

# $k$ -Schur functions indexed by a maximal rectangle

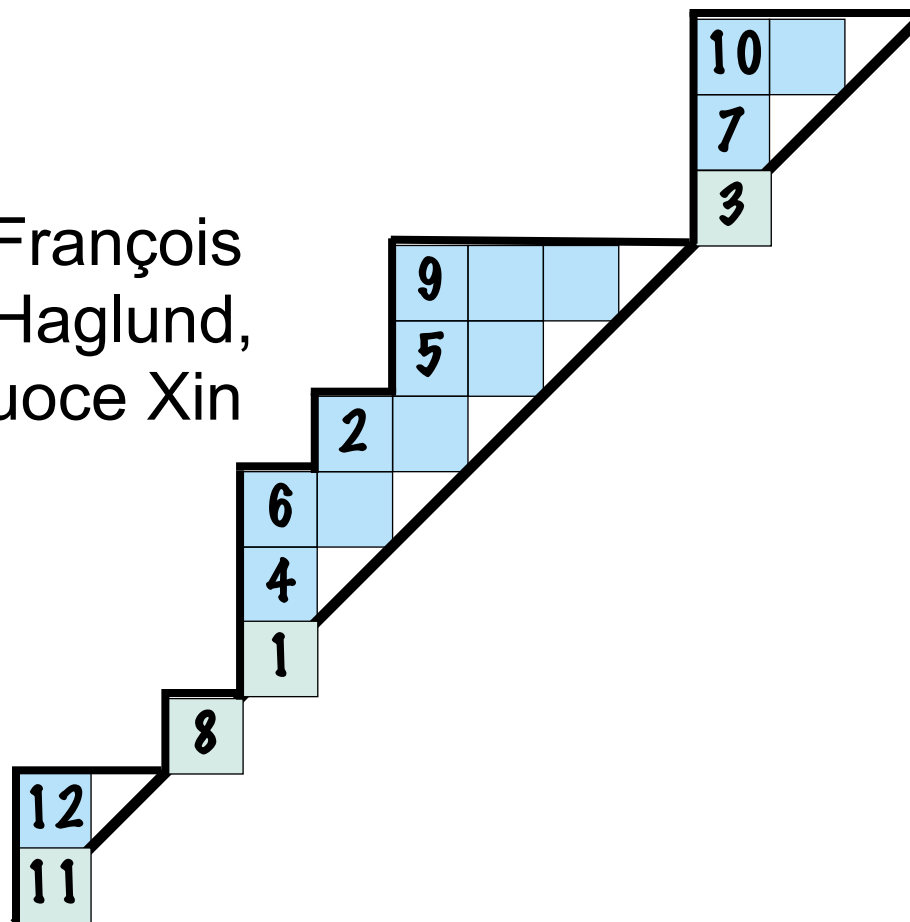
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joint work with Chris Berg, Nantel Bergeron, Hugh Thomas

# Advertisement:

## Progress on the Shuffle Conjecture

joint work with Nantel Bergeron, François Descouens, Adriano Garsia, Jim Haglund, Angela Hicks, Jennifer Morse, Guoce Xin

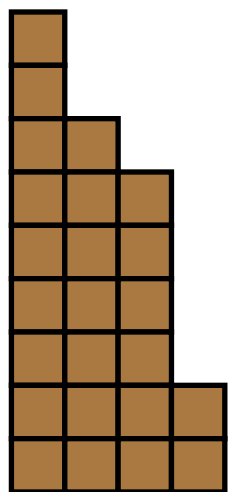


# Lapointe-Morse (2005)

$$\Lambda^{(k)} = \mathbb{Q}[h_1, h_2, \dots, h_k]$$

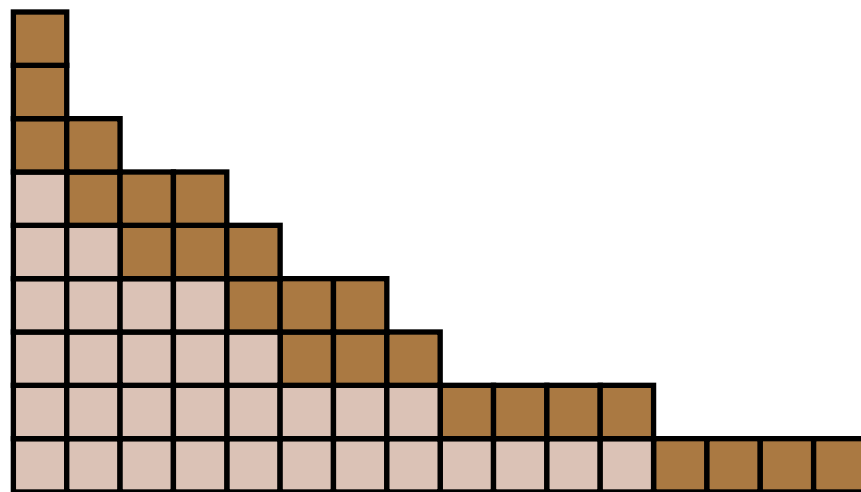
definition of a basis  $\{s_\lambda^{(k)}\}_\lambda$  indexed by partitions

$\lambda$  partitions  $\lambda_1 \leq k$



$$\lambda = p(\gamma)$$

$(k+1)$ -cores = partitions with  
no  $(k+1)$ -hooks



$$c(\lambda) = \gamma$$

Example:  
 $k = 4$

Affine symmetric group  $W$  of type  $\widetilde{A}_k$

$W$  generated by elements  $\{s_0, s_1, s_2, \dots, s_k\}$

$$s_i^2 = 1$$

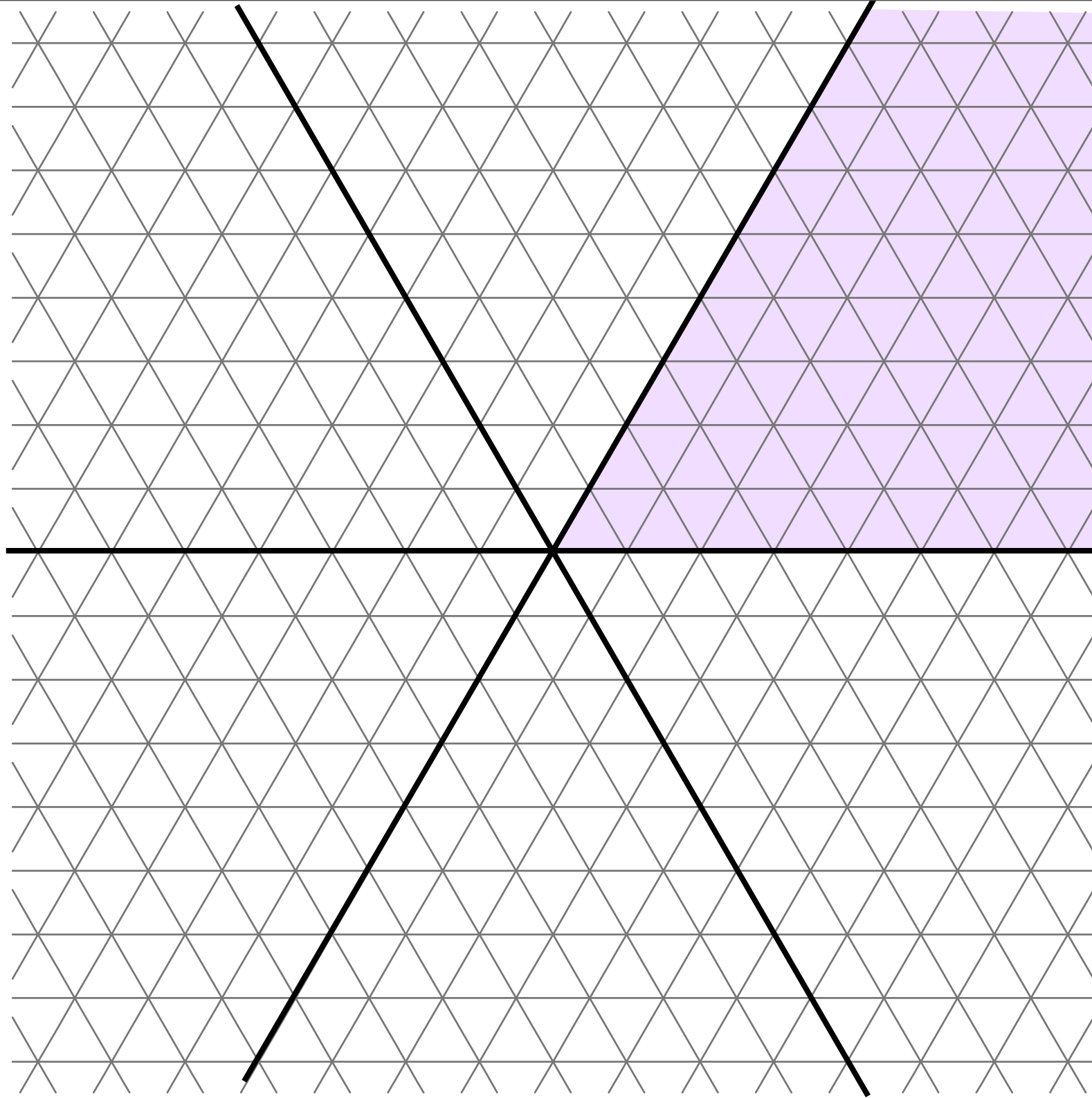
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad i, i+1 \pmod{k+1}$$

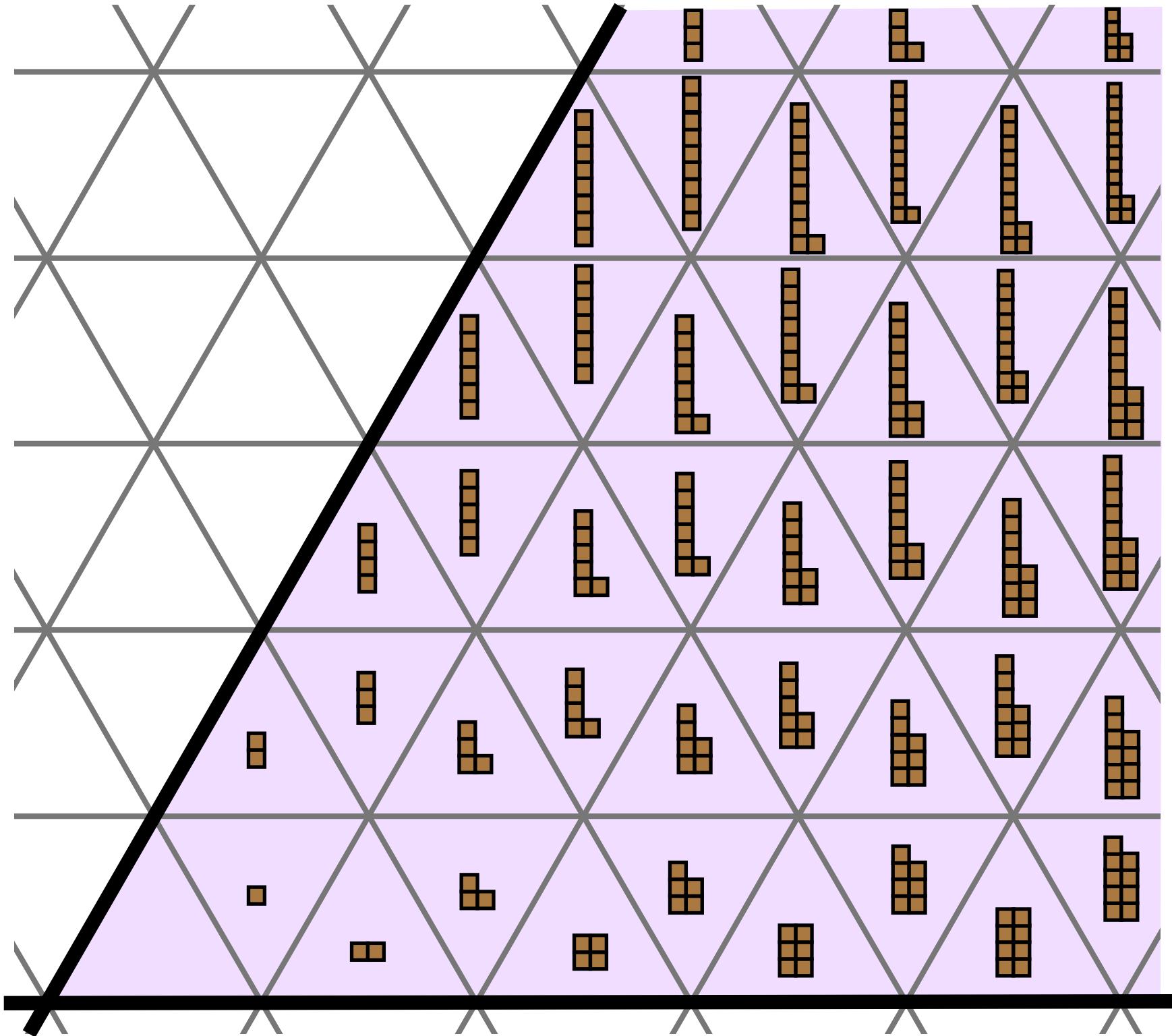
$$s_i s_j = s_j s_i \quad i - j \not\equiv k, 0, 1 \pmod{k+1}$$

