, c) + \sum_{i=1}^m c_i \log c_i - \sum_{i=1}^m c_i t_i \}$

Proof

Hence, for every $c_1, \ldots, c_m$ that $\sum_{i=1}^{m} c_i = n$

$$-\phi_A \left(\log \frac{c_1^{1-\frac{2}{p}}}{\theta_1^2}, \ldots, \log \frac{c_m^{1-\frac{2}{p}}}{\theta_m^2} \right) \leq$$

$$2 \log D(A, c) + \frac{2}{p} \sum_{i=1}^{m} c_i \log c_i + 2 \sum_{i=1}^{m} c_i \log \theta_i$$

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$$\frac{1}{\sqrt{\det\left(\sum_{i=1}^m \frac{c_i^{1-\frac{2}{p}}}{\theta_i^2} a_i \otimes a_i\right)}} \leq D(A,c)\prod_{i=1}^m (\theta_i c_i^{\frac{1}{p}})^{c_i}$$

Proof

For every positive $d_1, \ldots, d_m$ such that $\sum_{i=1}^m d_i = n$

$$-\phi_A\left(\log \frac{d_1^{1-\frac{2}{p}}}{\theta_1^2}, \ldots, \log \frac{d_m^{1-\frac{2}{p}}}{\theta_m^2}\right) =$$

$$\inf_{c:D(A,c)<\infty}\left\{2\log D(A,c) + \frac{2}{p}\sum_{i=1}^m c_i \log c_i + 2\sum_{i=1}^m c_i \log \theta_i\right.$$

$$\left.+\left(1-\frac{2}{p}\right)\sum_{i=1}^m c_i(\log c_i - \log d_i)\right\}$$

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$$\left. + \left(1 - \frac{2}{p}\right) \sum_{i=1}^m c_i (\log c_i - \log d_i) \right\}$$

$$\geq \inf_{c: D(A,c)<\infty} \left\{ 2 \log D(A,c) + \frac{2}{p} \sum_{i=1}^m c_i \log c_i + 2 \sum_{i=1}^m c_i \log \theta_i \right\}$$

Proof

Hence

$$\inf_{c_1,\dots,c_m}\{D(A,c)\prod_{i=1}^{m}(\theta_i c_i^{\frac{1}{p}})^{c_i}\} \leq \frac{1}{\sqrt{\det\left(\sum_{i=1}^{m}\frac{c_i^{1-\frac{2}{p}}}{\theta_i^2}a_i\otimes a_i\right)}}$$

for every $c_1,\dots,c_m$ such that $\sum_{i=1}^{m} c_i = n$

New estimates

For every $p \geq 2$ and for any $c_1, \ldots, c_m$ such that $\sum c_i = n$ the following inequalities hold

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$$|K_p^\circ| \geq |B_{p'}^n| \sqrt{\det\left(\sum_{i=1}^m \frac{c_i^{1-\frac{2}{p}}}{\theta_i^2} a_i \otimes a_i\right)} \left(\geq \frac{|B_{p'}^n|}{D(A,c)} \prod_{i=1}^m \left(\frac{1}{\theta_i c_i^{\frac{1}{p}}}\right)^{c_i}\right)$$

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$$|K_p^\circ| \geq |B_{p'}^n| \sqrt{\det\left(\sum_{i=1}^m \frac{c_i^{1-\frac{2}{p}}}{\theta_i^2} a_i \otimes a_i\right)} \left(\geq \frac{|B_{p'}^n|}{D(A,c)} \prod_{i=1}^m \left(\frac{1}{\theta_i c_i^{\frac{1}{p}}}\right)^{c_i}\right)$$

For $p = 2$ the bounds we obtained do not depend on the choice of c_i's, and is the exact value of $|K_2|$ and $|K_2^\circ|$

Best choice of c_i's

- If $c_1, \ldots, c_m$ minimize $D(A,c)\prod_{i=1}^m (\theta_i c_i^{\frac{1}{p}})^{c_i}$ they also minimize $-\phi_A\left(\log \frac{c_1^{1-\frac{2}{p}}}{\theta_1^2}, \ldots, \log \frac{c_m^{1-\frac{2}{p}}}{\theta_m^2}\right)$ and hence

$$c_i = \frac{c_i^{1-\frac{2}{p}}}{\theta_i^2} a_i \cdot \left(\sum_{j=1}^m \frac{c_j^{1-\frac{2}{p}}}{\theta_j^2} a_j \otimes a_j\right)^{-1} a_i.$$

and conversely

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Furthermore

$$T_p^t T_p = \sum_{i=1}^m \frac{c_i^{1-\frac{2}{p}}}{\theta_i^2} a_i \otimes a_i$$