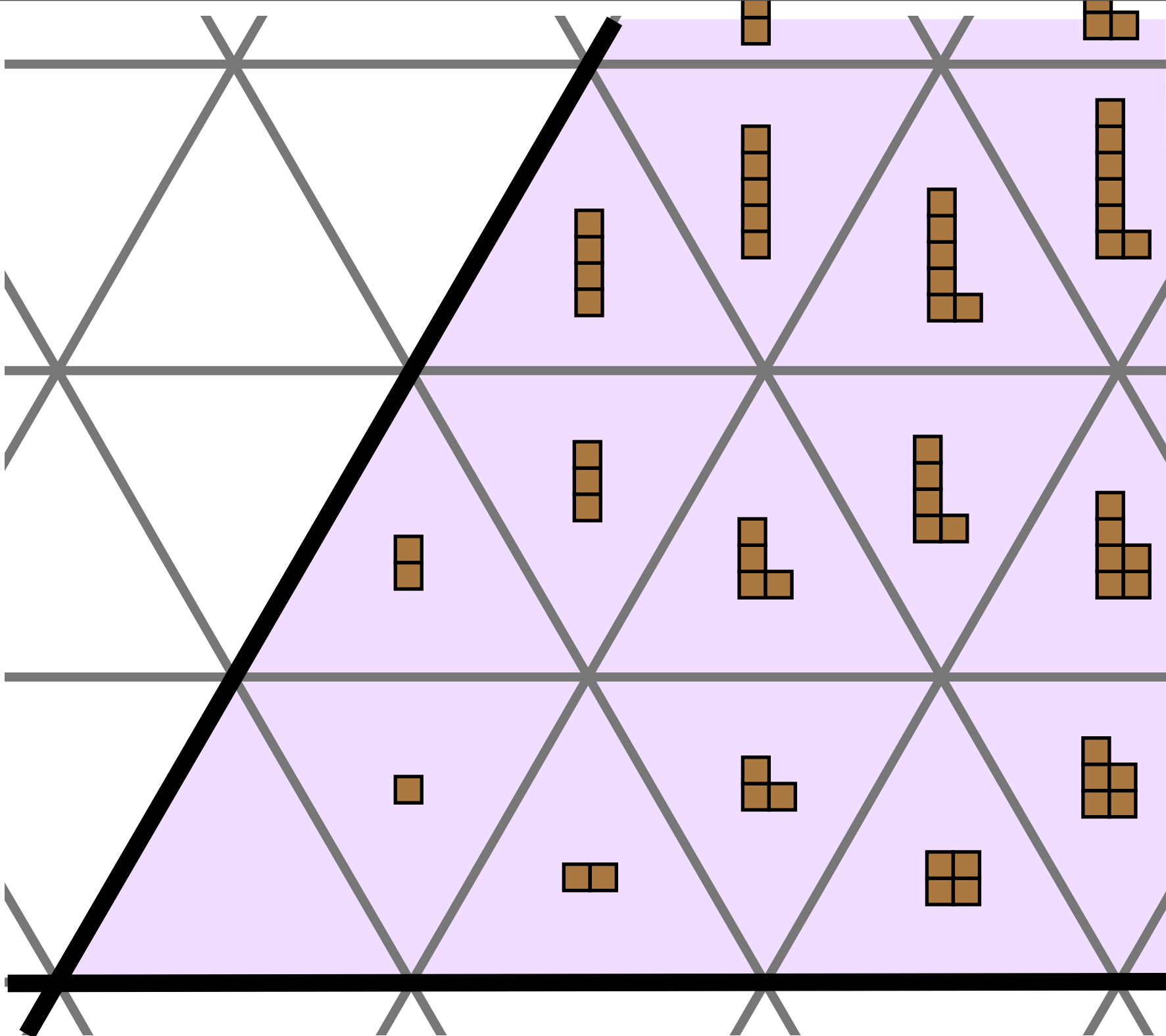
$W_0$  is the subgroup generated by  $\{s_1, s_2, \dots, s_k\}$

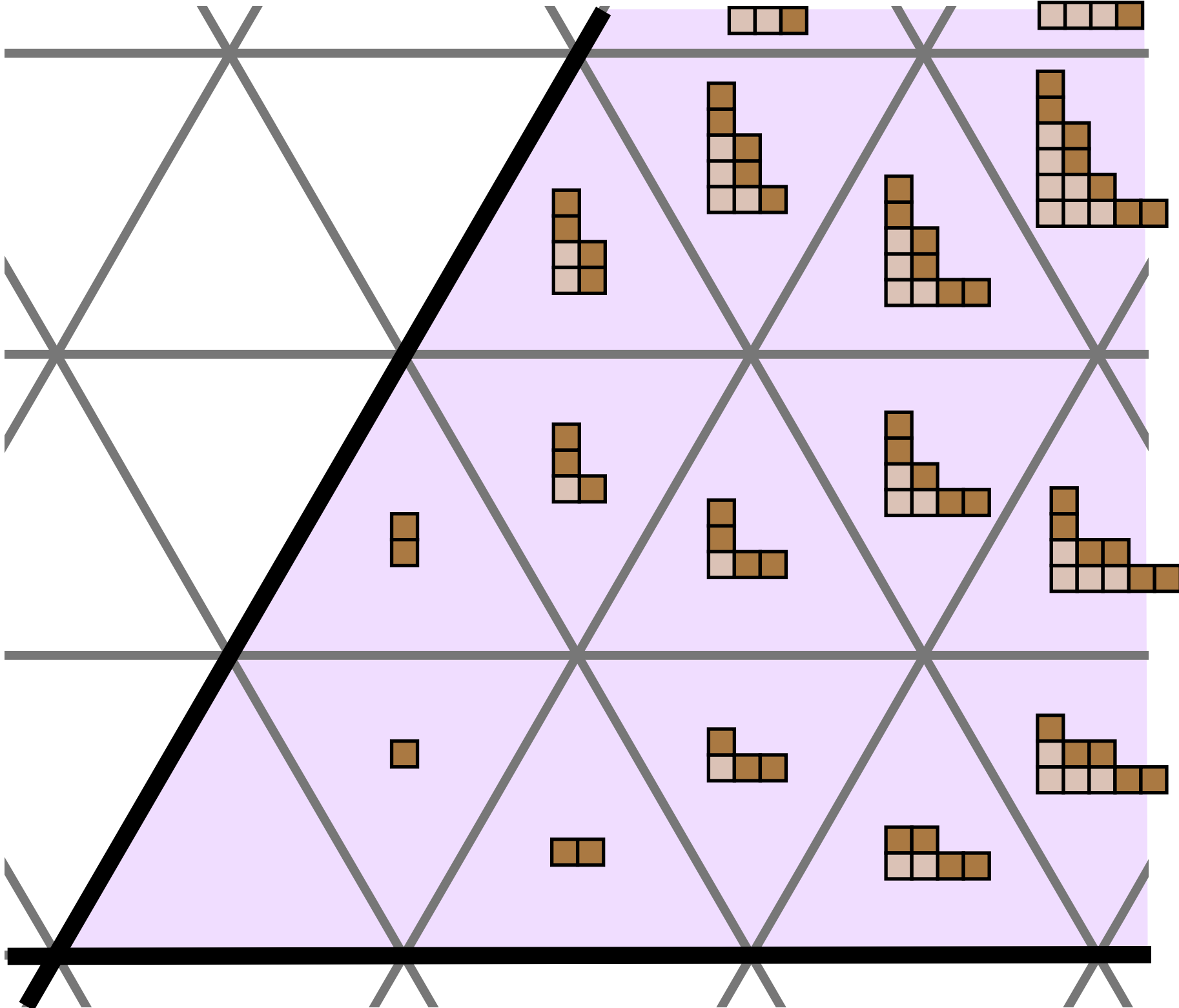
$W/W_0$  = cosets of  $W_0$  are in bijection with  
k-bounded partitions/(k+1)-cores

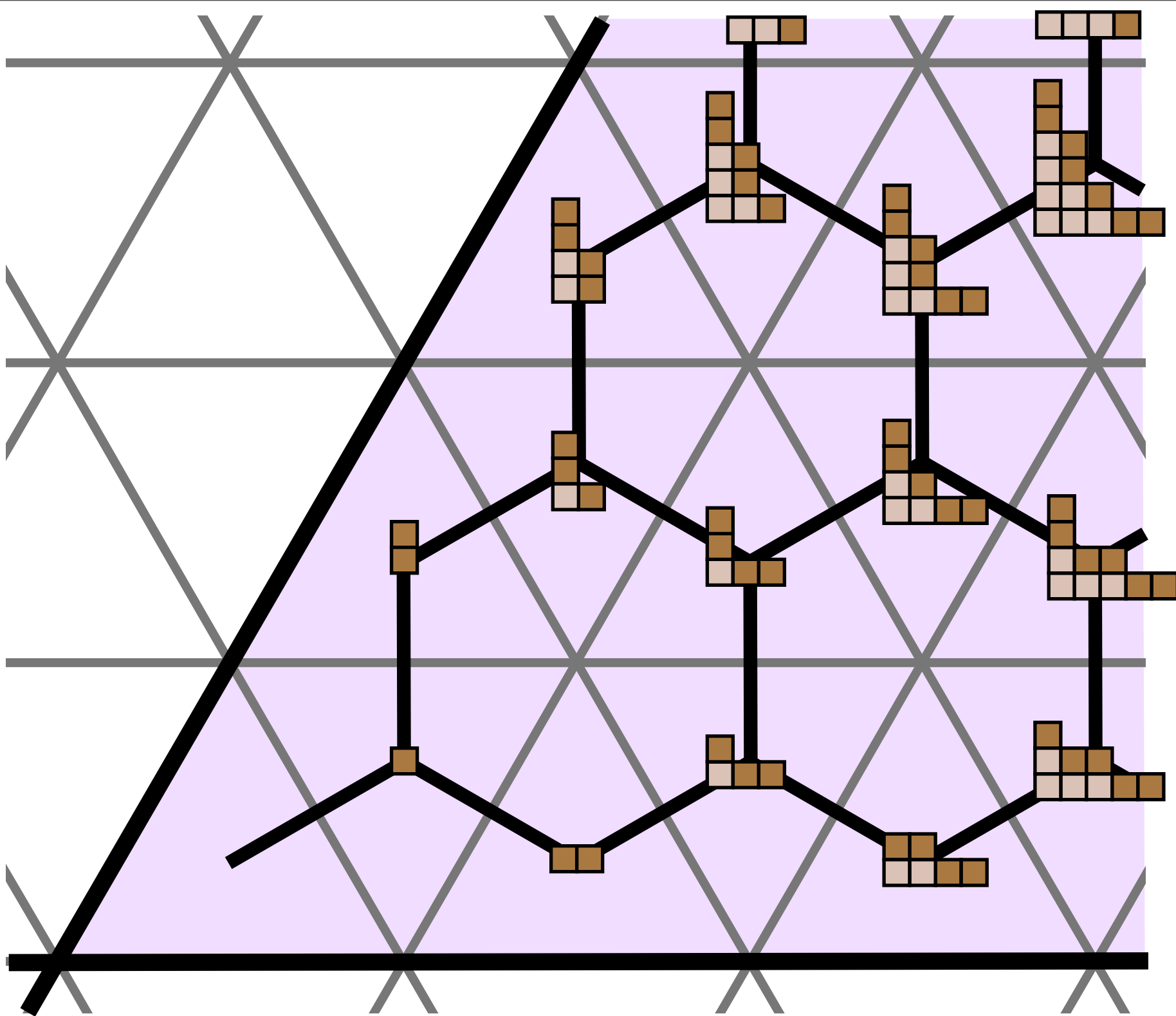
$k=2$

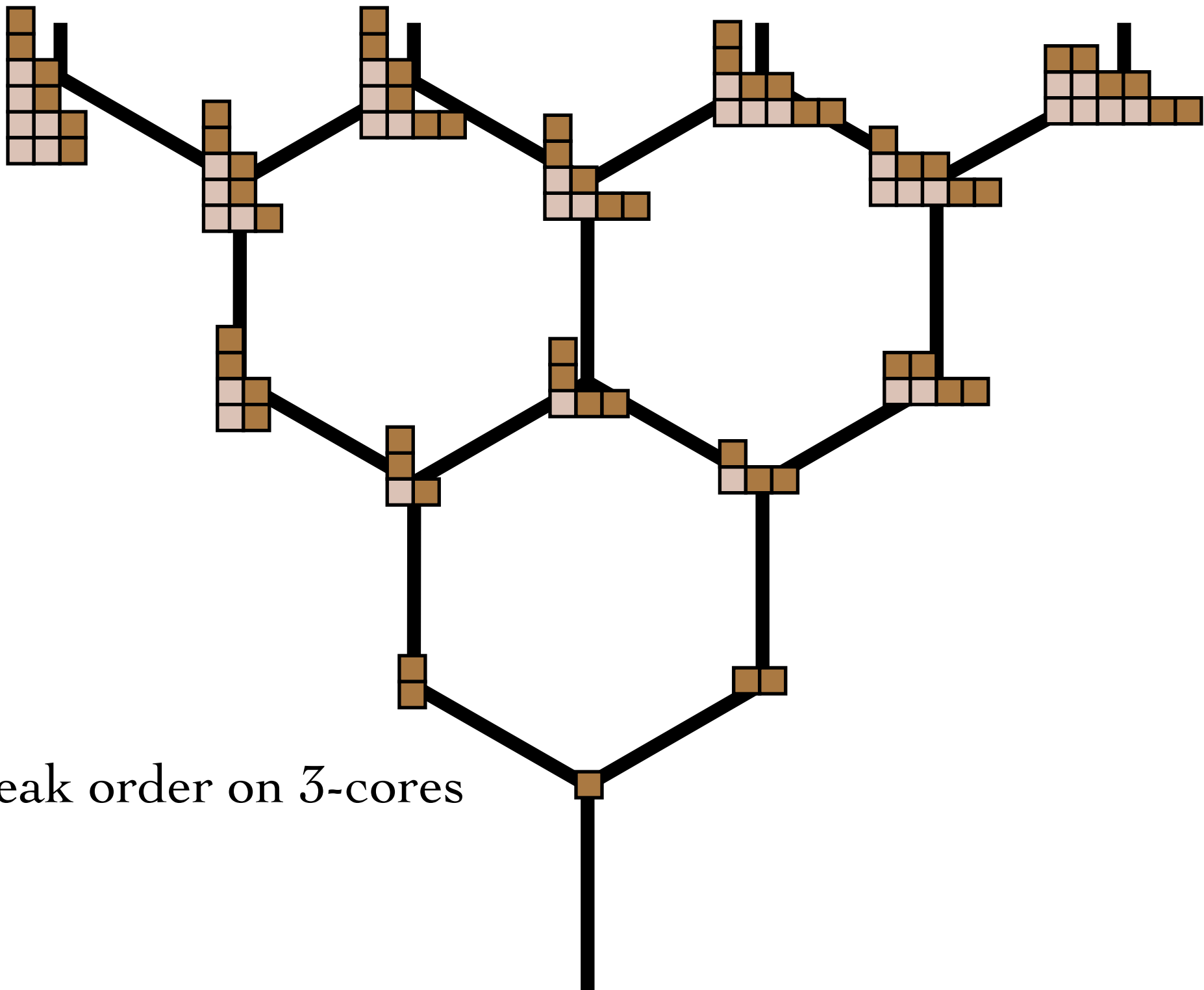




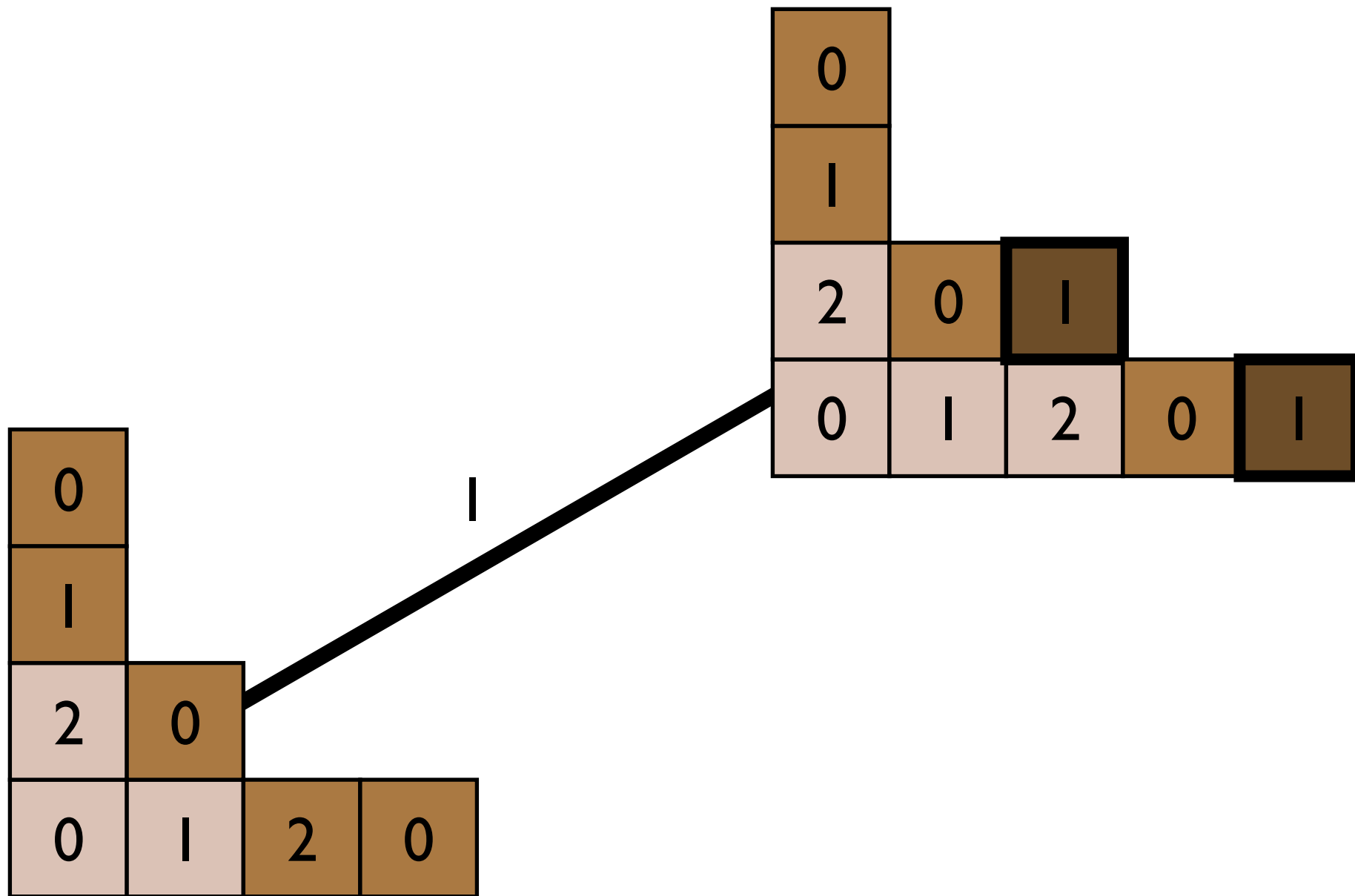




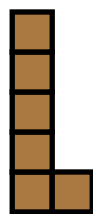
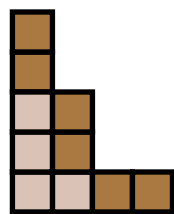




Weak order on 3-cores



3-cores/2-bounded partitions



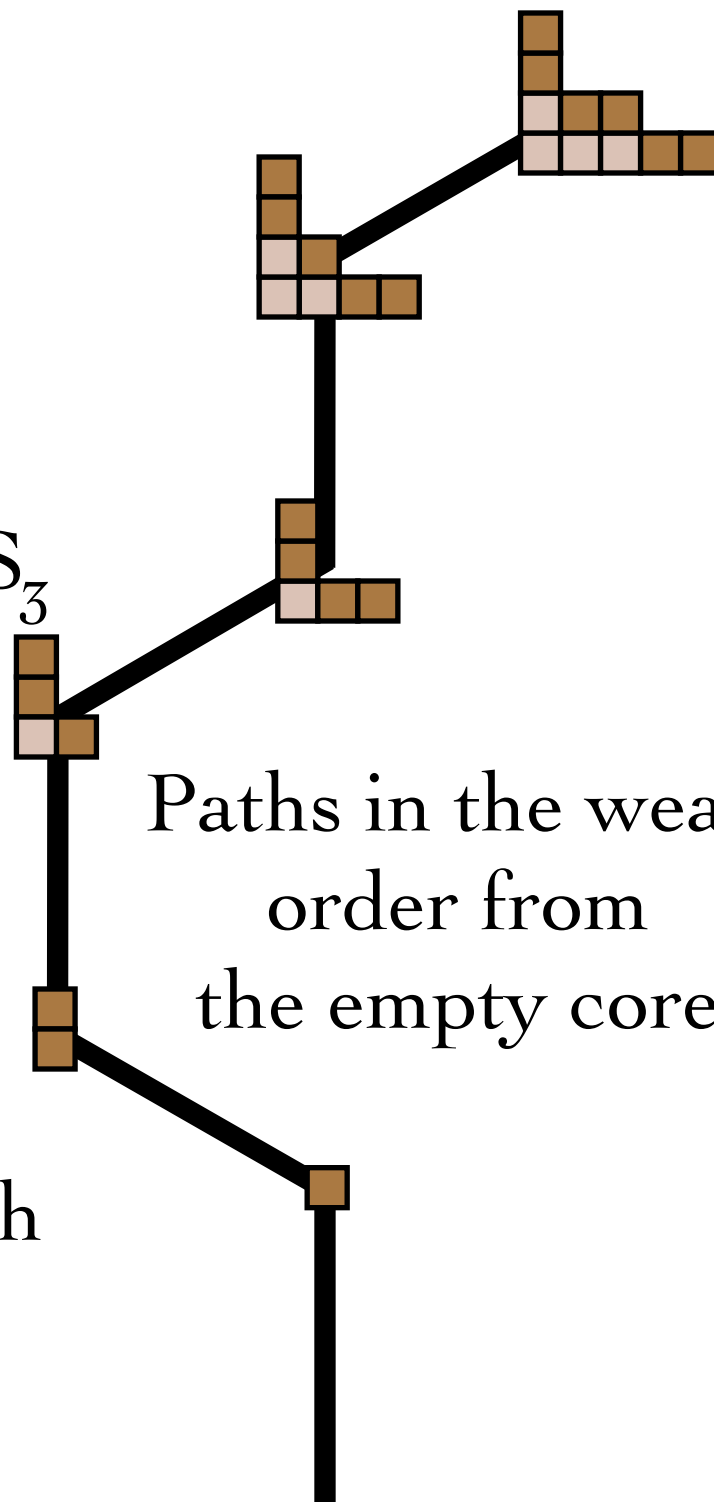
Cosets of  $S_3$  as subgroup of affine  $S_3$

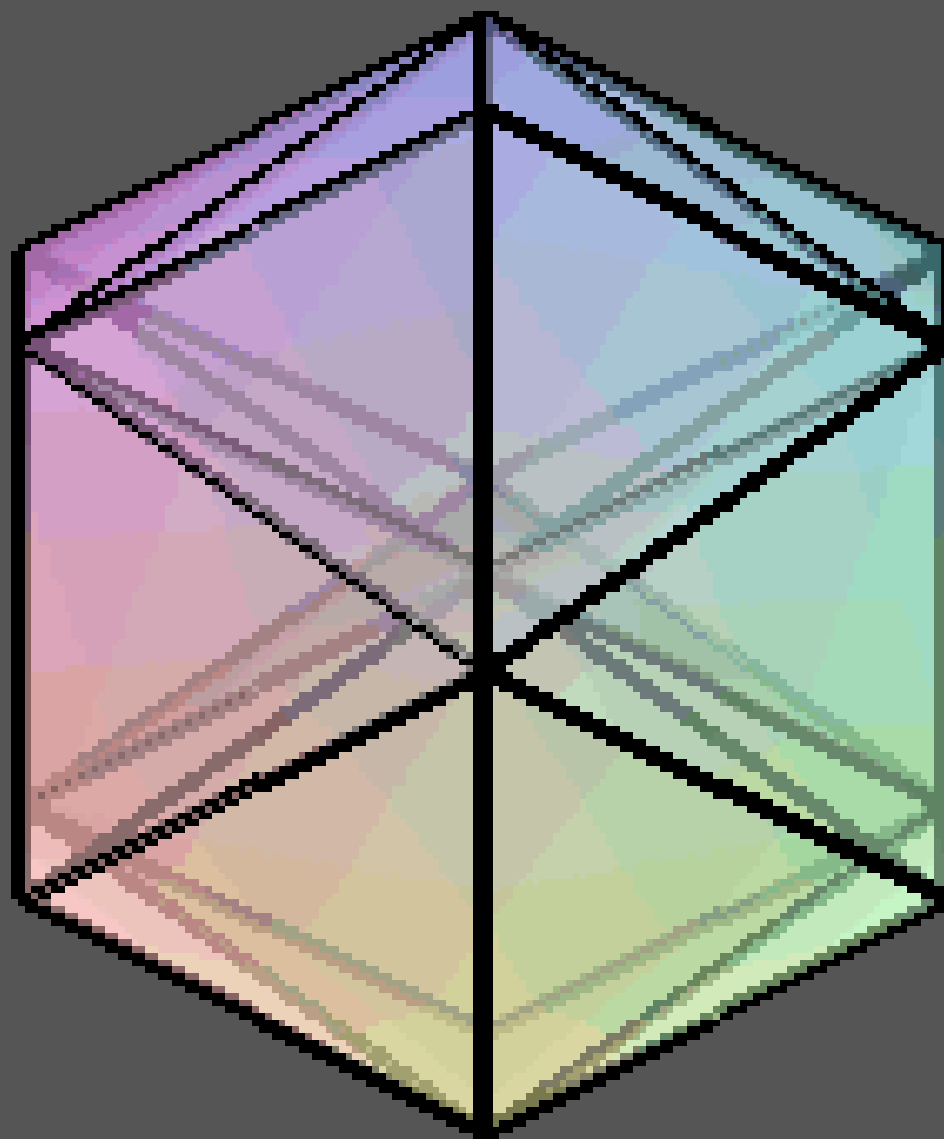
minimal length  
coset representatives

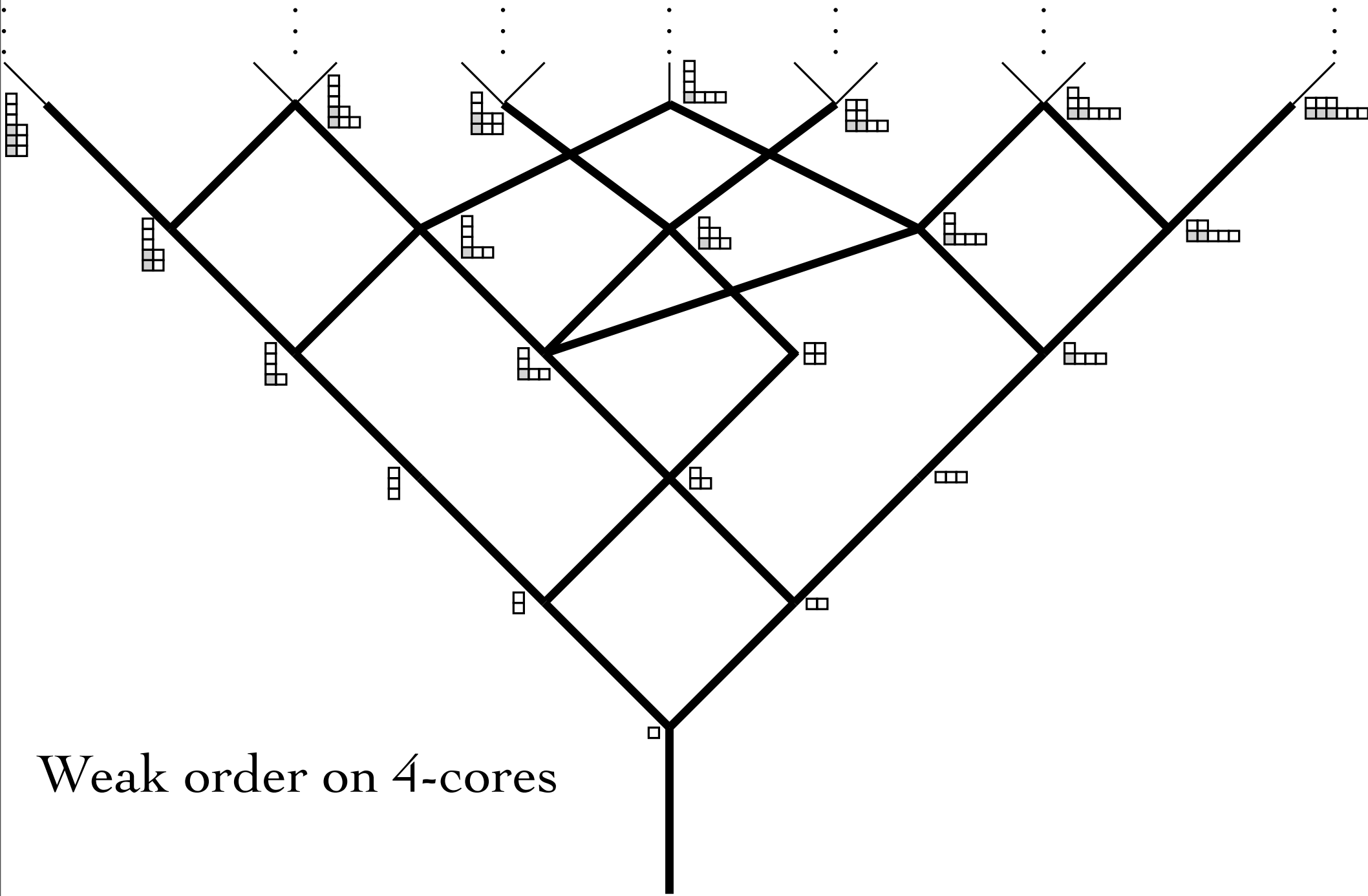
Reduced word for minimal length  
coset representatives

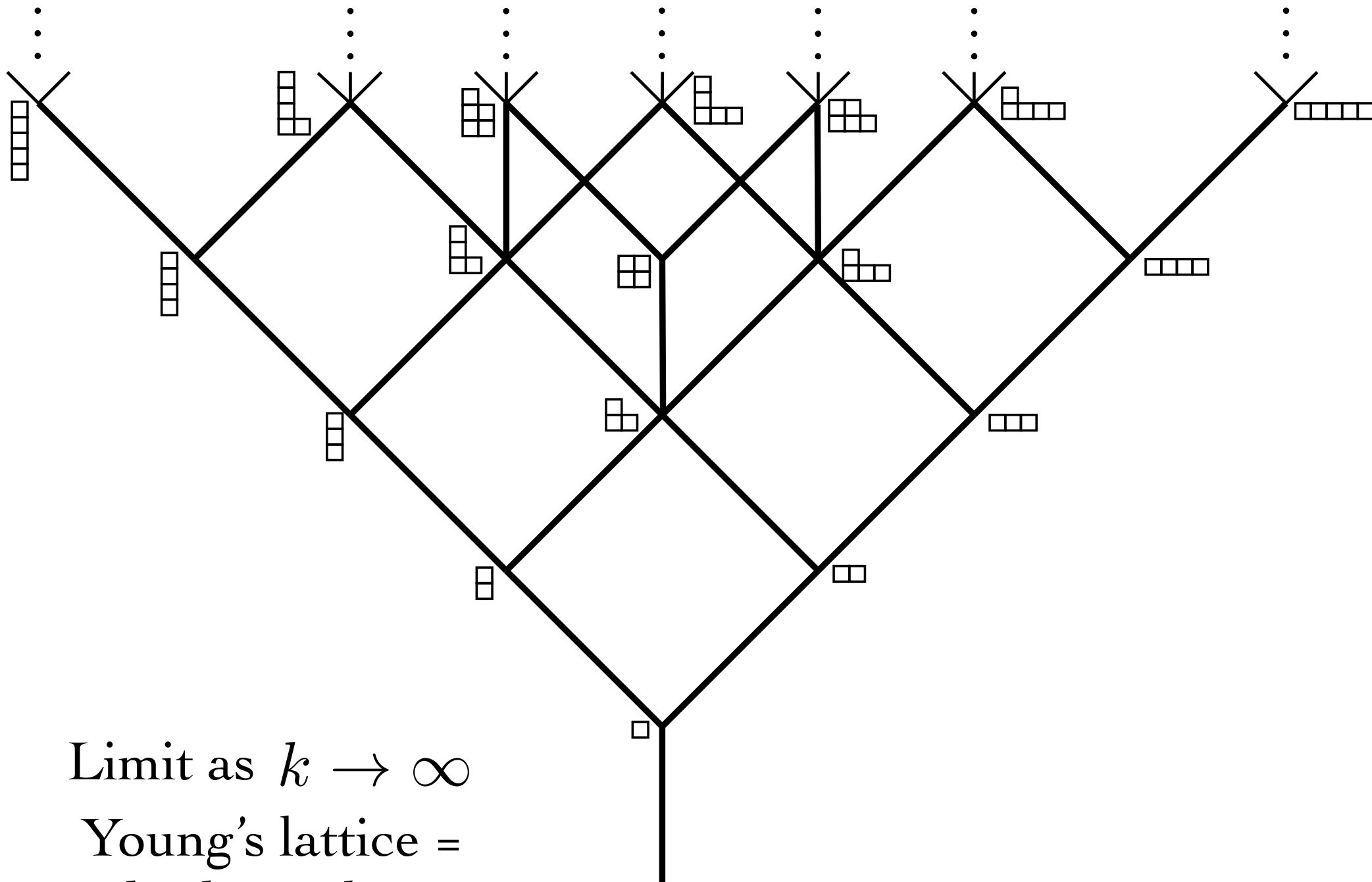
$$s_1 s_0 s_2 s_1 s_2 s_0$$

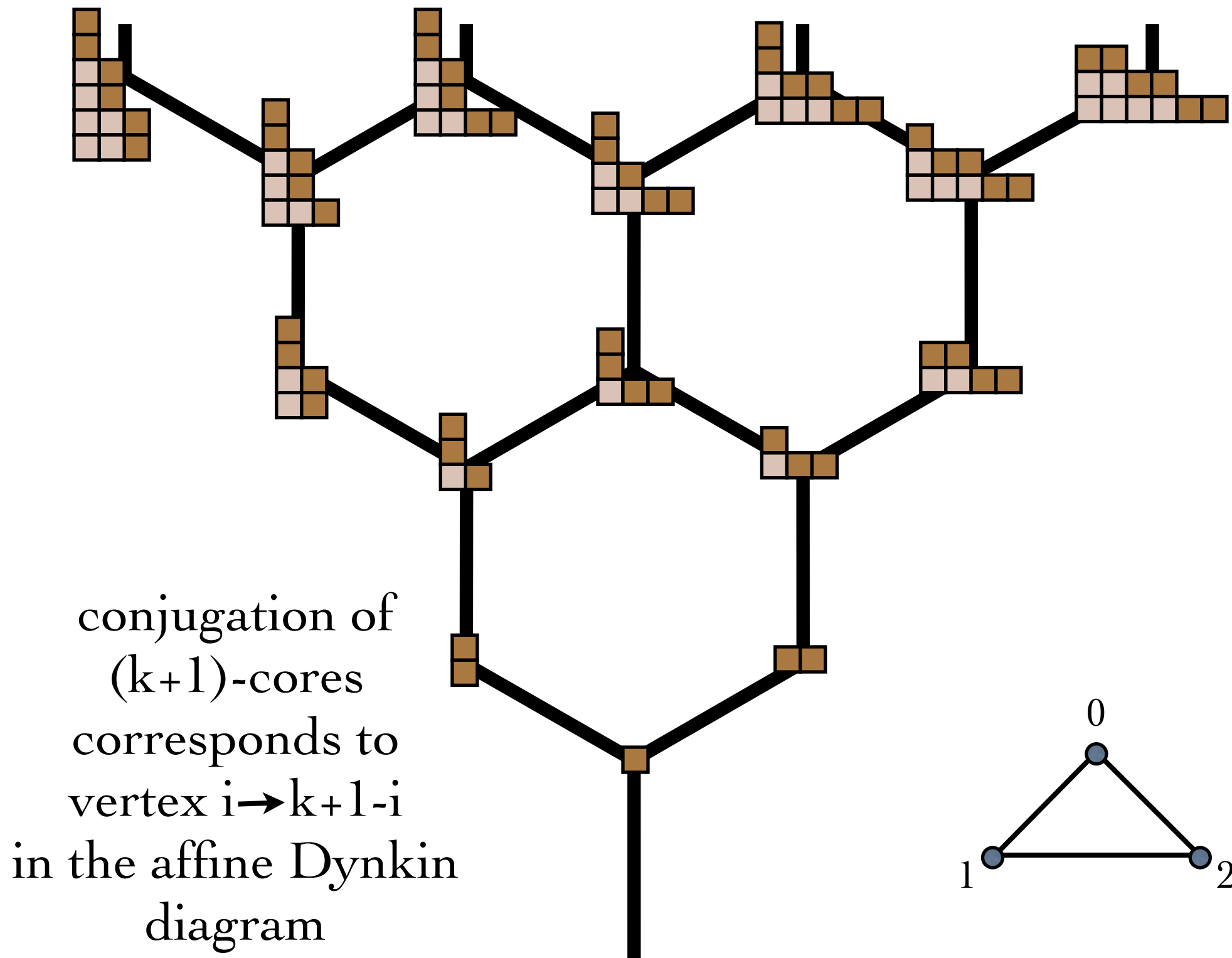
Paths in the weak  
order from  
the empty core

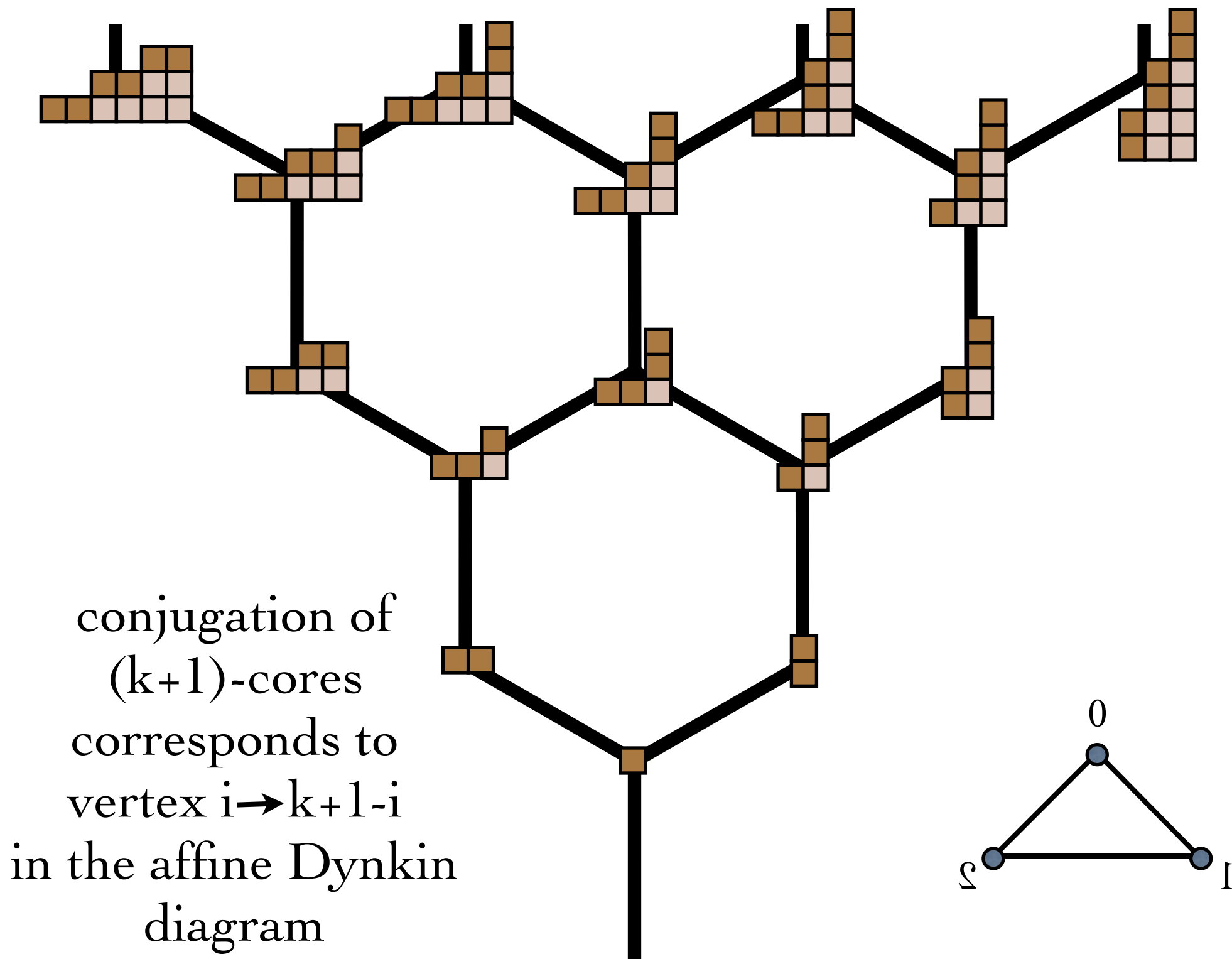












# Lapointe-Morse definition of k-Schur functions

$\{s_\lambda^{(k)}\}_\lambda$  basis of algebra  $\Lambda^{(k)} = \mathbb{Q}[h_1, h_2, \dots, h_k]$

satisfying

$$h_r s_\lambda^{(k)} = \sum_{\mu} s_\mu^{(k)}$$

$\mathbf{c}(\mu)/\mathbf{c}(\lambda)$  is a horizontal strip,  $\lambda \leq \mu$

$$|\mu| = |\lambda| + r$$

or

$\mu/\lambda$  and  $\mathbf{p}(\mathbf{c}(\mu)')/\mathbf{p}(\mathbf{c}(\lambda)')$  are horizontal strips

$$|\mu| = |\lambda| + r$$

This is a recursive definition because of triangularity considerations

**Example:**  $k=3$  to calculate  $s_{(2,2,1)}^{(3)}$

the 3-Pieri rule says:

$$h_2 s_{(2,1)}^{(3)} = s_{(2,2,1)}^{(3)} + s_{(3,1,1)}^{(3)}$$

We may assume (inductively) that expansions of  $s_{(3,1,1)}^{(3)}$  and  $s_{(2,1)}^{(3)}$  are known in terms of the generators

In particular, if hook  $\lambda$  is small (less or equal  $k$ ) then

$$s_{\lambda}^{(k)} = s_{\lambda}$$

Thomas Lam

Consider elements of the affine nil-Coxeter algebra

$$u_i^2 = 0$$

$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \quad i, i+1 \pmod{k+1}$$

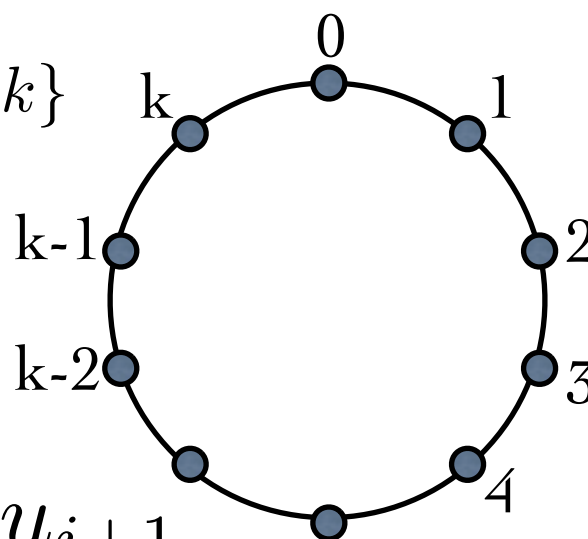
$$u_i u_j = u_j u_i \quad i - j \not\equiv k, 0, 1 \pmod{k+1}$$

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$$\mathbf{h}_r = \sum_{|A|=r} u_A \quad \begin{array}{l} 1 \leq r \leq k \\ A \subseteq \{0, 1, 2, \dots, k\} \end{array}$$

$u_A$  cyclically decreasing word  
with content  $A$

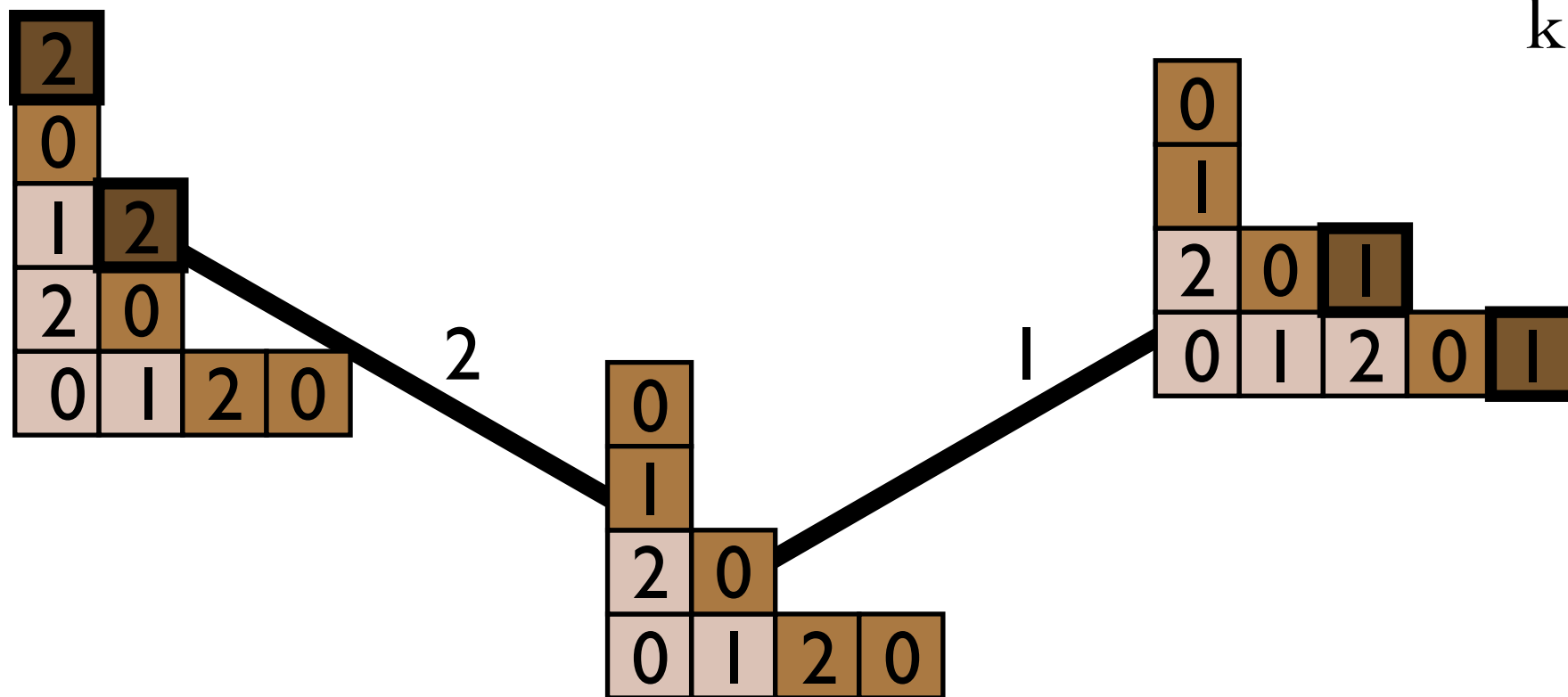
if  $i, i+1 \in A$ ,  $u_i$  comes before  $u_{i+1}$



Let  $\gamma$  be a  $(k+1)$ -core

$u_i$  acts on  $\gamma$  by adding  $i$ -addable corner if possible

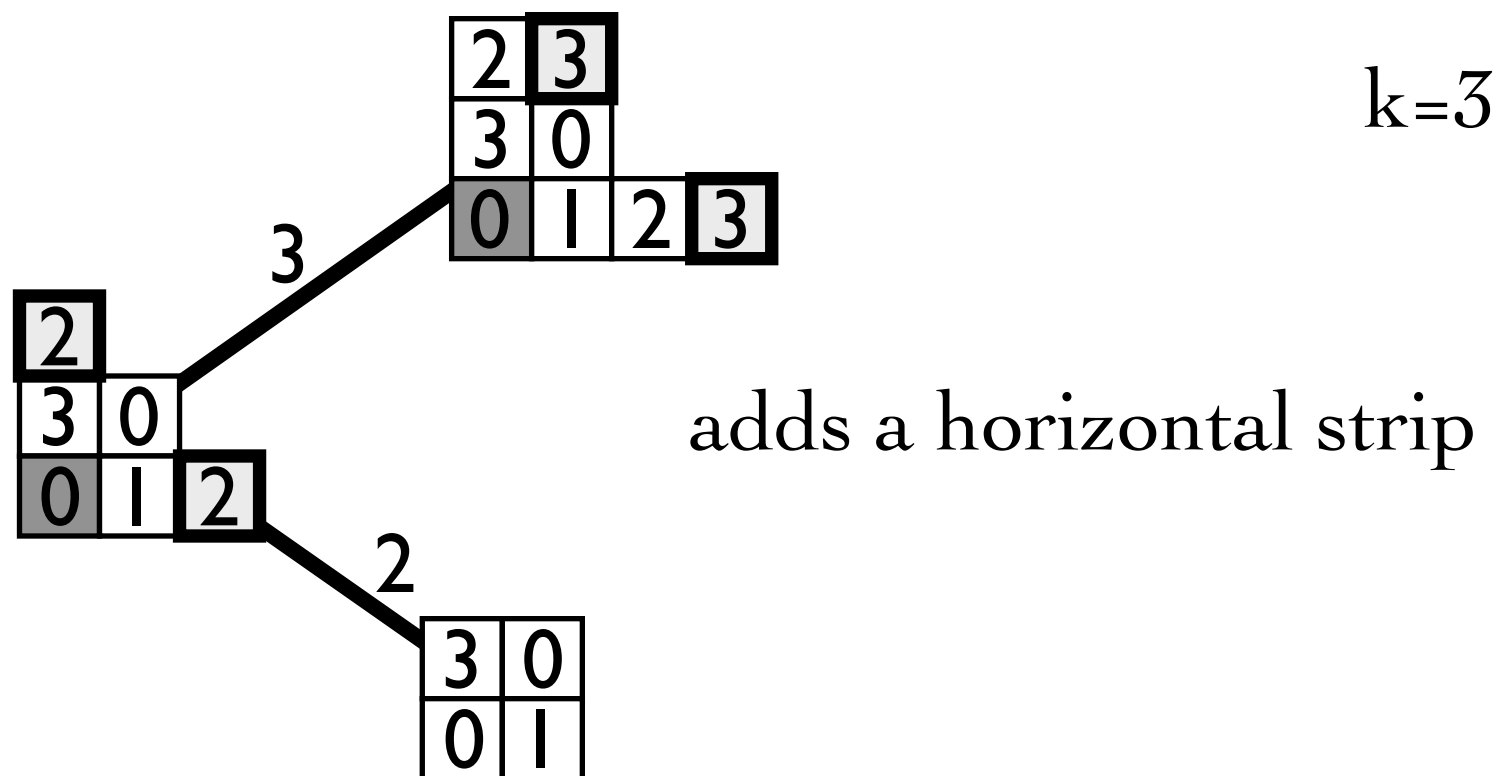
$k=2$



the result is 0 otherwise

$u_A$  cyclically decreasing word  
with content

if  $i, i + 1 \in A$   $u_i$  comes before  $u_{i+1}$



acting by all cyclically decreasing words adds  
all possible horizontal strips

$$\mathbf{h}_r(\gamma) = \sum_{\nu} \nu$$

$$\Lambda^{(k)} = \mathbb{Q}[h_1, h_2, \dots, h_k] \simeq \mathbb{Q}[\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_k]$$

$$u : \mathbb{Q}[h_1, h_2, \dots, h_k] \rightarrow \mathbb{Q}[\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_k]$$

$$\mathbf{s}_\lambda^{(k)} = u(s_\lambda^{(k)})$$

Say then that we determine:

$$\mathbf{s}_\lambda^{(k)} = \sum_w c_w w$$

$w$  is in the affine Nil-Coxeter algebra

$c_w$  coefficients

k-Littlewood-Richardson coefficients:

$$\mathbf{s}_{\lambda}^{(k)} \mathbf{s}_{\mu}^{(k)} = \sum_{\nu} c_{\lambda\mu}^{\nu(k)} \mathbf{s}_{\nu}^{(k)}$$

Viewing this in terms of actions on cores:

$$\mathbf{s}_{\mu}^{(k)} \emptyset = \mathbf{c}(\mu)$$

$$\mathbf{s}_{\lambda}^{(k)} \mathbf{c}(\mu) = \sum_{\nu} c_{\lambda\mu}^{\nu(k)} \mathbf{c}(\nu) \quad \text{with} \quad \mathbf{s}_{\lambda}^{(k)} = \sum_w c_w w$$

$c_{\lambda\mu}^{\nu(k)}$  is equal to  $c_w$  if there exists a  $w$  s.t.  
 $w\mathbf{c}(\mu) = \mathbf{c}(\nu)$

We haven't come up with a  $k$ -LR rule, but can reduce it to a more manageable problem

Let  $R$  be a rectangle with hook =  $k$

$$s_R s_{\lambda}^{(k)} = s_{R \cup \lambda}^{(k)}$$

$$s_{\lambda}^{(k)} = s_{R_1} s_{R_2} \cdots s_{R_d} s_{\tilde{\lambda}}^{(k)}$$

where each of the  $R_i$  are rectangles with hook =  $k$   
and the partition  $\tilde{\lambda}$  contains less than  $k+2-r$  parts of size  $r$


combinatorial formula #1

$$R = ((k + 1 - r)^r)$$

$v_\lambda$  is a reading of the  
grey = contents +  $k+1-r$   
white = contents

$$s_R = \sum_{\lambda \subseteq R} v_\lambda$$

**Example:**  $k=4$   $R=(2,2,2)$

3	4
4	0
0	1

3	4
4	0
2	1

3	4
4	0
2	3

3	4
1	0
2	1

3	4
1	0
2	3

0	4
1	0
2	1

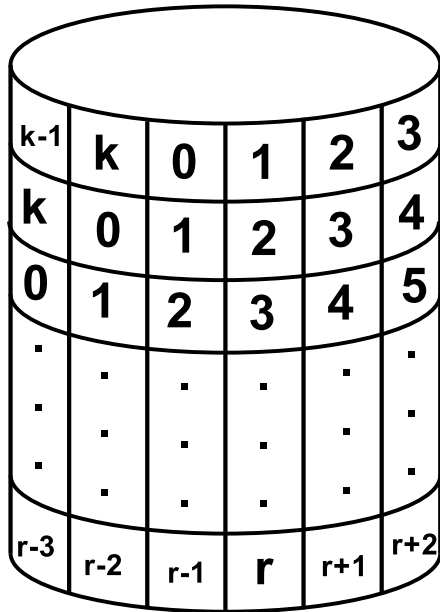
3	4
1	2
2	3

0	4
1	0
2	3

0	4
1	2
2	3

0	1
1	2
2	3

$$u_4 u_3 u_0 u_4 u_1 u_0 + u_2 u_4 u_3 u_0 u_4 u_1 + u_3 u_2 u_4 u_3 u_0 u_4 + u_1 u_2 u_4 u_3 u_0 u_1 + u_1 u_3 u_2 u_4 u_3 u_0 \\ + u_2 u_1 u_3 u_2 u_4 u_3 + u_0 u_1 u_2 u_4 u_0 u_1 + u_0 u_1 u_3 u_2 u_4 u_0 + u_0 u_2 u_1 u_3 u_2 u_4 + u_1 u_0 u_3 u_1 u_3 u_2$$



combinatorial formula #2

$$R = ((k + 1 - r)^r)$$

$$s_R = \sum_{|A|=k+1-r} u_A u_{A+1} u_{A+2} \cdots u_{A+r-1}$$

**Example:**  $k=4$   $R=(2,2,2)$

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

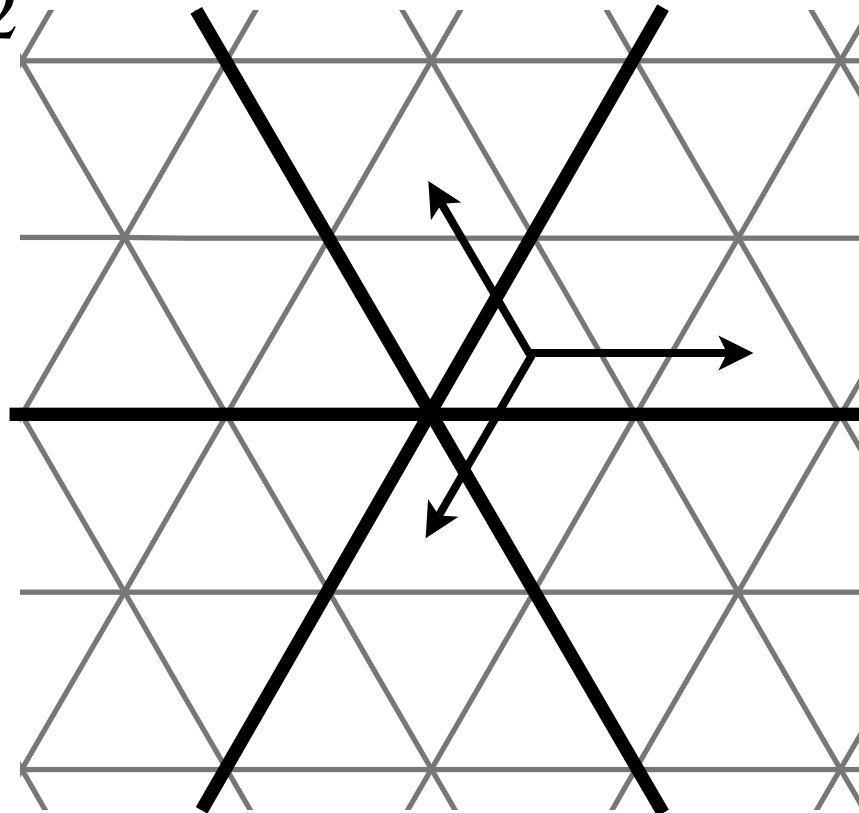
## More geometric formula

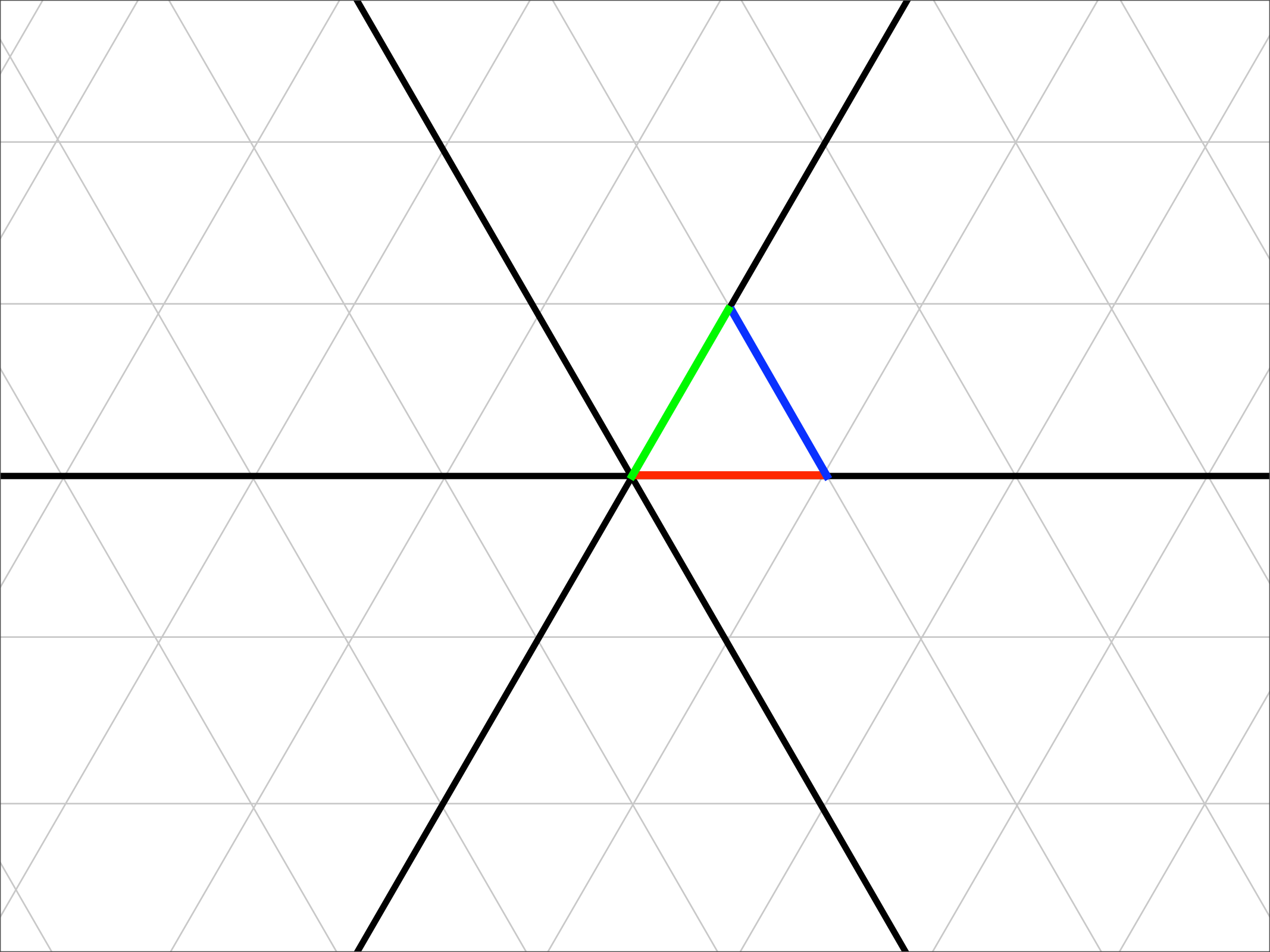
$$\Gamma = \{(a_1, a_2, \dots, a_{k+1}) : a_i \in \{0, 1\}, \sum a_i = k + 1 - r\}$$

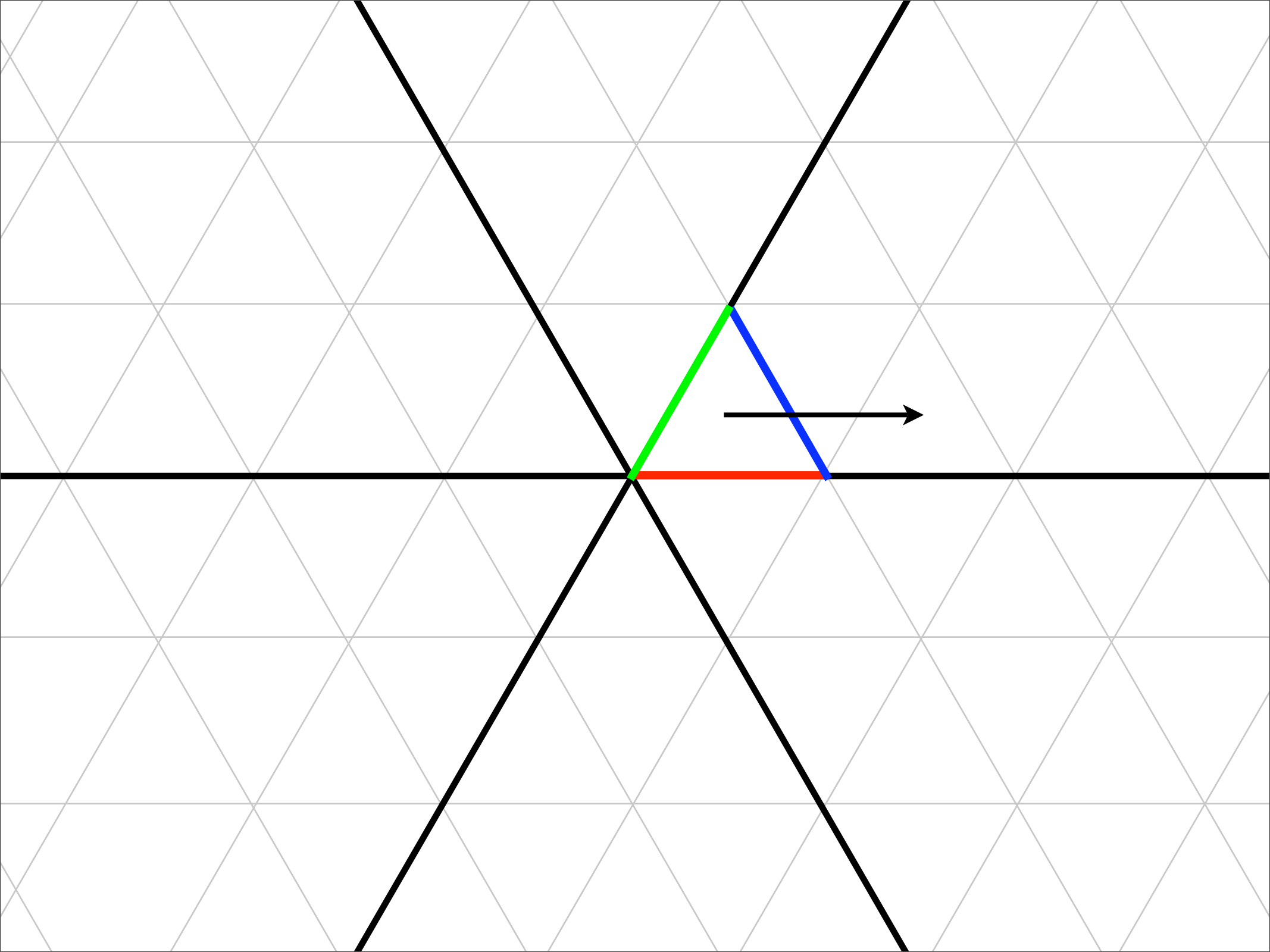
$$s_R = \sum_{\gamma \in \Gamma} \text{pseudo-translation by } \gamma$$

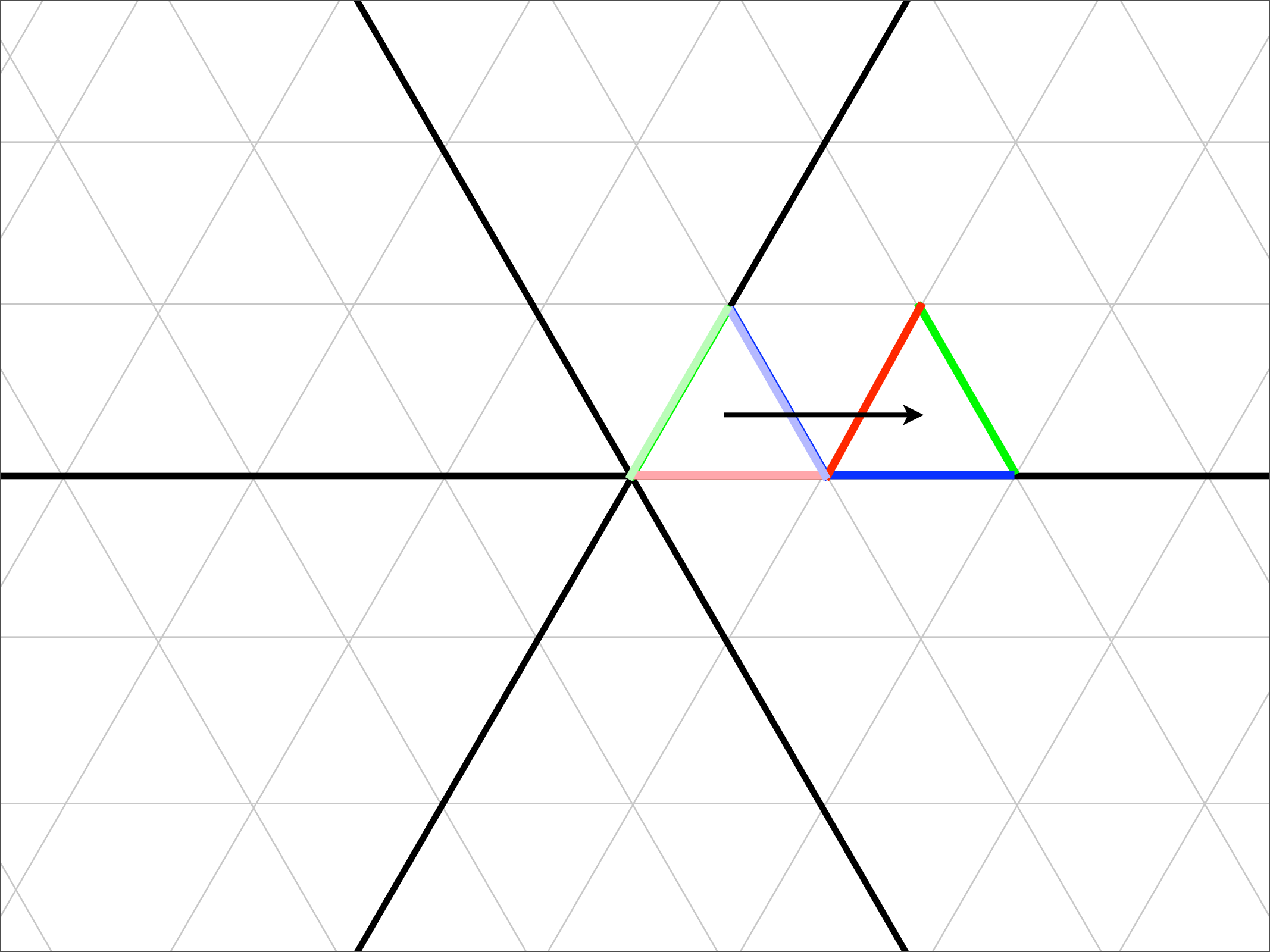
**Example:**  $k=2$

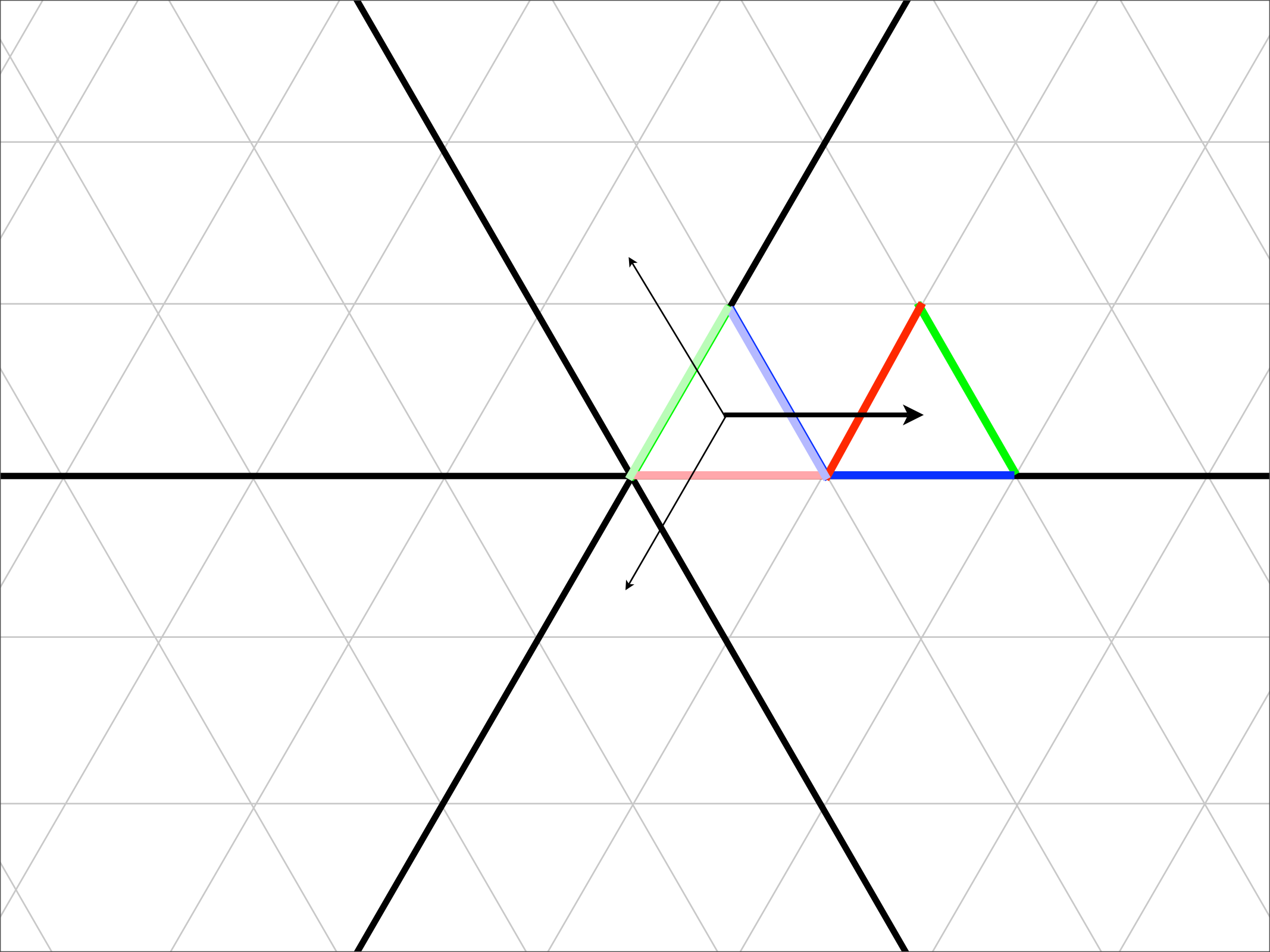
$$R = \begin{array}{|c|c|} \hline & \\ \hline \end{array}$$

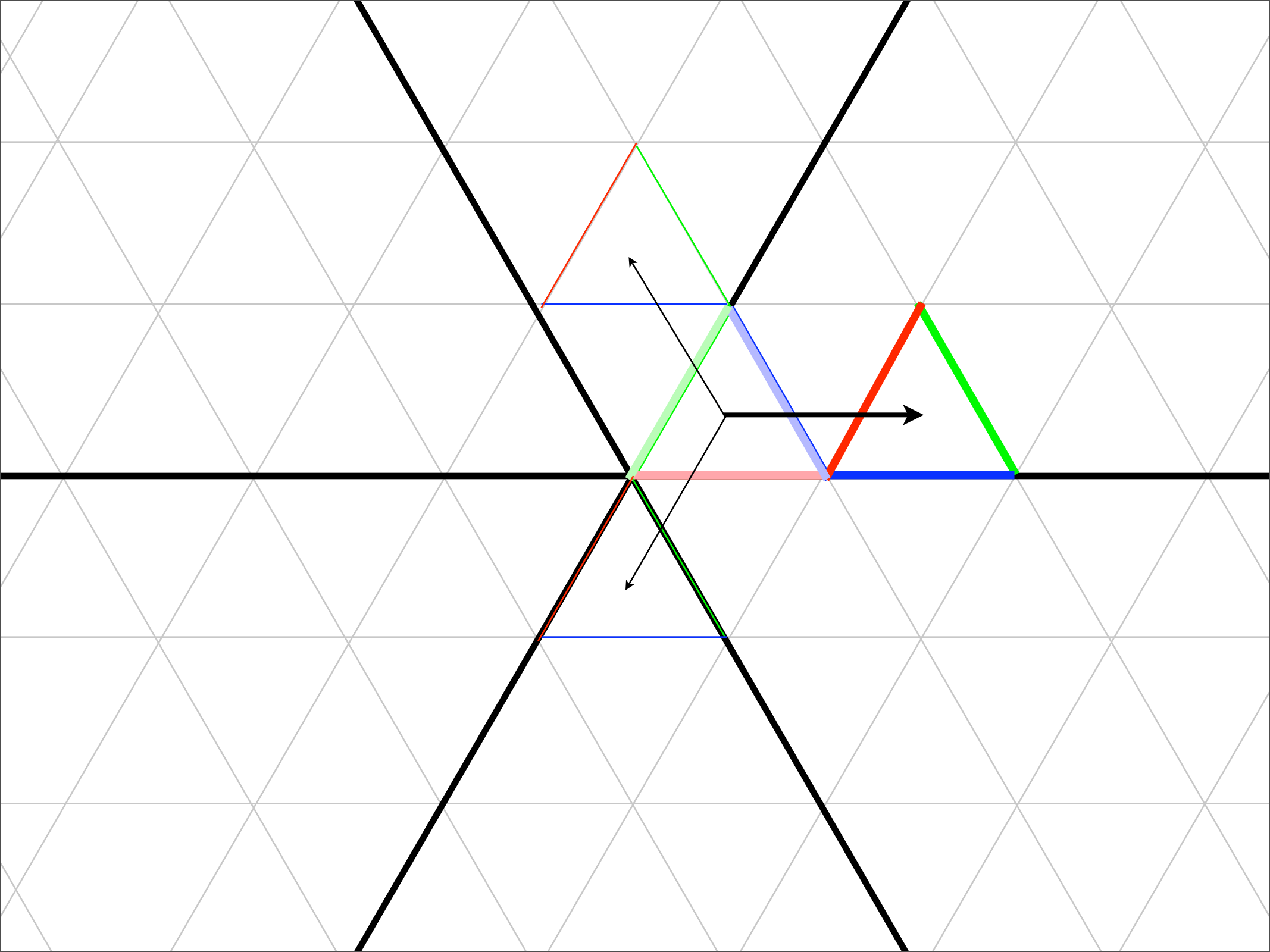


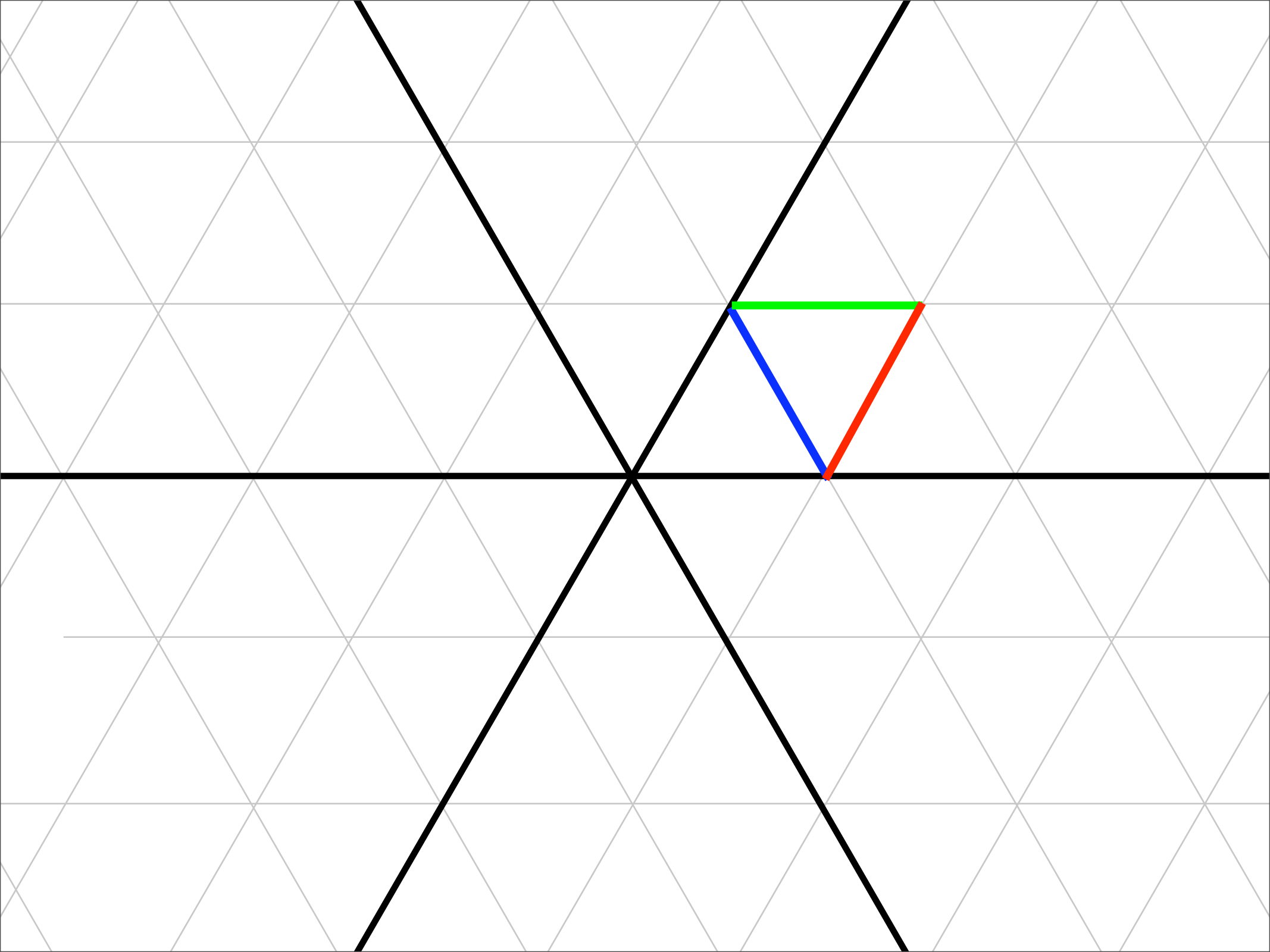


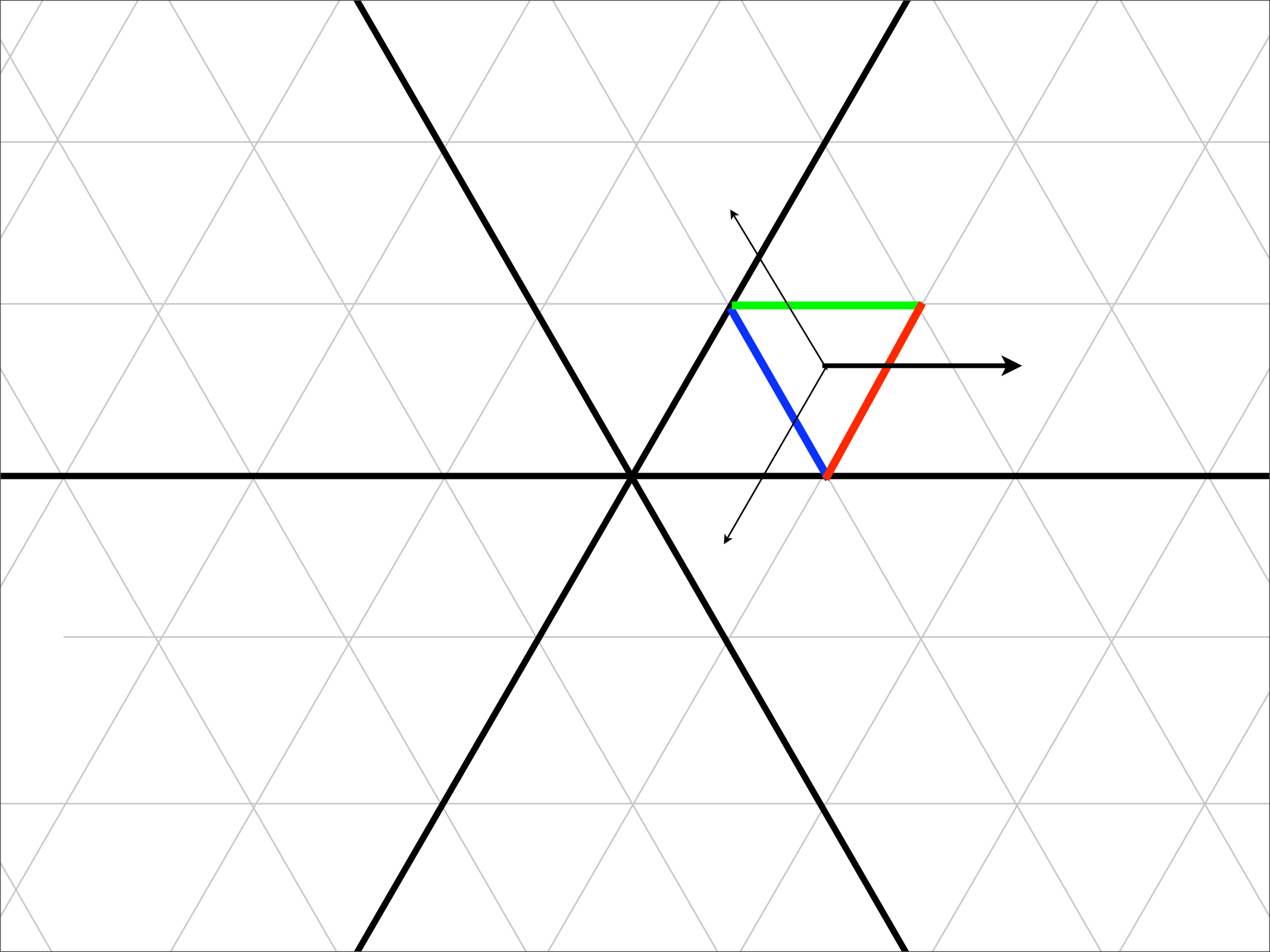


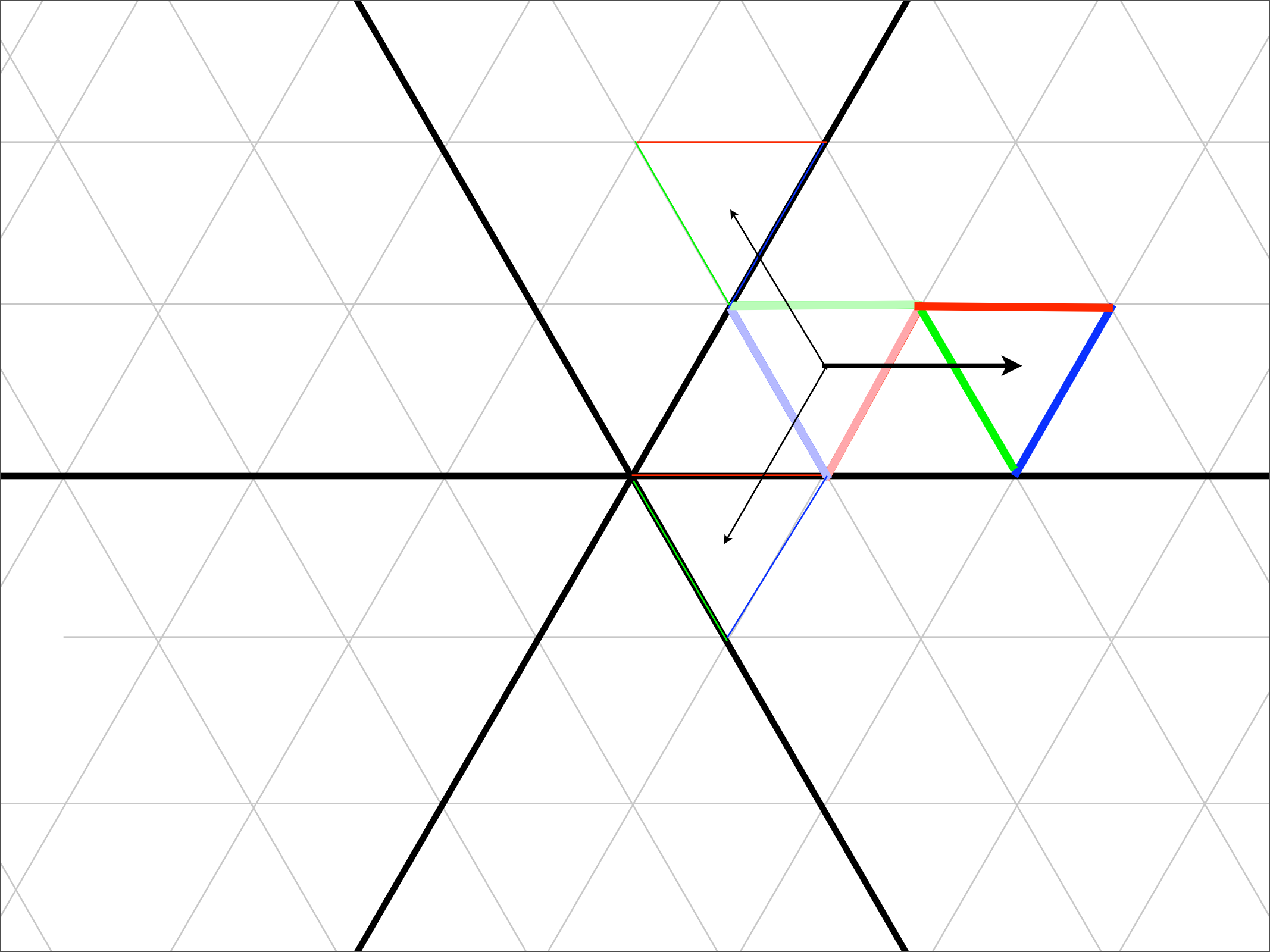












So then what remains to give an explicit  $k$ -Littlewood Richardson rule is to give more explicit formulas for  $S_{\tilde{\lambda}}^{(k)}$  where  $\tilde{\lambda}$  contains no rectangles with a  $k$ -hook.

For a fixed  $k$  there are  $k!$  such partitions.

END!